On Very True Operators and v-Filters

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Abstract: In this paper, based on Hájek, Vychodil, Rachunek and Šalounová's works, we study the concept of v-filters of residuated lattices with weak vt-operators, axiomatize very true operators, discuss filters and v-filters of residuated lattices with weak vt-operator, give the formulas for calculating the v-filters generated by subsets, and show that lattice of v-filters of a commutative residuated lattice with vt-operator is a complete Brouwerian lattice.

Key–Words: Fuzzy logic, Residuated lattice, Very true, Weak vt-operator, v-Filter

1 Introduction

Inspired by the considerations of Zadeh [19], Hájek in [8] formalized the fuzzy truth value "very true". He enriched the language of the basic fuzzy logic BL by adding a new unary connective vt and introduced the propositional logic BL_{vt}. The completeness of BL_{vt} was proved in [8] by using the so-called BL_{vt}-algebra, an algebraic counterpart of BL_{vt}. Recently, Vychodil [17] proposed an axiomatization of unary connectives like "slightly true" and "more or less true" and introduced $BL_{vt,st}$-logic which extends BL_{vt}-logic by adding a new unary connective "slightly true" denoted by "st". Noting that bounded commutative $R\ell$-monoids are algebraic structures which generalize, e.g. both BL-algebras and Heyting algebras (an algebraic counterpart of the intuitionistic propositional logic), Rachunek and Šalounová taken bounded commutative $R\ell$-monoids with a vt-operator as an algebraic semantics of a more general logic than Hájek's fuzzy logic and studied algebraic properties of $R\ell_{vt}$-monoids in [14].

Commutative residuated lattice [12] (i.e., integral commutative residuated l-monoid in [10]) is an important class of logical algebras, and the typical example of commutative residuated lattice is the interval $[0, 1]$ endowed with the structure induced by a left-continuous t-norm [7, 10]. The well-known commutative residuated lattices have Boolean algebras, Heyting algebras, MV-algebras, Gödel algebras, product algebras, BL-algebras, R_0-algebras [18], Bounded commutative $R\ell$-monoids [13, 14, 15], MTL-algebras [5], and so on. Many authors used commutative residuated lattices as the structures of truth degrees (e.g., see [1, 2, 12]). The filter theory plays an important role in studying these logical algebras and many authors discussed the notion of filters of these logical algebras (see [4, 6, 7, 10, 11, 13, 16]). From a logical point of view, a filter corresponds to a set of provable formulas. Sometimes, a filter is also called a deductive systems (see [16]).

In this paper, based on [8, 14, 17], we study the concept of v-filters of residuated lattices with weak vt-operators. In section 2, we axiomatize very true operators. In section 3, we briefly recall some definitions and results about residuated lattice and discuss filters and v-filters of residuated lattices with weak vt-operator. In section 4, we give the formulas for calculating the v-filters generated by subsets. In section 5, we show that lattice of v-filters of a commutative residuated lattice with vt-operator is a complete Brouwerian lattice.

The lattice properties required in this paper can be found in Birkhoff [3]. For the sake of simplicity, we denote by $\mathcal{N}^+$ the set of nonnegative integers.

2 Axiomatizing very true

In this section, we deal with propositional calculus. We enrich the language by the new unary connective vt and define the axioms of the logic BL_{vt} be those of BL (with the new notion of a formula) plus the following ones:

(VE1) $vt\varphi \to \varphi$,
(VE2) $vt(\varphi \to \psi) \to (vt\varphi \to vt\psi)$,
(VE3) $vt(\varphi \vee \psi) \to (vt\varphi \vee vt\psi)$.

The deduction rules are modus ponens and truth confirmation (a kind of necessitation): from φ infer $vt\varphi$.

Axiom (VE1) seems to be fully acceptable. Axiom (VE2) says that if both φ and $\varphi \to \psi$ are very true then so is ψ, which also appears reasonable. Concerning (VE3) (saying that if a disjunction $\varphi \vee \psi$ is very true then so is one of the disjuncts) the reader is asked to see the following definition and lemma showing that (VE3) is sound for each "natural" interpretation. Note that the disjunction $\vee$ is always interpreted as the lattice join, i.e. if your algebra of truth values is linearly ordered it is just maximum.

Let $*$ be a continuous t-norm and $\Rightarrow$ its residuum. A hedge vt is $*$-regular if

$vt(x \Rightarrow y) \leq vt(x) \Rightarrow vt(y)$

for all $x, y \in [0;1]$, ($\Rightarrow$ being the residuum of $*$.); i.e. if $*$ makes (VE2) a tautology (for each φ).

Call vt $*$-truth-stressing if it is $*$-regular, $vt(1) = 1$ and vt is subdiagonal.

Let $*$ be a continuous t-norm.

Lemma 1 *(1) BL_{vt} is $*$-sound for a hedge vt if and only if vt is $*$-truth stressing.*

(2) A $*$-truth stressing hedge is non-decreasing.

Proof: If vt is $*$-truth stressing then (VE1) and (VE2) are $*$-tautologies and the necessitation for vt is sound. Moreover, vt is non-decreasing: if $u \leq v$ then $(u \Rightarrow v) = 1$; thus $vt(u \Rightarrow v) = 1$ and hence $vt(u) \Rightarrow vt(v) = 1$; thus $vt(u) \leq vt(v)$. Thus also (VE3) is a $*$-tautology. If $u \leq v$ then $\max(u, v) = v$ and $\max(vt(u), vt(v)) = vt(v)$, thus

$vt(\max(u, v)) = vt(v) = \max(vt(u), vt(v))$.

Similarly for $v \leq u$. Thus in all cases $vt(\max(u, v)) = \max(vt(u), vt(v))$ and the formula $vt(\varphi \vee \psi) \equiv (vt\varphi \vee vt\psi)$ is a $*$-tautology.

Conversely, if BL_{vt} is $*$-sound for vt then evidently vt is $*$-truth-stressing. □

We may impose other conditions, e.g. continuity, injectivity (being one-to-one), etc. Let us go through some examples.

(1) In each t-norm logic one may define $vt\varphi$ to be $\varphi\&\varphi$ (written also φ^2) or, more generally, φ^n for $n \geq 1$. Then $vt(u) = u * u = u^{*2}$ or more generally $vt(u) = u^{*n}$. This "very true" is continuous.

