Convergence Analysis of a Streamline Diffusion Method for a Singularly Perturbed Convection-diffusion Problem

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Abstract: A streamline diffusion finite element method (SDFEM) is applied to a singularly perturbed convection-diffusion two-point boundary value problem in conservative form. The stability and accuracy of the SDFEM on arbitrary grids are studied. We derive the pointwise error estimates and the approximation of derivatives. These bounds are then made explicit for the particular cases of Shishkin-type meshes. Numerical experiments support our theoretical results.

Key–Words: Convection-diffusion, singular perturbation, streamline diffusion, Shishkin-type mesh

1 Introduction

Differential equations with a small parameter ε multiplying the highest order derivative terms are said to be singularly perturbed and normally boundary layers occur in their solutions. Singularly perturbed differential equations arise frequently in many applied areas which include fluid dynamics, quantumn mechanics, chemical reactions, and electrical networks. For the past two decades an extensive research has been made on numerical methods for the singularly perturbed differential equation, see [1,2] and reference their in .

It has been numerically observed that the streamline-diffusion finite element method (SDFEM) [3,4] often give a good and stable approximation of singularly perturbed boundary value problem if the grid is properly adapted to capture the singularity of the solution such as sharp layers. In this paper, we give a careful analysis of this phenomenon and develop a deeper understanding of the behavior of the SDFEM. The model problem we will study in this paper is a linear convection-diffusion problem in conservative form:

$$-\varepsilon u^{''}(x) - (b(x)u(x))^{'} = f(x),\ x \in (0,1), \quad (1)$$

$$u(0) = \gamma_0, \quad u(1) = \gamma_1, \quad (2)$$

where ε is a small positive parameter, $b(x)$ and $f(x)$ are sufficiently smooth, γ_0 and γ_1 are given constants, and for $0 \le x \le 1$ we assume that $b(x) \ge \beta > 0$. The solution $u(x)$ of (1)-(2) typically has a boundary layer at $x = 0$ and its derivatives can be bounded by

$$|u^{(k)}(x)| \le C(1 + \varepsilon^{-k} \exp(-\beta x/\varepsilon)) \quad (3)$$

for $k = 0, 1, 2, 3,\ x \in [0, 1]$, see [5].

The SDFEM, introduced first by Hughes and Brooks in [6], is one of such stabilized methods which combines good stability properties with high accuracy. Many convergence estimates of the SDFEM [7,8,9] have been done for quasiuniform meshes which show that the SDFEM is able to capture the main feature of the solution without layer-adapted meshes. However, very few ε uniform convergence results are obtained inside the boundary layer. We first analyze the SDFEM for the singularly perturbed problem (1)-(2) on arbitrary meshes. We derive the pointwise error estimates and the approximation of derivatives. These bounds are then made explicit for the Shishkin-type meshes.

An outline of our paper is as follows: In section 2 we will describe the SDFEM and solve the corresponding error equation to analyze the stability and accuracy of this scheme. In section 3 we derive the pointwise error estimates and the approximation of derivatives on Shishkin-type meshes. In section 4 we analyze the stability of the third-order problem. Finally numerical results that support our theoretical bounds are presented in section 5.

2 Stability analysis of the SDFEM

In this section, we will study the stability of the SDFEM applied to equation (1)-(2) on arbitrary grids.

Let $H^1 = \{v, v' \in L^2\}$ and $H_0^1 = \{v|v \in H^1, v(0) = v(1) = 0\}$. The weak solution to (1)-(2) is a function $u \in H^1$ satisfies $u(0) = \gamma_0, u(1) = \gamma_1$ and

$$a(u, v) = (f, v), \quad \forall v \in H_0^1, \tag{4}$$

where $(\cdot, \cdot)$ is the L^2 inner product and

$$\begin{aligned} a(u,v) &= \varepsilon(u', v') + (bu, v') \\ &\quad + \sum_{i=1}^{N-1} \int_{x_{i-1}}^{x_i} \delta_i (f - Lu) b v' \mathrm{d}x, \end{aligned}$$

where δ_i is a stabilization function in $[x_{i-1}, x_i]$. We will discuss the choice of δ_i later.

Here we assume that all integrals can be evaluated exactly. If this is not the case, then a suitable quadrature rule must be used. The existence and uniqueness of the weak solution are easy to establish.

For a positive integer N, Let $\Omega^N = \{x_i | 0 = x_0 < x_1 < \cdots < x_N = 1\}$ be an arbitrary grid. We denote by $\varphi_i(x)$ the nodal basis function at point x_i and the finite element space $V^N = \{v^N = \sum_{i=0}^N v_i^N \varphi_i(x)\}$. The finite element discretization of (4) is to find a $u^N \in V^N$ such that $u^N(0) = \gamma_0, u^N(1) = \gamma_1$ and

$$a(u^N, v^N) = (f, v^N), \quad \forall\, v^N \in V^N \cap H_0^1. \tag{5}$$

Let $e(x) = (u^I - u^N)(x) = \sum e_i \varphi_i$ with $e_i = e(x_i), i = 1, 2, \cdots, N-1$, where u^I denote the piecewise linear interpolation on the given mesh. Since $a(u - u^N, v^N) = 0$, we obtain the error equation

$$a(e, \varphi_i) = a(u^I - u, \varphi_i),\ i = 1, 2, \cdots, N-1, \tag{6}$$

$$e_0 = e_{N+1} = 0. \tag{7}$$

Let $a_{ij} = a(\varphi_j, \varphi_i)$ and $h_i = x_i - x_{i-1}$. A routine calculation shows that for $i = 1, 2, \cdots, N-1$

$$\begin{aligned} a(e, \varphi_i) &= a_{i,i-1} e_{i-1} + a_{i,i} e_i + a_{i,i+1} e_{i+1} \\ &\quad + \bar{f}_i - \bar{f}_{i+1}, \end{aligned}$$

where

$$\begin{aligned} a_{i,i-1} &= -\frac{\varepsilon}{h_i} + \frac{b_{i-1/2}}{2} - \frac{\delta_i}{h_i} b_{i-1} b_{i-1/2}, \\ a_{i,i} &= \frac{\varepsilon}{h_i} + \frac{\varepsilon}{h_{i+1}} + \frac{b_{i-1/2} - b_{i+1/2}}{2} \\ &\quad + \left(\frac{\delta_i b_{i-1/2}}{h_i} + \frac{\delta_{i+1} b_{i+1/2}}{h_{i+1}}\right) b_i, \\ a_{i,i+1} &= -\frac{\varepsilon}{h_{i+1}} - \frac{b_{i+1/2}}{2} - \frac{\delta_{i+1}}{h_{i+1}} b_{i+1} b_{i+1/2}, \\ \bar{f}_i &= h_i^{-1} \int_{x_{i-1}}^{x_i} \delta_i f(x) b(x) \mathrm{d}x, \\ b_{i-1/2} &= b\left(\frac{x_{i-1} + x_i}{2}\right). \end{aligned}$$

When ε is small relative to the local meshsize, a standard way of stabilizing this scheme is to choose δ_i according to the formula $\delta_i = h_i/(2b_{i-1})$. If the local meshsize is small enough—in particular, if $b_{i-1/2} h_i < 2\varepsilon$—then the standard Galerkin method works well, so it is possible to choose $\delta_i = 0$. Thus, to stabilize the scheme, we choose

$$\delta_i = \begin{cases} 0 & \text{if } b_{i-1/2} h_i < 2\varepsilon, \\ h_i/(2b_{i-1}) & \text{if } b_{i-1/2} h_i \geq 2\varepsilon. \end{cases} \tag{8}$$

Lemma 1. The error equation (6)-(7) can be written as

$$A^N e_i - A^N e_{i+1} = r_i - r_{i+1},\ 1 \leq i < N, \tag{9}$$

$$e_0 = e_N = 0, \tag{10}$$

where

$$\begin{aligned} A^N e_i &= \left(\frac{\varepsilon}{h_i} + \frac{b_{i-1/2}}{2} + \frac{\delta_i}{h_i} b_i b_{i-1/2}\right) e_i - \left(\frac{\varepsilon}{h_i}\right. \\ &\quad \left. - \frac{b_{i-1/2}}{2} + \frac{\delta_i}{h_i} b_{i-1} b_{i-1/2}\right) e_{i-1} \end{aligned} \tag{11}$$

and

$$\begin{aligned} r_i &= h_i^{-1} \left[\int_{x_{i-1}}^{x_i} (u^I - u)(x) b(x) \mathrm{d}x \right. \\ &\quad + \int_{x_{i-1}}^{x_i} \delta_i \varepsilon u'' b(x) \mathrm{d}x \\ &\quad \left. - \int_{x_{i-1}}^{x_i} \delta_i (b(x)(u^I - u)(x))' b(x) \mathrm{d}x \right]. \end{aligned} \tag{12}$$

Proof. Clearly,

$$\begin{aligned} & A^N e_i - A^N e_{i+1} = a_{i,i-1} e_{i-1} - a_{i,i} e_i \\ & + a_{i,i+1} e_{i+1} = a(e, \varphi_i) - (\bar{f}_i - \bar{f}_{i+1}) \\ & = a(u^I - u, \varphi_i) - (\bar{f}_i - \bar{f}_{i+1}). \end{aligned}$$

Note that $\int_0^1 (u^I - u)' \varphi_i' \mathrm{d}x = 0$,

$$a(u^I - u, \varphi_i) = \int_{x_{i-1}}^{x_{i+1}} b(x)(u^I - u)(x)\varphi_i'(x)\mathrm{d}x$$
$$+ \int_{x_{i-1}}^{x_i} \delta_i(f - L(u^I - u))b(x)\varphi_i'(x)\mathrm{d}x$$
$$+ \int_{x_i}^{x_{i+1}} \delta_{i+1}(f - L(u^I - u))b(x)\varphi_i'(x)\mathrm{d}x$$
$$= r_i - r_{i+1} + (\bar{f}_i - \bar{f}_{i+1})$$

and the desired result follows from this.

It is easy to see that $A^N e_i = r_i + C$ with an appropriate constant C such that $e_0 = e_N = 0$. However it is difficult to determine C explicitly. Instead we use the following splitting of e_i.

Lemma 2.

$$e_i = W_i - \frac{V_i}{V_N} W_N, \tag{13}$$

where V is the solution of the difference equation

$$A^N V_i = 1, \quad i = 1, 2, \cdots, N, \quad V_0 = 0,$$

and W is the solution of the difference equation

$$A^N W_i = r_i, \quad i = 1, 2, \cdots, N, \quad W_0 = 0.$$

Proof. It is clear that $e_i = W_i - CV_i$. Since $e_N = 0$, we get $C = W_N/V_N$.

The matrix associated with A^N is a bidiagonal M-matrix. Consequently one can use suitable barrier functions and the definitions of $\{V_i\}$ and $\{W_i\}$ to show that

$$0 \le V_i \le 1, \qquad |W_i| \le \|r\|_\infty V_i \tag{14}$$

for $i = 0, 1, 2, \cdots, N$. Thus, we have

$$|e_i| \le |W_i| + |\frac{W_N}{V_N} V_i| \le 2\|r\|_\infty \tag{15}$$

for $i = 1, 2, \cdots, N$.

Furthermore,

$$A^N e_i = A^N W_i - \frac{W_N}{V_N} A^N V_i = r_i - \frac{W_N}{V_N}. \tag{16}$$

From (16) and (14) we have

$$|A^N e_i| \le 2\|r\|_\infty \quad \text{for} \quad i = 1, 2, \cdots, N. \tag{17}$$

Since

$$\begin{aligned} A^N e_i &= \varepsilon D^- e_i + b_{i-1/2} \frac{e_i + e_{i-1}}{2} \\ &\quad + \delta_i b_{i-1/2} \frac{b_i e_i - b_{i-1} e_{i-1}}{h_i}, \end{aligned}$$

we obtain

$$\varepsilon |D^- e_i| \le C\|r\|_\infty \quad \text{for} \quad i = 1, 2, \cdots, N, \tag{18}$$

where we have used (17) and (15).

Now we can bound the pointwise errors in the computed solution and the ε-weighted errors $\varepsilon D^-(u_i - u_i^N)$.

Theorem 1. There exist constants C such that

$$|u_i - u_i^N| + \varepsilon |D^-(u_i - u_i^N)| \le C\|r\|_\infty \tag{19}$$

for $i = 1, 2, \cdots, N$.

Proof. From (15) we have

$$|u_i - u_i^N| = |u_i^I - u_i^N| \le C\|r\|_\infty \tag{20}$$

for $i = 1, 2, \cdots, N$.

Similarly, from (18) we have

$$\begin{aligned} \varepsilon |D^-(u_i - u_i^N)| &\le \varepsilon |D^-(u_i - u_i^I)| + \varepsilon |D^- e_i| \\ &\le C\|r\|_\infty \end{aligned} \tag{21}$$

for $i = 1, 2, \cdots, N$.

Combining (20) with (21), we get the desired results.