(2) Take $vt(u) = u \cdot u$ (product of reals, real square). For $\prod$ it is covered by (1); for Ł the axioms are tautologies ($(1 - u + v)^2 \leq 1 - u^2 + v^2$ for $0 \leq v \leq u \leq 1$); and so are for G ($(x \Rightarrow y)^2 = x^2 \Rightarrow y^2$ for Gödel implication).

(3) Note that if we take Łukasiewicz square $\max(0, 2u - 1)$ then the axioms fail to be tautologies for $\prod$ but are tautologies for G.

(4) Let $vt(u) = k \cdot u$ for $u < 1, vt(1) = 1(0 \leq k \leq 1$. This is a truth stresser for Ł,G, $\prod$. Note that choosing $k = 0$ we get Baaz's connective Δ ($\Delta\varphi$ says "φ is absolutely true").

Lemma 2 *(1) If $\vdash \varphi \to \psi$ then $\vdash vt\varphi \to vt\psi$.*

(2) The following formulas are provable in BL_{vt}:
(a) $\neg vt(\overline{0})$,
(b) $(vt\varphi \& vt\psi) \to vt(\varphi\&\psi)$,
(c) $vt(\varphi \vee \psi) \equiv (vt\varphi \vee vt\psi)$.

Proof: (1) Follows by applying the necessitation and the axiom (VE2).

Let us prove (2).

$\vdash vt(\overline{0}) \to \overline{0}$ by (VE1), from

$\vdash \varphi \to (\psi \to (\varphi\&\psi))$

follows

$\vdash vt\varphi \to (vt\psi \to vt(\varphi\&\psi))$.

Clearly $\vdash vt\varphi \to vt(\varphi \vee \psi)$ and

$\vdash vt\psi \to vt(\varphi \vee \psi)$,

thus $\vdash vt\varphi \vee vt\psi \to vt(\varphi \vee \psi)$.

The converse implication is our axiom (VE3). □

Remark. It is easy to prove $vt(\varphi \wedge \psi) \to (vt\varphi \wedge vt\psi)$; the converse implication is also provable as we shall show later.

We introduce an auxiliary notation: $\tau\varphi$ stands for $vt(\varphi\&\varphi)$, $\tau^n\varphi$ stands for $\tau(\tau \cdots \tau(\varphi) \cdots)$ (n copies of τ).

One easily shows the following:

Lemma 3 *(1) $BL_{vt} \vdash \tau^{n+1}\varphi \to \tau^n\varphi$.*

(2) $BL_{vt} \vdash \tau\varphi \to vt\varphi$, $\tau\varphi \to (\varphi\&\psi)$.

(3) $BL_{vt} \vdash \tau(\varphi \vee \psi) \to (\tau\varphi \vee \tau\psi)$.

Theorem 4 *Let T be a theory over BL_{vt}, let φ, ψ be formulas.*

$T \cup \{\varphi\} \vdash \psi$ if and only if $T \vdash \tau^n\varphi \to \psi$ for some n.

Proof: As usual, let us check the deduction rules.

If $T \vdash \tau^n\varphi \to \alpha$ and $T \vdash \tau^n\varphi \to (\alpha \to \beta)$ then

$T \vdash (\tau^n\varphi\&\tau^n\varphi) \to \beta$,

thus

$T \vdash \tau^{n+1}\varphi \to \beta$.

Similarly, if $T \vdash \tau^n\varphi \to \beta$ then

$T \vdash vt\tau^n\varphi \to vt\beta$,

thus

$T \vdash \tau^{n+1}\varphi \to vt\beta$. □

To prove completeness let us define a BL_{vt}-algebra to be an algebra $L = (L, \cap, \cup, \star, \Rightarrow, 0, 1, v)$ which is a BL-algebra expanded by an unary operation v satisfying, for all x, y,

$v(1) = 1$,

$v(x) \leq x$,
$v(x \Rightarrow y) \leq (v(x) \Rightarrow v(y))$,
$v(x \cup y) \leq v(x) \cup v(y)$.

Clearly each t-algebra (given by a continuous t-norm on $[0, 1]$ together with a truth-stresser is a BL_{vt}-algebra, BL_{vt}-algebras form a variety and BL_{vt} is sound for BL_{vt}-algebras. The completeness proof is standard and relies on the following lemma:

Lemma 5 *If T is a theory and α a formula such that $T \vdash \alpha$ doesn't hold, then there is a complete extension T' of T such that $T' \vdash \alpha$ doesn't hold.*

Proof: One successively handles all pairs φ, ψ of formulas and shows: of $T' \supseteq T$ and $T' \vdash \alpha$ doesn't hold, then $T' \cup \{(\varphi \to \psi)\} \vdash \alpha$ doesn't hold or $T' \vdash \alpha$ doesn't hold, then $T' \cup \{(\psi \to \varphi)\} \vdash \alpha$ doesn't hold.

Indeed, if both theories prove α then for some n, $T' \vdash (\tau^n(\varphi \to \psi) \vee \tau^n(\psi \to \varphi)) \to \alpha$, hence

$T' \vdash \tau^n((\varphi \to \psi) \vee (\psi \to \varphi)) \to \alpha$

and thus

$T' \vdash \alpha$ since obviously

$BL_{vt} \vdash \tau^n((\varphi \to \psi) \vee (\psi \to \varphi))$. □

Note that the algebra of classes T-equivalent formulas is well defined since $[\varphi]_T = [\psi]_T$ implies $[vt\varphi]_T = [vt\psi]_T$ and is a BL_{vt}-algebra.

It is linearly ordered if and only if T is complete. Thus we have the usual.

Theorem 6 (Completeness Theorem) *Let T be a theory over BL_{vt}, φ a formula. The following are equivalent:*

(1) T proves φ over BL_{vt}.

(2) For each (linearly ordered) BL_{vt}-algebra L and each L-model e of T, $e_L = 1$ (φ is L-true in e).

Corollary 7 *(1) BL_{vt} proves $vt(\varphi \wedge \psi \equiv (vt\varphi \wedge vt\psi)$*

Since the formula is L-true in each L-evaluation for each linearly ordered L.

(2) BL_{vt} is a conservative extension of BL.

Indeed, if $BL \vdash \varphi$ doesn't hold where φ does not contain vt then there is a linearly ordered BL-algebra L such that φ is not an L-tautology. Expand L to a BL_{vt}-algebra, e.g. by defining $vt(u) = 0$ for $u < 1$, $vt(1) = 1$. Thus $BL_{vt} \vdash \varphi$ doesn't hold.