3 Analysis on Shishkin-type meshes

In this section let N be an even integer. We shall consider a mesh Ω^N that is equidistant in $[x_{N/2}, 1]$ but graded in $[0, x_{N/2}]$, where we choose the transition point $x_{N/2}$ as Shishkin does:

$$x_{N/2} = \tau = \frac{2\varepsilon}{\beta} \ln N. \tag{22}$$

On $[0, x_{N/2}]$ let our mesh be given by a mesh-generating function φ, with $\varphi(0) = 0$ and $\varphi(1/2) = \ln N$, where φ is continuous, monotonically increasing and piecewise continuously differentiable. Then our mesh is

$$x_i = \begin{cases} \frac{2\varepsilon}{\beta} \varphi(t_i) \;\; t_i = i/N, & 0 \le i \le N/2, \\ 1 - (1 - \frac{2\varepsilon}{\beta} \ln N) \frac{2(N-i)}{N}, & N/2 < i \le N. \end{cases}$$

We define a new function ψ by $\psi(t) = \exp(-\varphi(t)),\ t \in [0, 1/2]$. This function is monotonically decreasing with $\psi(0) = 1$ and $\psi(1/2) = N^{-1}$. Examples of the mesh-characterizing function ψ are

$$\psi(t) = 1 - 2(1 - N^{-1})t$$

for Bakhvalov-Shishkin mesh and

$$\psi(t) = e^{-2(\ln N)t}$$

for standard Shishkin mesh.

For Shishkin-type meshes we have the following general result [10].

Lemma 3. Let us assume that the mesh-generating function φ is piecewise differentiable and that it satisfies the condition

$$\max_{x\in[0,1/2]} \varphi^{'}(x) = \max_{x\in[0,1/2]} \frac{|\psi^{'}|}{\psi} \leq CN. \qquad (23)$$

Then

$$\begin{aligned} \vartheta_k(\Omega^N) &= \max_{i=1,\cdots,N} \int_{x_{i-1}}^{x_i} [1 + \\ &\quad +\varepsilon^{-1}\exp(-\beta x/(k\varepsilon))]\mathrm{d}x \\ &\leq C\{\varepsilon + N^{-1}\max_{x\in[0,1/2]} |\psi^{'}(x)|\} \qquad (24) \end{aligned}$$

for $k = 1, 2, \cdots$.

The following interpolation error estimate for Shishkin-type meshes is well known; see for example [11].

lemma 4. Assume that the piecewise differentiable mesh generating function φ satisfies (23). Then the interpolation error for linear interpolation on the Shishkin-type meshes satisfies

$$|(u-u^I)(x)| \leq \begin{cases} C(N^{-1}\max|\psi^{'}|)^2, & x \in [0, x_{N/2}], \\ CN^{-2}, & x \in [x_{N/2}, 1]. \end{cases}$$

The next lemma gives us a useful estimate for r_i on Shishkin-type meshes.

Lemma 5. Assume that the condition (23) holds true. Then on Shishkin-type meshes we have

$$|r_i| \leq C(N^{-1}\max|\psi^{'}|)^2 \quad \text{for} \quad i = 1, 2, \cdots, N.$$

Proof. From (12) we have

$$\begin{aligned} |r_i| &\leq |h_i^{-1}\int_{x_{i-1}}^{x_i} (u^I - u)(x)b(x)\mathrm{d}x| \\ &\quad +|h_i^{-1}\int_{x_{i-1}}^{x_i} \delta_i\varepsilon u^{''}(x)b(x)\mathrm{d}x| \\ &\quad +|h_i^{-1}\int_{x_{i-1}}^{x_i} \delta_i(b(x)(u^I-u)(x))^{'}b(x)\mathrm{d}x| \\ &\leq C\max_{x_{i-1}\leq x\leq x_i} |(u^I-u)(x)| \\ &\quad +C\delta_i\varepsilon h_i^{-1}\int_{x_{i-1}}^{x_i}(1+\varepsilon^{-2}\exp(-\beta x/\varepsilon))\mathrm{d}x \\ &\quad +C\delta_i h_i^{-1}|b_i(u^I-u)(x_i) \\ &\quad -b_{i-1}(u^I-u)(x_{i-1})| \\ &\leq C\max_{x_{i-1}\leq x\leq x_i} |(u^I-u)(x)| + C\varepsilon\delta_i \\ &\quad +C\delta_i h_i^{-1}\exp(-\beta x_{i-1}/\varepsilon), \end{aligned}$$

where we have used (3).

Thus,using the lemma 4 and (8) we obtain

$$|r_i| \leq C(N^{-1}\max|\psi^{'}|)^2 \quad \text{for} \quad i = 1, 2, \cdots, N,$$

where we have used (22).

With the interpolation error estimates, we can get the convergence approximation of the SDFEM .

Theorem 2. Assume that the condition (23) holds true. Then on Shishkin-type meshes we have the following error estimates:

$$\begin{aligned} &|u_i - u_i^N| + \varepsilon|D^-(u_i - u_i^N)| \\ &\leq C(N^{-1}\max|\psi^{'}|)^2, \qquad (25) \end{aligned}$$

and

$$\max_{x_{i-1}\leq x\leq x_i} \varepsilon|D^-u_i^N - u^{'}(x)| \leq CN^{-1}\max|\psi^{'}|, (26)$$

$$\varepsilon|D^-u_i^N - u^{'}(x_{i-1/2})| \leq C(N^{-1}\max|\psi^{'}|)^2. \quad (27)$$

Proof. The first result follows immediately from Theorem 1 and Lemma 5.

Next, using a Taylor expansion for u and $u^{'}$ about x_i, we get

$$\begin{aligned} &\max_{x_{i-1}\leq x\leq x_i} \varepsilon|D^-u_i - u^{'}(x)| \leq C\varepsilon\int_{x_{i-1}}^{x_i} |u^{''}(x)|\mathrm{d}x \\ &\leq C\int_{x_{i-1}}^{x_i}(1+\varepsilon^{-1}\exp(-\beta x/\varepsilon))\mathrm{d}x \\ &\leq N^{-1}\max|\psi^{'}|, \end{aligned}$$

where we have used (3) and (24). Combining this inequality with the first result, we obtain the second result.