Caution: We get strong completeness of stronger logics $Ł_{vt}$, G_{vt}, $\prod_{vt}$ with respect to models over MV-algebras, G-algebras, $\prod$-algebras as corollaries. Can we get standard completeness, i.e. if Γ stands for Ł, G, $\prod$ and $[0, 1]_\Gamma$ is the standard t-algebra given by the respective continuous t-norm, is it true that φ is provable in the logic Γ_{vt} if and only if for each Γ-truth stresser v and each $[0, 1]$-evaluation e, φ is $([0, 1]_\Gamma, v)$-true in e?

Imitating the corresponding proofs of standard completeness of Γ, the problem is: if L is a linearly ordered Γ_{vt}-algebra and X is a finite subset of L (containing $0_L, 1_L$), can we find a finite $Y \subseteq [0, 1]$ (containing 0, 1), a Γ-truth stresser vt and a 1-1 mapping $f : X \to Y$ which is a partial isomorphism, i.e. for $x, y, z \in X$, f preserves $x \star y = z, x \to y = z, x \leq y$, plus $vt(x) = y$?

The answer is easy for G: take the partial isomorphism $f : X \to Y$ preserving $*, \Rightarrow$ and $\leq$, this gives you finitely many pairs (y_i, z_i) of elements of Y determining finitely many conditions $vt(y_i) = z_i$ (among them $vt(0) = 0, vt(1) = 1$). Clearly, $z_i \leq y_i$ and you can just take for vt the piecewise linear function connecting neighboured points (y_i, z_i). It is subdiagonal, non-decreasing, $vt(1) = 1$ and that is all since it follows that vt is G-regular: for $> y$, either $vt(x) > vt(y)$ and $vt(x \Rightarrow y) = vt(y) = vt(x) \Rightarrow vt(y)$ or $vt(x) = vt(y)$ and then $vt(x) \Rightarrow vt(y) = 1$.

For Ł, $\prod$ the situation is more complicated and the question of standard completeness seems to remain an open problem.

Of course if we restrict ourselves to a fixed definable truth stresser, e.g. postulating $vt\varphi \equiv (\varphi \& \chi)$ then standard completeness of this extension of Γ_{vt} follows from standard completeness of Γ.

Let $BL\forall_{vt}$ stand for the extension of the basic fuzzy predicate logic $BL\forall$ by the hedge (connective) vt and the corresponding axioms (VE1)-(VE3) for it; semantics over an arbitrary BL_{vt}-algebra is defined in the obvious way.

Theorem 8 (Deduction Theorem) *Let T be a theory over $BL\forall_{vt}$ and let φ, ψ be formulas. $T \cup \{\varphi\} \vdash \psi$ if and only if for some n,*

$T \vdash \tau^n\varphi \to \psi$,

where τ is as above, $\tau\varphi$ is $vt(\varphi \& \varphi)$).

Theorem 9 (Completeness Theorem) *Let T be a theory over $BL\forall_{vt}$, φ a formula. The following are equivalent:*

(1) $T \vdash \varphi$.

(2) For each linearly ordered BL_{vt}-algebra L and each L-model M of T, $||\varphi||^L_M = 1$, i.e. $||\varphi||_{M^L,\, e} = 1$ for each evaluation e of object variables.

Let us stress that an L-model of T is a safe L-interpretation of T in which all axioms of T are true. The proof is by inspecting, note that the present version of the deduction theorem is to be used.

The analogous completeness theorem for $Ł\forall_{vt}$, $G\forall_{vt}$, $\prod\forall_{vt}$ follows immediately.

Similarly to $G\forall$, we have a standard completeness theorem for $G\forall_{vt}$.

Theorem 10 *Over $G\forall_{vt}$, $T \vdash \varphi$ if and only if φ is true in each $([0,1])_{G,vt})$-model of T, for each G-truth stresser vt.*

Proof: If $T \vdash \varphi$ doesn't hold, we get a countable (L, v)-model M of T with $||\varphi||_M^{(L,v)} < 1$. We may assume that L is a subalgebra of $[0,1]_G$ and the identical embedding preserves all sups and infs existing in L. Now v is a non-decreasing subdiagonal function on L and $v(1) = 1$. We have to extend v to a non-decreasing subdiagonal function on $[0,1]$, but this is an easy exercise; for example put, for $x \in [0,1]$, $w(x) = \sup\{v(y) \mid y \in L \& y \leq x\}$. This gives $w(x) = v(x)$ for $\in L$ and clearly is subdiagonal and total on $[0,1]$. □

Remark: Note that, in general, vt need not commute with quantifiers, i.e. if vt is Baaz's $\triangle$, $r_P(a) < 1$ for each $a \in M$ but $\sup_a r_P(a) = 1$ then $||(\exists x)vtP(x)||_M = 0$, but $||vt(\exists)P(x)||_M = 1$.

Similarly for $\forall$ and a truth stresser with is not continuous from above.

Theorem 11 *If vt is a continuous truth stresser, then for each continuous t-norm $*$, the formulas $(\forall x)vt\varphi \equiv vt(\forall x)\varphi$ and $(\exists x)vt\varphi \equiv vt(\exists x)\varphi$ are $([0,1]_{*,vt}$-tautologies.*

Proof: Clearly $BL_{vt} \vdash vt(\forall x)\varphi \to vt\varphi$, hence

$BL_{vt} \vdash (\forall x)(vt\forall x\varphi \to vt\varphi)$

and hence

$BL_{vt} \vdash vt(\forall x)\varphi \to (\forall x)vt\varphi$.

We show the tautologicity of the converse implications assuming that vt is interpreted by a continuous truth stresser vt. But then for each (nonempty) $A \subseteq [0,1]$ we get $vt(\inf A) = \inf vt(A)$ and $vt(\sup A) = \sup(vt(A))$, here $vt(A) = \{vt(a) \mid a \in A\}$. □

Finally, let us consider Pavelka Rational (predicate) Logic $RPL\forall_{vt}$. Extend $RPL\forall$ by the connective vt interpreted by a fixed continuous truth stresser vt such that $vt(r)$ is rational for each rational $r \in [0,1]$, extend the axiom by (VE1)-(VE3) plus the book-keeping axioms $vt\overline{r} \equiv \overline{vt(r)}$ for each rational r.

As usual, given a theory T, define the truth-degree $||\varphi||_T$ of a formula φ to be $\inf\{||\varphi||_M^{([0,1]_{*,vt}} \mid M$ a $([0,1]_*, vt)$-model of $T\}$ and define the provability degree $|\varphi|_T$ to be $\sup\{r \in [0,1], r$ rational $|T \vdash \overline{r} \to \varphi\}$.

Theorem 12 (Pavelka Completeness) *Under the present notation (T a theory over RPL_{vt}, φ a formula),*

$|\varphi|_T = ||\varphi||_T$.

The proof consists again in checking the proof for $RPL_\forall$, the only thing to be added is, assuming T' a complete extension of T, to show that the provability degree commutes with vt, i.e. $vt(|\varphi|_{T'}) = |vt\varphi|_{T'}$.

Remark: The paper [9] deals with a system of axioms introduced by Yashin in the context of intuitionistic logic and admitting an interpretation as an axiomatization of "more or less true" over Gödel logic (but not e.g. over Łukasiewicz logic). The comparison of that system with the present one could be (modestly) interesting.

3 Filters and v-filters of residuated lattices with weak vt-operator

In this section, we briefly recall some definitions and results about filters of a residuated lattice and discuss v-filters of residuated lattices with weak vt-operator.

Definition 13 [7, 12, 16] *A commutative residuated lattice $L = (L, \leq, \wedge, \vee, \cdot, \to, 0, 1)$ is a lattice L containing the least element 0 and the largest element 1, and endowed with two binary operations $\cdot$ (called product) and $\to$ (called residuum) such that*

(1) $\cdot$ is associative, commutative and isotone and, for all elements $x \in L$, $x \cdot 1 = x$,

(2) for all $x, y, z \in L$, the Galois correspondence

$$x \cdot y \leq z \text{ if and only if } x \leq y \to z$$

holds.

Commutative residuated lattices are known also under other names, e.g Höhle [10] calls them integral, residuated, commutative l-monoids.

We adopt the usual convention of representing the monoid operation by juxtaposition, writing ab for $a \cdot b$, and set $x^0 = 1$, $x^n = x^{n-1}x$ for any $n \geq 1$.

Definition 14 [15] *A bounded commutative $R\ell$-monoids is a commutative residuated lattce $L = (L, \leq, \wedge, \vee, \cdot, \to, 0, 1)$ satisfying the divisibility condition:*

(3) $x(x \to y) = x \wedge y$ for any $x, y \in L$.

In fact, the notion of a bounded commutative $R\ell$-monoids is a duplicate name for a commutative residuated lattice satisfying divisibility condition or for a divisible commutative residuated lattice.

Definition 15 [5] *A MTL-algebra is a commutative residuated lattce $L = (L, \leq, \wedge, \vee, \cdot, \rightarrow, 0, 1)$ satisfying the identity of pre-linearity:*

(4) $(x \rightarrow y) \vee (y \rightarrow x) = 1$ for any $x, y \in L$.

MTL-algebras is an algebraic counterpart of the so-called Monoidal t-norm logic [5] (MTL, for short).

Let us define on any commutative residuated lattice L the unary operation $^-$ (negation) by $x^- = x \rightarrow 0$.

A commutative residuated lattce L is

(a) a BL-algebra if and only if L satisfies divisibility condition and the identity of pre-linearity;

(b) an MV-algebra if and only if L fulfils the double negation law $x^{--} = x$;

(c) a Heyting algebra if and only if the operations $\cdot$ and $\wedge$ coincide on L.

Now it is obvious that an $R\ell$-monoid is an MTL-algebra if and only if it is a BL-algebra. The facts can be verified as for the BL case, since the pre-linearity condition is not involved, and hence the proofs are omitted.

Lemma 16 *In any bounded commutative $R\ell$-monoid M we have for any $x, y, z \in M$,*

(1) $x \leq y \Leftrightarrow x \rightarrow y = 1$.

(2) $x \leq y \Rightarrow x \cdot z \leq y \cdot z$.

(3) $x \leq y \Rightarrow z \rightarrow x \leq z \rightarrow y, y \rightarrow z \leq x \rightarrow z$.

(4) $x \rightarrow x = 1, 1 \rightarrow x = x, x \rightarrow 1 = 1$.

(5) $y \leq x \rightarrow y$.

(6) $x \leq x^{--}, x^- = x^{---}$.

(7) $x \leq y \Rightarrow y^- \leq x^-$.

(8) $(x \vee y)^- = x^- \wedge y^-$.

(9) $(x \wedge y)^- = x^{--} \wedge y^{--}$.

(10) $(x \cdot y)^- = y \rightarrow x^- = y^{--} \rightarrow x^- = x \rightarrow y^- = x^{--} \rightarrow y^-$.

(11) $(x \rightarrow y)^{--} = x^{--} \rightarrow y^{--}$.

In the sequel, unless otherwise stated, L always represents any given commutative residuated lattice with maximal element 1 and minimal element 0.

Definition 17 [16] *A filter of L is a subset $F \subseteq L$ with the properties*

(F1) $1 \in F$;

(F2) if $a \in F$ and $a \leq b, b \in L$, then $b \in F$;

(F3) if $a, b \in F$, then $ab \in F$.

Denote by $\mathcal{F}$ the set of all filters of L. Clearly, $\{1\}$ and L are, respectively, the smallest filter and the greatest filter of L. It is easy to see that a nonempty subset F of L is a filter of L if and only if it satisfies (F2) and (F3). Moreover, the following result gives an equivalent version of the concept of filters.

Theorem 18 [16] *A nonempty subset F of L is a filter of L if and only if it satisfies the following conditions:*

(F1) $1 \in F$;

(F4) $x \in F, x \rightarrow y$ imply $y \in F$.

Let $F \in \mathcal{F}$. If $x, x \rightarrow y \in F$, then it follows from Theorem 5 that $y \in F$. Thus, from a logical point of view, a filter corresponds to a set of provable formulas. Sometimes, a filter is also called a deductive system (see [16]) or **ds** in short.

Definition 19 [8, 14, 17] *A mapping $v : L \rightarrow L$ is called a weak vt-operator (wvt-operator in brief) on L if for any $x, y \in L$:*

(1) $v(1) = 1$,

(2) $v(x) \leq x$, i.e., v is subdiagonal,

(3) $v(x \rightarrow y) \leq v(x) \rightarrow v(y)$.