Finally, we use Taylor expansions for u and $u^{'}$ about x_i to obtain

$$\begin{aligned} &\varepsilon|\frac{u_i - u_{i-1}}{h_i} - u^{'}_{i-1/2}| \\ &\leq \frac{3\varepsilon}{2}\int_{x_{i-1}}^{x_i} |u^{'''}(t)|(t-x_{i-1})\mathrm{d}t \\ &\leq \frac{3\varepsilon}{2}\int_{x_{i-1}}^{x_i} (t-x_{i-1})(1+\varepsilon^{-3}\exp(-\beta t/\varepsilon))\mathrm{d}t \end{aligned}$$

by (3). To bound the right-hand side we use the inequality in [12]

$$\int_{x_{i-1}}^{x_i} g(\xi)(\xi - x_{i-1})\mathrm{d}\xi \leq \frac{1}{2}\{\int_{x_{i-1}}^{x_i} g(\xi)^{1/2}\}^2$$

which holds true for any positive monotonically decreasing function g on $[x_{i-1}, x_i]$. This can be easily verified by considering the two integrals as functions of the upper integration limit. We get

$$\varepsilon|\frac{u_i - u_{i-1}}{h_i} - u'_{i-1/2}| \leq C[\int_{x_{i-1}}^{x_i} (1 + \varepsilon^{-1}\exp(-\beta t/(2t)))\mathrm{d}t]^2 \leq C(N^{-1}\max|\psi'|)^2$$

by (24). Combining this inequality with the first result, we obtain the third result.

4 Analysis the stability of the third-order problem

In this section, we treat the following stability of discrete scheme for the third-order singularly perturbed ordinary differential equations

$$-\varepsilon y'''(x) - a(x)y''(x) + b(x)y'(x) - c(x)y(x) = f(x), \ \ x \in D, \quad (28)$$

$$y(0) = p, \ y'(0) = q, \ y'(1) = r, \quad (29)$$

where $0 < \varepsilon \ll 1$ is a small positive parameter,$a(x), b(x), c(x)$ and $f(x)$ are sufficiently smooth functions satisfying the following conditions:

$$a(x) \geq \alpha > 0,$$
$$b(x) \geq 0,$$
$$0 \geq c(x) \geq -\gamma, \ \ \gamma > 0,$$
$$\alpha - \gamma(1 + 3\eta) \geq \eta' > 0 \text{ for some } \eta \text{ and } \eta',$$

with $D = (0, 1), D_0 = (0, 1], \bar{D} = [0, 1]$ and $y \in C^{(3)}(D) \cap C^{(1)}(\bar{D})$.

The aim of this section is to illustrate an application of a priori estimates of the solutions of discrete problems, which are obtained using Green's function, to analyze the accuracy of finite difference schemes in the discrete maximum norm.

The singularly perturbed boundary value problem (28)-(29) can be transformed into an equivalent problem of the form

$$\mathbf{Ay} = \mathbf{F} \Longleftrightarrow \begin{cases} P_1\mathbf{y} \equiv y_1'(x) - y_2(x) = 0, \\ P_2\mathbf{y} \equiv -\varepsilon y_2''(x) - a(x)y_2'(x) \\ +b(x)y_2(x) + c(x)y_1(x) = f(x), \\ y_1(0) = p, \ y_2(0) = q, \ y_2(1) = r, \end{cases} \quad (30)$$

where $\mathbf{y} = (y_1, y_2)$.

Lemma 6. (Maximum principle [13]) Consider the boundary value problem (30). Assume that $P_1\mathbf{u} \geq 0, P_2\mathbf{u} \geq 0$ in D,$u_1(0) \geq 0, u_2(0) \geq 0$, and $u_2(1) \geq 0$. Then $\mathbf{u}(x) \geq 0$ in $[0, 1]$. Here $\mathbf{u}(x) = (u_1(x), u_2(x))$ for all $x \in \bar{D}$.

Lemma 7. (Stability result [13]) Consider the boundary value problem (30). If $\mathbf{y}$ is a smooth function, then

$$\|\mathbf{y}(x)\| \leq C\max\{|y_1(0)|, |y_2(0)|, |y_2(1)|, \max_{x\in\bar{D}}|P_1\mathbf{y}|, \max_{x\in\bar{D}}|P_2\mathbf{y}|\}$$

for all $x \in \bar{D}$, where $\|\mathbf{y}(x)\| = \max\{|y_1(x)|, |y_2(x)|\}$.

The construction of layer-adapted meshes and the analysis of numerical methods for singularly perturbed problems require precise knowledge about the behavior of the derivatives of the exact solution.The following lemma provides that information.

Lemma 8. If $a(x), b(x), c(x)$ and $f(x) \in C^{(j)}(\bar{D})$,then the solution $\mathbf{y}(x)$ of (28)-(29) has the representation $\mathbf{y} = \mathbf{v} + \mathbf{w}$ on $[0, 1]$, where the smooth part $\mathbf{v}$ satisfies

$$P_1\mathbf{v}(x) = 0, P_2\mathbf{v}(x) = f(x)$$

and

$$\|\mathbf{v}^{(k)}(x)\| \leq C, \ \text{ for all } \ k \leq j \ , \ x \in \bar{D},$$

while the layer part $\mathbf{w}$ satisfies

$$P_1\mathbf{w}(x) = 0, P_2\mathbf{w}(x) = 0,$$
$$\|\mathbf{w}(0)\| \leq C, \|\mathbf{w}(1)\| \leq C\exp(-\alpha/\varepsilon)$$

and

$$|w_1^{(k)}(x)| \leq C\varepsilon^{1-k}\exp(-\alpha x/\varepsilon),$$
$$|w_2^{(k)}(x)| \leq C\varepsilon^{-k}\exp(-\alpha x/\varepsilon)$$

for all $k \leq j, x \in \bar{D}$.

Proof. Following the method of proof used in [1] and using Lemma 6 we can derive the desired estimates.

Now we consider the upwind difference scheme

$$P_1^N\mathbf{y}_i^N \equiv Dy_{1,i}^N - y_{2,i}^N = 0, \quad (31)$$

$$P_2^N\mathbf{y}_i^N \equiv -\varepsilon D^+D^-y_{2,i}^N - a_iDy_{2,i}^N + b_iy_{2,i}^N + c_iy_{1,i}^N = f_i, \ i = 1, 2, \cdots, N-1, \quad (32)$$

$$y_{1,0}^N = p, \ y_{2,0}^N = q, \ y_{2,N}^N = r, \quad (33)$$

where

$$D^+v_i = \frac{v_{i+1} - v_i}{h_{i+1}}, \ D^-v_i = \frac{v_i - v_{i-1}}{h_i},$$

$$Dv_i = \frac{v_{i+1} - v_i}{\hbar_i} \ \text{ and } \ \hbar_i = \frac{h_i + h_{i+1}}{2}, \hbar_0 = h_1.$$

Analogous to the continuous problem (30), we can give results for the discrete problem.