Moreover, if a wvt-operator v satisfies for any $x, y \in L$

(4) $v(x \vee y) \leq v(x) \vee v(y)$,

then v is called a vt-operator on L, if a wvt-operator v satisfies for any $x \in L$

(5) $v(v(x)) = v(x)$,

then v is called a hedge (see [9]) on L.

Any commutative residuated lattice admits vt-operators, e.g. the identity and the globalization v, where $v(x) = 0$ for $x \neq 1$ and $v(1) = 1$. Globalization can be seen as an interpretation of a connective "absolutely/fully true".

BL_{vt}-algebras [8] and $R\ell_{vt}$-monoids ($R\ell_{wvt}$-monoids) [14] are, respectively, BL-algebras with vt-operator and $R\ell$-monoids with vt-operator (weak vt-operator).

Let v be a wvt-operator on L. For any natural number n, we define $x^{(n)}$ recursively as follows: $x^{(0)} = x$ and $x^{(n)} = v(x^{(n-1)})$, where $x \in L$.

Theorem 20 *Let v be a weak vt-operator on L and $x, y, x \in L, n \in \mathcal{N}^+$. Then*

(1) $v(0)^{(n)} = 0$,

(2) $x \leq y \Rightarrow x^{(n)} \leq y^{(n)}$,

(3) $v(x^-) \leq (v(x))^-$.

(4) $xy \leq z \Rightarrow x^{(n)} y^{(n)} \leq z^{(n)}$,

(5) $x^{(n)} y^{(n)} \leq (xy)^{(n)}$.

(6) $v(x) \cdot v(x \rightarrow y) \leq v(x \wedge y) \leq v(x) \wedge v(y)$.

Moreover, if v is a vt-operator on L, then

(7) $(x \vee y)^{(n)} = x^{(n)} \vee y^{(n)}$.

Proof: (1) By the definition, $v(0) \leq 0$, hence $v(0) = 0$. Thus,$v(0)^{(n)} = 0$.

(2) Let $x, y \in L$ and $x \leq y$. Then

$x \rightarrow y = 1$,

hence by conditions (3) and (1) of Definition 19, we get $v(x) \rightarrow v(y) = 1$, and thus $v(x) \leq v(y)$.

Therefore, $x^{(n)} \leq y^{(n)}$.

(3) Let $x \in L$. Then by condition (3) of Definition 19 and by (1),

$v(x^-) = v(x \to 0) \leq v(x) \to v(0)$
$= v(x) \to 0 = (v(x))^-$.

(4) Let $x \cdot y \leq z$. Then

$x \leq y \to z$,

so by (2) and (3),

$v(x) \leq v(y \to z) \leq v(y) \to v(z)$,

and from this,

$v(x) \cdot v(y) \leq v(z)$.

Moreover, $x^{(n)} y^{(n)} \leq z^{(n)}$.

(5) It follows from (4) for $z = x \cdot y$.

(6) By (5) and (2),

$v(x) \cdot v(x \to y) \leq v(x \cdot (x \to y))$
$= v(x \wedge y) \leq v(x) \wedge v(y)$.

(7) By (2), we have

$v(x) \vee v(y) \leq v(x \vee y)$,

hence by condition (4) of Definition 19,

$v(x \vee y) = v(x) \vee v(y)$.

Thus, we have that $(x \vee y)^{(n)} = x^{(n)} \vee y^{(n)}$. □

Definition 21 *Let v be a weak vt-operator on L and F a filter of L. If $v(x) \in F$ for every $x \in F$, then F is called a v-filter of L.*

Denote by $\mathcal{F}_v$ the set of all v-filters of L.

For any $F \in \mathcal{F}_v$, it is easy to see that $C_F = \{(x, y) \mid (x \to y), (y \to x) \in F\}$ is a congruence relation on L. Moreover, if $(a, b) \in C_F$, then it follows from Definition 8 that $v(x) \to v(y) \in F$ and $v(y) \to v(x) \in F$ and hence $(v(x), v(y)) \in C_F$. We call C_F the v-congruence relation induced by v-filter on L.

Let $L_F = \{\overline{a} \mid a \in L\}$, where $\overline{a} = \{b \in L \mid (a, b) \in C_F\}$. Define a quasi-order "$\leq_F$" as follows:

$$\overline{a} \leq_F \overline{b} \Leftrightarrow a \to b \in F.$$

Clearly, $\overline{a} = \overline{b} \Leftrightarrow \overline{a} \leq_F \overline{b}$ and $\overline{b} \leq_F \overline{a} \Leftrightarrow (a, b) \in C_F$; and $\leq \subseteq \leq_F$.

It is easy to verify that $L_F = (L_F, \leq_F, \wedge_F, \vee_F, \cdot_F, \to_F, \overline{0}, \overline{1})$ is also a commutative residuated lattice, where

$$\overline{a} \wedge_F \overline{b} = \overline{a \wedge b},\ \overline{a} \vee_F \overline{b} = \overline{a \vee b},$$
$$\overline{a} \cdot_F \overline{b} = \overline{ab},\ \overline{a} \to_F \overline{b} = \overline{a \to b},\ \forall a, b \in L.$$

Here we call L_F the quotient residuated lattice of L with respect to the v-filter F and denote it by L/F.

Theorem 22 *Let v be a vt-operator (a weak vt-operator) on L and F a v-filter of L. Denote by $v_F : L/F \to L/F$ the mapping such that $v_F(\overline{x}) = \overline{v(x)}$, for each $x \in L$. Then v_F is a vt-operator (a weak vt-operator) on L/F.*

Proof: Firstly we will show that v_F is a correctly defined mapping of L/F into L/F.

Let $x, y \in L$ and $\overline{x} = \overline{y}$. Then $(x, y) \in C_F$, i.e.

$(x \to y) \wedge (y \to x) \in F$,

and thus also $x \to y, y \to x \in F$. Since F is a v-filter, we get

$v(x \to y), v(y \to x) \in F$,

and hence by condition (3) of the definition of a vt-operator we obtain

$v(x) \to v(y), v(y) \to v(x) \in F$.

By Lemma 16(5),

$v(y) \to v(x) \leq (v(x) \to v(y)) \to (v(y) \to v(x))$,

hence

$(v(x) \to v(y)) \to (v(y) \to v(x)) \in F$,

and this means $(v(x), v(y)) \in C_F$, i.e. $v_F(\overline{x}) = v_F(\overline{y})$.

Now it is easy to verify that the mapping v_F is a vt-operator on L/F.

(1) $v_F(\overline{1}) = \overline{v(1)} = \overline{1}$.

(2) $v_F(\overline{x}) = \overline{v(x)} \leq \overline{x}$.

(3) $v_F(\overline{x} \to_F \overline{y}) = v_F(\overline{x \to y}) = \overline{v(x \to y)}$
$\leq \overline{(v(x) \to v(y))} = \overline{v(x)} \to_F \overline{v(y)}$
$= v_F(\overline{x}) \to_F v_F(\overline{y})$.

(4) $v_F(\overline{x} \vee_F \overline{y}) = v_F(\overline{x \vee y}) = \overline{v(x \vee y)}$
$\leq \overline{(v(x) \vee v(y)} = \overline{(v(x)} \vee_F \overline{v(y)}$
$= v_F(\overline{x}) \vee_F v_F(\overline{y})$. □

Theorem 23 *If (L, v) is an $R\ell$-monoid, then there is a one-to-one correspondence between its v-filters and v-congruences.*

Proof: (a) Let C be a v-congruence on (L, v) and let

$F_C = \overline{1} = \{x \in L \mid (x, 1) \in C\}$.

Then F_C is a filter of the $R\ell$-monoid L. Let us suppose that $x \in F_C$. Then $(x, 1) \in C$, hence $(v(x), 1) = (v(x), v(1) \in C$, and therefore $v(x) \in F_C$. That means F_C is a v-filter on (L, v).

(b) Let F be a v-filter of (L, v) and let C_F be the corresponding congruence on L, i.e. $(x, y) \in C_F$ if and only if $(x \to y) \wedge (y \to x) \in F$. Hence, if $(x, y) \in C_F$ then also $v((x \to y) \wedge (y \to x)) \in F$. Let $(x, y) \in C_F$. Then by property (3) of a vt-operator and Theorem 20(6),

$(v(x) \to v(y)) \wedge ((v(y) \to v(x))$
$\geq v(x \to y) \wedge v(y \to x)$
$\geq v((x \to y) \wedge (y \to x)) \in F$,

hence $(v(x) \to v(y)) \wedge ((v(y) \to v(x)) \in F$, and this means $(v(x), v(y)) \in C_F$.

Therefore C_F is a v-congruence on (L, v). □

Now we will deal with $R\ell_{wvt}$-monoids (L, v) satisfying the identity

(P) $v(x \to y) \vee v(y \to x) = 1$.

Theorem 24 *If L is an $R\ell$-monoid, then there is a wvt-operator v on l satisfying (P) if and only if L is a BL-algebra.*

Proof: (a) Let L be an $R\ell$-monoid which is not a BL-algebra. Then there exist $x, y \in L$ such that

$(x \to y) \vee (y \to x) \neq 1.$

Hence for any wvt-operator v on L,

$v(x \to y) \vee v(y \to x) \leq (x \to y) \vee (y \to x) < 1,$

therefore (P) fails.

(b) Let L be a BL-algebra. Then for each vt-operator v on L and $x, y \in L$,

$v(x \to y) \vee v(y \to x)$
$= v((x \to y) \vee (y \to x)) = v(1) = 1,$

thus L satisfies (P). □

An $R\ell_{vt}$-monoid (L, v) is called an $R\ell_{vt}$-chain if the $R\ell$-monoid L is linearly ordered. By [13], the class of BL-algebras coincides with the class of (bounded commutative) $R\ell$-monoids which are representable as subdirect products of $R\ell$-chains. Hence, among others, every $R\ell_{vt}$-chain is in fact a BL_{vt}-chain. We will prove that every BL_{vt}-algebra is a subdirect product not only of BL-chains (i.e., as a BL-algebra in the corresponding signature), but, moreover, it is also such a product of BL_{vt}-chains (in the extended signature).

Recall that a filter F of an $R\ell$-monoid L is called prime if $F = G \cap H$ implies $F = G$ or $F = H$ for any filters G and H of L.

A prime filter is called minimal if it is a minimal element in the sets of prime filters of L ordered by set inclusion. By Zorn's lemma, every prime filter of L contains a minimal prime filter. For any $a \in L$ put

$a^{\perp} = \{x \in L \mid x \vee a = 1\}.$

If F is a prime filter of L, then

$x \vee y = 1$ implies $x \in F$ or $y \in F$

for each $x, y \in L$ and then the quotient $R\ell$-monoid L/F is linearly ordered.

If F is a minimal prime filter of L, then

$F = \cup\{a^{\perp} \mid a \in L - F\}.$

Theorem 25 *Every BL_{vt}-algebra is a subdirect product of BL_{vt}-chains.*

Proof: It is obvious that it suffices to prove that every BL_{vt}-algebra is isomorphic to subdirect product of BL_{vt}-chains. Since any BL-algebra L is representable as a subdirect product of BL-chains, the intersection of all minimal prime filters of $L$is equal to $\{1\}$. Hence, it remains to show that every minimal prime filter of L is a v-filter.

Let F be a minimal prime filter of L. Then

$F = \cup\{a^{\perp} \mid a \in L - F\}.$

Let $x \in F$. Then there exists $a \notin F$ such that $x \vee a = 1$, hence

$1 = v(1) = v(x \vee a) = v(x) \vee v(a).$

Since $a \notin F$, we get $v(a) \notin F$, therefore $v(x) \in F$ since F is a prime filter. □

Every vt-operator on an $R\ell$-monoid L is, by the definition and Theorem 20(2), a subdiagonal and monotone self mapping of L. Now, we use vt-operators to introduce derived self-mappings of L that are, among others, superdiagonal and monotone, and in the case of MV-algebras they have the properties of unary connectives "very false".

If L is a $R\ell$-monoid and $f : L \to L$, then we denote by f^- the mapping of L into L such that for any $x \in L$,

$f^-(x) = (f(x^-))^-.$

Let us consider the standard MV-algebra $[0, 1] = \Gamma(R, 1)$. It is known that the mapping $v : [0, 1] \to [0, 1]$ such that $v(x) = x^2$ is a vt-operator on $[0, 1]$. Then

$v^- : [0, 1] \to [0, 1]$

is the mapping such that $v^-(x) = 2x - x^2$ for each $x \in [0, 1]$.

We say that an $R\ell$-monoid L is normal if L satisfies the identity

$(x \cdot y)^{--} = x^{--} \cdot y^{--}.$

Remark: Every BL-algebra and every Heyting algebra is normal, hence the variety of normal $R\ell$-monoids is considerably wide.