Lemma 9. (Discrete maximum principle [13]) Consider the discrete problem (31)-(33). If $y_{1,0} \geq 0, y_{2,0} \geq 0, y_{2,N} \geq 0, P_1^N \mathbf{y}_i \geq 0$ for $i = 0, 1, \cdots, N-1$, and $P_2^N \mathbf{y}_i \geq 0$ for $i = 1, 2, \cdots, N-1$, then $\mathbf{y}_i \geq 0$ for $i = 0, 1, \cdots, N$.

Lemma 10. (Stability result) If $\mathbf{y}_i$ is any mesh function, then

$$\begin{aligned}|y_{1,i}| &\leq C\max\{|y_{1,0}|, \max_{1\leq i\leq N-1}|P_1^N\mathbf{y}_i|\},\\ |y_{2,i}| &\leq C\max\{|y_{2,0}|, |y_{2,N}|, \max_{1\leq i\leq N-1}|P_2^N\mathbf{y}_i|\}\end{aligned}$$

for $i = 1, \cdots, N$.

For any mesh function w^N, we use $\|\cdot\|_\infty$ for the standard maximum norm, and we define a discrete L_1 norm by

$$\|w^N\|_1 = \sum_{i=1}^{N-1} \hbar_i |w_i^N|.$$

We also define the scalar product in $\mathbb{R}^{N+1}$ by

$$(v^N, w^N) = \sum_{j=1}^{N-1} v_j^N w_j^N \hbar_j, \ \ \forall v^N, w^N \in \mathbb{R}^{N-1}.$$

Consider the Green's function $\mathbf{G}^N(x_i, \xi_j)$ of problem (31)-(33). As a function of x_i for fixed ξ_j this function is defined by the relations

$$P_1^N \mathbf{G}^N(x_i, \xi_j) = 0\ ,\ \ x_i \in D_0^N, \xi_j \in D^N, \quad (34)$$

$$P_2^N \mathbf{G}^N(x_i, \xi_j) = \delta^N(x_i, \xi_j),$$
$$x_i \in D^N, \xi_j \in D^N, \quad (35)$$

$$G_1^N(0, \xi_j) = G_2^N(0, \xi_j)$$
$$= G_2^N(1, \xi_j) = 0,\ \xi_j \in D^N, \quad (36)$$

where

$$\delta^N(x_i, \xi_j) = \begin{cases} \hbar_i^{-1} & \text{for } x_i = \xi_j, \\ 0 & \text{for } x_i \neq \xi_j. \end{cases}$$

It is easy to see that using Green's function, we can give the following formula for the solution of problem (34)-(35)

$$y_{1,i}^N = \sum_{j=1}^{i-1} y_{2,i}^N + y_{1,0}^N, \quad (37)$$

$$y_{2,i}^N = \sum_{j=1}^{N-1} G_2^N(x_i, \xi_j) f_j \hbar_j\ ,\ x_i \in D^N. \quad (38)$$

Indeed,taking into account (34)-(35), we obtain

$$\begin{aligned}&(G_2^N(x_i, \xi_j), f_j) = (G_2^N(x_i, \xi_j), -\varepsilon D^+D^- y_{2,j}^N\\ &-a_j D y_{2,j}^N + b_j y_{2,j}^N + c_j y_{1,j}^N)\\ &= (P_2^N \mathbf{G}^N, y_{2,j}^N) + (G_2^N(x_i, \xi_j), c_j y_{1,j}^N)\\ &-(G_1^N(x_i, \xi_j), c_j y_{2,j}^N)\\ &= (\delta^N(x_i, \xi_j), y_{2,j}^N) + (D^+D^- G_1^N(x_i, \xi_j), c_j y_{1,j}^N)\\ &-(G_1^N(x_i, \xi_j), c_j y_{2,j}^N)\\ &= y_{2,i}^N + (G_1^N(x_i, \xi_j), c_j D^+D^- y_{1,j}^N)\\ &-(G_1^N(x_i, \xi_j), c_j y_{2,j}^N) = y_{2,i}^N, \ \text{for } \ x_i \in D^N.\end{aligned}$$

The Green's function $\mathbf{G}^N(x_i, \xi_j)$ as the function of a variable ξ_j for fixed x_i is the solution of the adjoint problem:

$$P_1^{N,*} \mathbf{G}^N(x_i, \xi_j) = 0\ ,\ \xi_j \in D_0^N, x_i \in D^N, \quad (39)$$

$$P_2^{N,*} \mathbf{G}^N(x_i, \xi_j) = \delta^N(x_i, \xi_j),$$
$$\xi_j \in D^N, x_i \in D^N, \quad (40)$$

$$G_1^N(x_i, 0) = G_2^N(x_i, 0)$$
$$= G_2^N(x_i, 1) = 0\ ,\ x_i \in D^N. \quad (41)$$

This arises from the following arguments: using (38),(34) and (35), and the fact that $P_1^{N,*}, P_2^{N,*}$ is adjoint to P_1^N, P_2^N respectively,we have

$$P_1^{N,*} \mathbf{G}^N(x_i, \xi_j) = 0$$

and

$$\begin{aligned}y_{2,i}^N &= \sum_{j=1}^{N-1} G_2^N(x_i, \xi_j) f_j \hbar_j\\ &= \sum_{j=1}^{N-1} G_2^N(x_i, \xi_j) P_2^N \mathbf{y}_j^N \hbar_j\\ &= \sum_{j=1}^{N-1} P_2^{N,*} \mathbf{G}^N(x_i, \xi_j) y_{2,j}^N \hbar_j\\ &\Longrightarrow P_2^{N,*} \mathbf{G}^N(x_i, \xi_j) = \delta^N(x_i, \xi_j),\end{aligned}$$

where we have used (39)-(40).

Lemma 11. The Green's function $\mathbf{G}^N(x_i, \xi_j)$ is nonnegative and bounded uniformly in ε:

$$0 \leq \mathbf{G}^N(x_i, \xi_j) \leq \frac{1}{\alpha - \gamma}.$$

Proof. From Lemma 9, we can easily get the nonnegativity of the Green's function.