Theorem 26 *Let (L, v) be an $R\ell_{vt}$-monoid. Then we have for any $x, y, z \in L$,*

(1) $v^-(0) = 0,\ v^-(1) = 1$,
(2) $x \leq v^-(x)$,
(3) $x \leq y$ implies $v^-(x) \leq v^-(y)$,
(4) $vv^-(x \wedge y) \leq v^-(x) \wedge v^-(y)$,
(5) $v^-(x \vee y) \geq v^-(x) \vee v^-(y)$,
(6) $v^-(x \to y^{--}) \leq v(x) \to v^-(y)$,
(7) $x \cdot y \leq z$ implies $v^-(x) \cdot v(y) \leq v^-(z)$,
(8) $v^-(x) \cdot v(y) \leq v^-(x \cdot y)$,
(9) $v^-(x) \cdot v(x \to y) \leq v^-(x \wedge y)$,
(10) If L is normal, then
$v^-(x \to y) \leq v(x^{--}) \to v^-(y)$
$\leq v(x) \to v^-(y)$,
(11) If L is an MV-algebra, then
$v^-(x \to y) \leq v(x) \to v^-(y)$.

Proof: (1) $v^-(0) = (v(0^-))^- = 1^- = 0$,

$v^-(1) = (v(1^-))^- = (v(0))^- = 0^- = 1.$

(2) $v^-(x) = ((v(x^-))^- \geq x^{--} \geq x.$

(3) $x \leq y \Rightarrow x^- \geq y^- \Rightarrow v(x^-) \geq v(y^-)$
$\Rightarrow (v(x^-))^- \leq (v(y^-))^- \Rightarrow v^-(x) \leq v^-(y).$

(4) and (5). They follow from (3).

(6) We have

$v^-(x \to y^{--}) = v^-((x \cdot y^-)^-)$
$= (v((x \cdot y^-)^{--}))^-,$

$v(x) \to v^-(y) = v(x) \to (v(y^-))^-$
$= ((v(x) \cdot v(y^-))^- \geq (v(x \cdot y^-))^-$
$\geq (v((x \cdot y^-)^-)^-,$
hence
$v^-(x \to y^{--}) \leq v(x) \to v^-(y).$

(7) By 3 and 6, we get
$x \cdot y \leq z \ \Rightarrow\ x \leq y \to z$
$\Rightarrow\ v^-(x) \leq v^-(y \to z) \leq v^-(y \to z^{--})$
$\Rightarrow\ v^-(x) \leq v(y) \to v^-(z)$
$\Rightarrow\ v^-(x) \cdot v(y) \leq v^-(z).$

(8) It follows from 7.

(9) $v^-(x) \cdot v(x \to y) \leq v^-(x \cdot (x \to y))$
$= v^-(x \wedge y).$

(10) $x \to y \leq y^- \to x^-$, thus from the normality of L we get
$v^-(x \to y) = (v((x \to y)^-))^-$
$\leq (v((y^- \to x^-)^-))^- = (v((y^- \cdot x)^{--}))^-$
$= (v(y^- \cdot x^{--}))^-.$
Furthermore,
$(v(y^-) \cdot v(x^-))^- = v(x^{--}) \to (v(y^-))^-$
$= v(x^-) \to v^-(y),$
and
$(v(y^- \cdot x^-))^- \leq (v(y^-) \cdot v(x^-))^-,$
we obtain
$v^-(x \to y) \leq v(x^{--}) \to v^-(y).$

(11) It follows from 6 as well as from 10. □

If L is an MV-algebra, then L satisfies the double negation law $x^{--} = x$, hence there exist on L both a t-norm $*$ and its associated residuum $\to$ and a t-conorm $\oplus$ and its associated residuum, say $\ominus$.

Consequently, on any MV-algebra L one can define not only vt-operators but also dual operators, vf-operators (vf= very false). We will show that every vt-operator on L determines a vf-operator on L.

As usual, put
$x \oplus y = (x^- \cdot y^-)^-$ and $x \ominus y = x \cdot y^-$,
for any $x, y \in L$.

Theorem 27 *If L is an MV-algebra and v is a vt-operator on L, then v^- is a vf-operator on L, i.e. for each $x, y \in L$ it holds:*
(1) $v^-(0) = 0$,
(2) $v^-(x) \geq x$,
(3) $v^-(x) \ominus v^-(y) \leq v^-(x \ominus y)$,
(4) $v^-(x \wedge y) = v^-(x) \wedge v^-(y)$.

Proof: It remains to verify properties (3) and (4).

(3) Using 8 from Theorem 26, we get
$v^-(x) \ominus v^-(y) = (v(x^-)^- \ominus (v(y^-))^-$
$= (v(x^-))^- \cdot (v(y^-))^{--} = (v(x^-))^- \cdot v(y^-)$
$= v^-(x) \cdot v(y^-) \leq v^-(x \cdot y^-) = v^-(x \ominus y).$

(4) Since any MV-algebra satisfies de Morgan laws for the lattice operations, we have
$v^-(x \wedge y) = (v((x \wedge y)^-))^-$
$= (v(x^- \vee y^-))^- = (v(x^-) \vee v(y^-))^-$
$= (v(x^-))^- \wedge (v(y^-))^- = v^-(x) \wedge v^-(y).$ □

Remark: Very recently, Vychodil [17] has introduced the notion of a $BL_{vt,st}$-algebra in order to study the so called truth depressing hedges on BL-algebras.

An algebra $(L, \vee, \wedge, \cdot, \to, v, s, 0, 1)$ is called a $BL_{vt,st}$-algebra if $L = (L, \vee, \wedge, \cdot, \to, 0, 1)$ is a BL-algebra, v is a wvt-operator on L and $s : L \to L$ is a mapping such that
(1) $s(0) = 0$,
(2) $x \leq s(x)$,
(3) $v(x \to y) \leq s(x) \to s(y)$.

We will show that if L is a BL-algebra and v is a wvt-operator on L, then
$(L, \vee, \wedge, \cdot, \to, v, v^-, 0, 1)$
is a $BL_{vt,st}$-algebra.

Let $x, y \in L$. Then,
$v^-(x) \to v^-(y) = (v^-(x^-))^- \to (v(y^-))^-$
$\geq v(y^-) \to v(x^-) \geq v(y^- \to x^-) \geq v(x \to y).$

4 Generated v-filters

In this section, we give the formula for calculating the v-filters generated by subsets of L.