We now wish to prove the upper bound.Let the point $\xi_{j_0} \in D^N$ be such that

$$\max_{\xi_j \in D^N} G_2^N(x_i, \xi_j) = G_2^N(x_i, \xi_{j_0}) \ , \ x_i \in D^N.$$

Multiply (40) by $\hbar_j$ and sum with respect to j from 1 to j_0.Taking into account that $G_2^N(x_i, 0) = 0$, we obtain

$$\begin{aligned}
&\sum_{j=1}^{j_0} P_2^{N,*}\mathbf{G}^N(x_i,\xi_j)\hbar_j = -\varepsilon D_\xi^+ G_2^N(x_i,\xi_{j_0}) \\
&+\varepsilon D_\xi^- G_2^N(x_i,\xi_1) \\
&+a_{j_0} G_2^N(x_i,\xi_{j_0}) + \sum_{j=1}^{j_0}(b_j G_2^N(x_i,\xi_j)\hbar_j \\
&+c_j G_1^N(x_i,\xi_j)\hbar_j). \qquad (42)
\end{aligned}$$

Because of the choice of ξ_{j_0},

$$\begin{aligned}
&D_\xi^+ G_2^N(x_i,\xi_{j_0}) = (G_2^N(x_i,\xi_{j_0+1}) \\
&-G_2^N(x_i,\xi_{j_0}))h_{j_0+1} \le 0, \qquad (43)
\end{aligned}$$

and as $G_2^N(x_i, \xi_j)$ is nonnegative then

$$D_\xi^- G_2^N(x_i,\xi_1) = G_2^N(x_i,\xi_1)h_1^{-1} \ge 0. \qquad (44)$$

On the other hand, from (39) we can get

$$\sum_{k=0}^{j-1} G_2^N(x_i,\xi_k)\hbar_k = G_1^N(x_i,\xi_j).$$

So

$$G_1^N(x_i,\xi_j) \le G_2^N(x_i,\xi_{j_0}). \qquad (45)$$

Combining (42)-(45), we obtain

$$(\alpha-\gamma)G_2^N(x_i,\xi_{j_0}) \le \sum_{j=1}^{j_0} \delta^N(x_i,\xi_j)\hbar_j \le 1. \quad (46)$$

Also, from (39) and (42) we have

$$G_1^N(x_i,\xi_j) = \sum_{k=1}^{j-1} G_2(x_i,\xi_j)\hbar_k. \qquad (47)$$

From (46) and (47) we can obtain the desired results.

Lemma 12. The operator P_2^N satisfies

$$\|y_2^N\|_\infty \le \frac{1}{\alpha-\gamma}\|P_2^N \mathbf{y}^N\|_1.$$

proof. The proof follows directly from the representation of the solution in (38) and Lemma 11.

Let $\mathbf{y}_i^N$ be the solution of the discrete problem (31)-(33) and $\mathbf{y}_i$ be the values of the solution of the original continuous problem at the nodes of mesh $\bar{D}^N$. Then $\mathbf{z}_i = \mathbf{y}_i^N - \mathbf{y}_i$ is the accuracy of the solution.Substituting $\mathbf{y}_i^N = \mathbf{z}_i + \mathbf{y}_i$ into (31)-(32). We see that $\mathbf{z}_i$ is the solution of the following problem

$$\begin{aligned}
P_1^N\mathbf{z}_i &= -P_1^N\mathbf{y}_i = -D^+y_{1,i} + y_{2,i} \equiv \psi_{1,i}, &(48)\\
P_2^N\mathbf{z}_i &= f_i - P_2^N\mathbf{y}_i = f_i + \varepsilon D^+D^- y_{2,i} \\
&+ a_iD^+y_{2,i} - b_i y_{2,i} - c_i y_{1,i} \equiv \psi_{2,i}, &(49)\\
z_{1,0} &= z_{2,0} = z_{2,N} = 0. &(50)
\end{aligned}$$

Using (30), we have one more representation

$$\begin{aligned}
&\psi_{1,i} = -(D^+y_{1,i} - y_{1,i}^{'}), \\
&\psi_{2,i} = \varepsilon(D^+D^-y_{2,i} - y_{2,i}^{''}) + a_i(D^+y_{2,i} - y_{2,i}^{'}).
\end{aligned}$$

We now estimate the truncation error ψ_i on the Bakhvalov-Shishkin mesh.

Lemma 13. The following estimates for the truncation error hold true:

$$|\psi_1(x_i)| = Ch_{i+1}\varepsilon^{-1}\exp(-\alpha x_i/\varepsilon) \le CN^{-1}$$

for $i = 0, 1, \cdots, N-1$,

$$|\psi_2(x_i)| \le C(h_{i+1} + N^{-1}\varepsilon^{-1}\exp(-\frac{\alpha x_i}{2\varepsilon})$$

for $i = 1, 2, \cdots, N/2-1$,

$$|\psi_2(x_i)| \le C(h_{i+1} + \varepsilon^{-2}(h_i + h_{i+1})\exp(-\frac{\alpha x_{i-1}}{\varepsilon}))$$

for $i = N/2+1, \cdots, N-1$,

$$|\psi_2(x_i)| \le C(h_{i+1} + h_i\exp(-\frac{\alpha x_i}{\varepsilon}) + 1)$$

for $i = N/2, N/2+1$,

Proof. For $i = 0, 1, \cdots, N-1$ we use a Taylor expansion for $x = x_i$ to get

$$\begin{aligned}
&|\psi_{1,i}| = \frac{1}{2}h_{i+1}|y_1^{''}(\xi_i)| \\
&\le Ch_{i+1}\varepsilon^{-1}\exp(-\alpha x_i/\varepsilon) \le CN^{-1} \quad (51)
\end{aligned}$$

for $\xi_i \in (x_i, x_{i+1})$, where we have used

$$\frac{h_i}{\varepsilon}\exp(-\alpha x_i/\varepsilon) \le CN^{-1} \ \text{ for } \ i = 1, 2, \cdots, N/2.$$

Recalling the decomposition of Lemma 8, we have

$$\begin{aligned}
|\psi_{2,i}| &= |f_i - P_2^N\mathbf{y}_i| \le |P_2\mathbf{v}_i - P_2^N\mathbf{v}_i| \\
&+ |P_2\mathbf{w}_i - P_2^N\mathbf{w}_i| \qquad (52)
\end{aligned}$$

for $i = 1, 2, \cdots, N-1$.

For the smooth part, we have

$$|P_2\mathbf{v}_i - P_2^N\mathbf{v}_i| \le 2\varepsilon \int_{x_{i-1}}^{x_{i+1}} |v_2^{'''}(t)|\mathrm{d}t + a_i \int_{x_i}^{x_{i+1}} |v_2^{''}(t)|\mathrm{d}t \le Ch_{i+1} \quad (53)$$

for $i = 1, 2, \cdots, N-1$.

For the truncation error of the method with respect to the layer part $\mathbf{w}$ we have

$$|P_2\mathbf{w}_i - P_2^N\mathbf{w}_i| \le 2\varepsilon \int_{x_{i-1}}^{x_{i+1}} |w_2^{'''}(t)|\mathrm{d}t + a_i \int_{x_i}^{x_{i+1}} |w_2^{''}(t)|\mathrm{d}t \le C\varepsilon^{-2} \int_{x_{i-1}}^{x_{i+1}} \exp(-\alpha t/\varepsilon)\mathrm{d}t \quad (54)$$

for $i = 1, 2, \cdots, N-1$. Let $x_i = \frac{2\varepsilon}{\alpha}\varphi(t) = -\frac{2\varepsilon}{\alpha}\ln[1 - 2(1 - N^{-1})t]$ and $t_i = \varphi^{-1}(x_i)$ for $i = 1, 2, \cdots, N/2$. Then

$$\begin{aligned} &|P_2^N(\mathbf{w}_i - \mathbf{w}_i^N)| \\ &\le C\varepsilon^{-1} \int_{t_{i-1}}^{t_{i+1}} \exp(-2\varphi(t))\varphi^{'}(t)\mathrm{d}t \\ &\le C\varepsilon^{-1} \int_{t_{i-1}}^{t_{i+1}} \exp(\varphi(t))\mathrm{d}t \\ &\le C\varepsilon^{-1} \int_{t_{i-1}}^{t_{i+1}} \exp(-\frac{\alpha x_{i-1}}{2\varepsilon})\mathrm{d}t \\ &\le CN^{-1}\varepsilon^{-1} \exp(-\frac{\alpha x_{i-1}}{2\varepsilon}) \\ &\le CN^{-1}\varepsilon^{-1} \exp(-\frac{\alpha x_i}{2\varepsilon}) \end{aligned} \quad (55)$$

for $i = 1, 2, \cdots, N/2-1$, and

$$|P_2\mathbf{w}_i - P_2^N\mathbf{w}_i| \le C\varepsilon^{-2}(h_i + h_{i+1}) \exp(-\alpha x_{i-1}/\varepsilon) \quad (56)$$

for $i = N/2+1, \cdots, N-1$.

Next we estimate $|P_2\mathbf{w}_{N/2} - P_2^N\mathbf{w}_{N/2}|$.

$$\begin{aligned} &|P_2\mathbf{w}_{N/2} - P_2^N\mathbf{w}_{N/2}| = |P_2^N\mathbf{w}_{N/2}| \\ &= |\varepsilon D^+D^- w_{2,N/2} + a_{N/2}D^+ w_{2,N/2} \\ &- b_{N/2}w_{2,N/2} - c_{N/2}w_{1,N/2}| \\ &\le \frac{1}{\hbar_{N/2}} |\varepsilon(D^+ w_{2,N/2} - D^- w_{2,N/2}) \\ &+ a_{N/2}(w_{2,N/2+1} - w_{2,N/2})| + C \\ &= \frac{1}{\hbar_{N/2}}[\varepsilon(w_2^{'}(\xi_{N/2}) - w_2^{'}(\xi_{N/2-1})) \\ &+ a_{N/2}(w_{2,N/2+1} - w_{2,N/2})] + C \\ &\le C(\hbar_{N/2}^{-1} \exp(-\frac{\alpha x_{N/2-1}}{\varepsilon}) + 1). \end{aligned} \quad (57)$$

Using similar reasoning, we obtain the following estimate

$$\begin{aligned} &|P_2\mathbf{w}_{N/2+1} - P_2^N\mathbf{w}_{N/2+1}| \\ &\le C(1 + h_{N/2+1}^{-1} \exp(-\frac{\alpha x_{N/2}}{\varepsilon})). \end{aligned} \quad (58)$$

Combining (52)-(58) we can complete the local estimate of $\psi_{2,i}$.

We can now derive our main result.

Theorem 3. The error of the difference scheme on the Bakhvalov-Shishkin mesh satisfies

$$\|\mathbf{y}_i - \mathbf{y}_i^N\| \le CN^{-1} \text{ for } i = 0, 1, \cdots, N,$$

where $\|\mathbf{y}_i\| = \max\{|y_{1,i}|, |y_{2,i}|\}$ for $i = 0, 1, \cdots, N$.

By (38) and Lemma 12, we have the following a priori estimate for the accuracy $z_{2,i} = y_{2,i}^N - y_{2,i}$ of the solution in terms of the truncation error $\psi_{2,i}$

$$|y_{2,i} - y_{2,i}^N| \le C\|\psi_{2,i}\|_1, \ \ i = 1, 2, \cdots, N-1. \quad (59)$$

Using Lemma 13, we obtain

$$\begin{aligned} \|\psi_{2,i}\|_1 &= \sum_{i=1}^{N/2-1} |\psi_{2,i}|\hbar_i + |\psi_{2,N/2}|\hbar_{N/2} \\ &+ |\psi_{2,N/2+1}|\hbar_{N/2+1} + \sum_{i=N/2+2}^{N-1} |\psi_{2,i}|\hbar_i \\ &\le C(\sum_{i=1}^{N/2-1} h_{i+1}\hbar_i + \hbar_{N/2} + \hbar_{N/2+1} \\ &+ \sum_{i=N/2+2}^{N-1} h_{i+1}\hbar_i) \\ &+ CN^{-1}\varepsilon^{-1} \sum_{i=1}^{N/2-1} \exp(-\frac{\alpha x_i}{2\varepsilon})\hbar_i \\ &+ C(\exp(-\frac{\alpha x_{N/2-1}}{\varepsilon}) + \exp(-\frac{\alpha x_{N/2}}{\varepsilon})) \\ &+ C\varepsilon^{-2} \sum_{i=N/2+2}^{N-1} (h_i + h_{i+1})\hbar_i \exp(-\frac{\alpha x_{i-1}}{\varepsilon}) \\ &\le CN^{-1}. \end{aligned} \quad (60)$$

Combining (59) and (60) we get

$$|y_{2,i} - y_{2,i}^N| \le CN^{-1} \text{ for } i = 0, 1, \cdots, N. \quad (61)$$

From Lemma 10 we have

$$|y_{1,i} - y_{1,i}^N| \le C|\psi_{1,i}| \le CN^{-1} \quad (62)$$

for $i = 0, 1, \cdots, N$.

By (61) and (62) we get the desired results.

5 Numerical experiments

In this section we verify experimentally the theoretical results obtained in the preceding section.

Example 1. Consider the problem

$$-\varepsilon u''(x) - u'(x) = -2, \quad x \in (0,1), \quad (63)$$
$$u(0) = 0, \quad u(1) = 1. \quad (64)$$

The exact solution is given by

$$u(x) = \frac{\exp(-x/\varepsilon) - \exp(-1/\varepsilon)}{1 - \exp(-1/\varepsilon)} + 2x - 1.$$

Example 2. Consider the problem

$$-\varepsilon u''(x) - ((1+x)u(x))' = f(x),\ 0 < x < 1, \quad (65)$$

$$u(0) = u(1) = 0, \quad (66)$$

where $f(x)$ is chosen such that

$$u(x) = \frac{1 - \exp(-x/\varepsilon)}{1 - \exp(-1/\varepsilon)} - x$$

is the exact solution.

For our tests we take $\varepsilon = 10^{-8}$ which is a sufficiently small choice to bring out the singularly perturbed nature of the problems. In order to evaluate the integrals in (5), we apply the standard midpoint rule

$$\int_{x_{j-1}}^{x_j} \Psi(x)\mathrm{x} \sim (x_j - x_{j-1})\Psi(x_{j-1/2}).$$

We measure the accuracy of the pointwise error estimates and the approximation of derivatives in the discrete maximum norm $\|\cdot\|_\infty$, respectively. We also present the convergence rates of these errors as N increases with ε fixed. These rates are computed in the usual way; for example, the convergence rates r^N of the pointwise errors are computed using the following formula:

$$r^N = \log_2\left(\frac{\|u-u^N\|_\infty}{\|u-u^{2N}\|_\infty}\right).$$

The numerical results (Tables 1-12) are clear illustrations of the convergence estimate of Theorem 2. They indicate that the theoretical results are fairly sharp.

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Table 1: The pointwise error estimates of the SDFEM on the standard Shishkin mesh for Example 1

	$\|u-u^N\|_\infty$	
N	*error*	*rate*
32	5.3118e-3	1.458
64	1.9338e-3	1.534
128	6.6775e-4	1.597
256	2.2076e-4	1.648
512	7.0444e-5	1.687
1024	2.1876e-5	-

Table 2: The pointwise approximation of derivatives of the SDFEM on the standard Shishkin mesh for Example 1

	$\max_{i=1,\cdots,N-1} \varepsilon\|D^- u_i^N - u'(x_i)\|$	
N	*error*	*rate*
32	1.7282e-1	0.604
64	1.1367e-1	0.696
128	7.0164e-2	0.759
256	4.1459e-2	0.802
512	2.3775e-2	0.832
1024	1.3355e-2	-

Table 3: The approximation of derivatives of the SDFEM on the standard Shishkin mesh for Example 1

	$\max_{i=1,\cdots,N-1} \varepsilon\|D^- u_i^N - u'(x_{i-1/2})\|$	
N	*error*	*rate*
32	1.5995e-2	1.266
64	6.6510e-3	1.421
128	2.4844e-3	1.532
256	8.5934e-4	1.610
512	2.8147e-4	1.667
1024	8.8663e-5	-

Table 4: The pointwise error estimates of the SDFEM on the Bakvalov-Shishkin mesh for Example 1

N	$\|u-u^N\|_\infty$ error	rate
32	3.5144e-3	2.044
64	8.5215e-4	2.034
128	2.0803e-4	2.024
256	5.1143e-5	2.016
512	1.2645e-5	2.010
1024	3.1394e-6	-

Table 5: The pointwise approximation of derivatives of the SDFEM on the Bakvalov-Shishkin mesh for Example 1

N	$\max_{i=1,\cdots,N-1} \varepsilon\|D^- u_i^N - u'(x_i)\|$ error	rate
32	5.1387e-2	0.977
64	2.6099e-2	0.990
128	1.3141e-2	0.995
256	6.5952e-3	0.997
512	3.3037e-3	0.999
1024	1.6535e-3	-

Table 6: The approximation of derivatives of the SDFEM on the Bakvalov-Shishkin mesh for Example 1

N	$\max_{i=1,\cdots,N-1} \varepsilon\|D^- u_i^N - u'(x_{i-1/2})\|$ error	rate
32	2.8134e-3	1.984
64	7.1101e-4	1.998
128	1.7795e-4	2.003
256	4.4411e-5	2.002
512	1.1085e-5	2.002
1024	2.7678e-6	-

Table 7: The pointwise error estimates of the SDFEM on the standard Shishkin mesh for Example 2

N	$\|u-u^N\|_\infty$ error	rate
32	4.8669e-3	1.441
64	1.7928e-3	1.501
128	6.3324e-4	1.581
256	2.1163e-4	1.636
512	6.8100e-5	1.678
1024	2.1286e-5	-

Table 8: The pointwise approximation of derivatives of the SDFEM on the standard Shishkin mesh for Example 2

N	$\max_{i=1,\cdots,N-1} \varepsilon\|D^- u_i^N - u'(x_i)\|$ error	rate
32	1.7220e-1	0.602
64	1.1348e-1	0.695
128	7.0111e-2	0.758
256	4.1444e-2	0.802
512	2.3772e-2	0.832
1024	1.3354e-2	-

Table 9: The approximation of derivatives of the SDFEM on the standard Shishkin mesh for Example 2

N	$\max_{i=1,\cdots,N-1} \varepsilon\|D^- u_i^N - u'(x_{i-1/2})\|$ error	rate
32	1.5370e-2	1.251
64	6.4587e-3	1.410
128	2.4307e-3	1.524
256	8.4511e-4	1.605
512	2.7779e-4	1.663
1024	8.7729e-5	-

Table 10: The pointwise error estimates of the SD-FEM on the Bakvalov-Shishkin mesh for Example 2

	$\|u-u^N\|_\infty$	
N	*error*	*rate*
32	2.8897e-3	2.078
64	6.8429e-4	1.951
128	1.7700e-4	1.976
256	4.5007e-5	1.988
512	1.1348e-5	1.994
1024	2.8489e-6	-

Table 11: The pointwise approximation of derivatives of the SDFEM on the Bakvalov-Shishkin mesh for Example 2

	$\max\limits_{i=1,\cdots,N-1} \varepsilon\|D^-u_i^N - u'(x_i)\|$	
N	*error*	*rate*
32	5.1087e-2	0.974
64	2.6001e-2	0.988
128	1.3113e-2	0.993
256	6.5881e-3	0.997
512	3.3019e-3	0.998
1024	1.6530e-3	-

Table 12: The approximation of derivatives of the SD-FEM on the Bakvalov-Shishkin mesh for Example 2

	$\max\limits_{i=1,\cdots,N-1} \varepsilon\|D^-u_i^N - u'(x_{i-1/2})\|$	
N	*error*	*rate*
32	2.6371e-3	2.012
64	6.5393e-4	2.007
128	1.6270e-4	2.005
256	4.0541e-5	2.004
512	1.0108e-5	2.002
1024	2.5228e-6	-

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