Theorem 28 *If $F_j \in \mathcal{F}_v (j \in J)$, where J is any index set, then $\bigcap_{j \in J} F_j \in \mathcal{F}_v$.*

Proof: Straightforward. □

Given $A \subseteq L$, it follows from Theorem 10 that $\bigcap\{F \in \mathcal{F}_v \mid A \subseteq F\}$ is the smallest v-filter of L containing A. Here we call this v-filter the v-filter of L generated by A, and denote it by $< A >_{F_v}$.

We denote $< \{a_1, a_2, \ldots, a_n\} >_{F_v}$ by $< a_1, a_2, \ldots, a_n >_{F_v}$ for short.

It is easy to verify that for any subsets A, B of L,
(1) $< 1 >_{F_v} = \{1\}$ and $< \emptyset >_{F_v} = \{1\}$,
(2) $A \subseteq B \ \Rightarrow\ < A >_{F_v} \subseteq < B >_{F_v}$,
(3) if $A \in \mathcal{F}_v$, then $< A >_{F_v} = A$.

Below, we give the formula for calculating $< A >_{F_v}$.

Theorem 29 *If A is a nonempty subset of L, then*

$$< A >_{F_v} = \{x \in L \mid x \geq a_1^{(m_1)} \cdots a_s^{(m_s)},\ a_1, \ldots, a_s \in A,\ m_1, \ldots, m_s \in \mathcal{N}^+\}.$$

Proof: Denote

$$B = \{x \in L \mid x \geq a_1^{(m_1)} \cdots a_s^{(m_s)},\ a_1, \ldots, a_s \in A,\ m_1, \ldots, m_s \in \mathcal{N}^+\}.$$

We first prove that B is a v-filter of L. Clearly, $1 \in B$. Let $x \in B$ and $x \le y$, $y \in L$. Then there are $a_1, ..., a_s \in A$ and $m_1, \dots, m_s \in \mathcal{N}^+$ such that $x \ge a_1^{(m_1)} \cdots a_s^{(m_s)}$ and hence $y \ge a_1^{(m_1)} \cdots a_s^{(m_s)}$, i.e., $y \in B$. Now let $y \in B$. Then there are $b_1, ..., b_t \in A$ and $n_1, \dots, n_t \in \mathcal{N}^+$ such that $y \ge b_1^{(n_1)} \cdots b_t^{(n_t)}$ and so

$$\begin{aligned} xy &\ge (a_1^{(m_1)} \cdots a_s^{(m_s)})(b_1^{(n_1)} \cdots b_t^{(n_t)}) \\ &= a_1^{(m_1)} \cdots a_s^{(m_s)} b_1^{(n_1)} \cdots b_t^{(n_t)}, \end{aligned}$$

i.e., $xy \in B$. Moreover, it follows from Theorem 7 that

$$\begin{aligned} v(x) &\ge v(a_1^{(m_1)} \cdots a_s^{(m_s)}) \ge v(a_1^{(m_1)}) \cdots v(a_s^{(m_s)}) \\ &= a_1^{(m_1+1)} \cdots a_s^{(m_s+1)}, \end{aligned}$$

i.e., $v(x) \in B$. Thus, B is a v-filter of L.

Finally we prove that B is the least v-filter containing A. Clearly, $A \subseteq B$. Now assume that $A \subseteq F \in \mathcal{F}_v$ and let $x \in B$. Then there exist $a_1, ..., a_s \in A$ and $m_1, \dots, m_s \in \mathcal{N}^+$ such that $x \ge a_1^{(m_1)} \cdots a_s^{(m_s)}$. Applying Definitions 4 and 8, we obtain $a_1^{(m_1)} \cdots a_s^{(m_s)} \in F$ and $x \in F$. This shows $B \subseteq F$. Therefore B is the least v-filter containing A, i.e., $B =< A >_{F_v}$. □

For any $a_1, ..., a_n \in L$, $a_n \to (\cdots \to (a_1 \to x) \cdots) = 1$ if and only if $a_n \le a_{n-1} \to (\cdots \to (a_1 \to x) \cdots)$ if and only if $x \ge a_1 \cdots a_{n-1} a_n$. Thus, we have an equivalent form of Theorem 11.

Theorem 30 *Let A be a nonempty subset of L. Then*

$$\begin{aligned} &< A >_{F_v} \\ &= \{x \in L \mid a_1^{(m_1)} \to (\cdots (a_s^{(m_s)} \to x) \cdots) = 1, \\ &\qquad a_1, \dots, a_s \in A,\ m_1, \dots, m_s \in \mathcal{N}^+\}. \end{aligned}$$

As a corollary of Theorem 11, we have the following theorem.

Theorem 31 *If $F \in \mathcal{F}_v$ and $a \in L$. then*

$$\begin{aligned} &< F \cup \{a\} >_{F_v} = \{x \in L \mid x \ge a^{(m_1)} \cdots a^{(m_s)} f, \\ &f \in F, m_1, \dots, m_s \in \mathcal{N}^+\}. \end{aligned}$$

In particular,

$$\begin{aligned} &< a >_{F_v} = \{x \in L \mid x \ge a^{(m_1)} \cdots a^{(m_s)}, \\ &m_1, \dots, m_s \in \mathcal{N}^+\}. \end{aligned}$$

We denote $< F \cup \{a\} >_{F_v}$ by $F(a)$ for convenience. Clearly, $F(a \vee b) \subseteq F(a) \cap F(b)$.

5 Lattice of v-filters

Noting that $\{1\}$ and L are, respectively, the smallest element and the greatest element of $\mathcal{F}_v$, we see that $\mathcal{F}_v$ is a complete lattice. For any $F_1,\ F_2 \in \mathcal{F}_v$, $F_1 \cap F_2$ is the greatest lower bound of F_1 and F_2. We denote by $F_1 \vee_{\mathcal{F}_v} F_2$ the least upper bound of F_1 and F_2. Obviously, $F_1 \vee_{\mathcal{F}_v} F_2 =< F_1 \cup F_2 >_{F_v}$.

Theorem 32 *If $F_1,\ F_2 \in \mathcal{F}_v$, then*

$$F_1 \vee_{\mathcal{F}_v} F_2 = \{x \in L \mid x \ge ab,\ a \in F_1,\ b \in F_2\}.$$

Proof: Suppose that $F_1,\ F_2 \in \mathcal{F}_v$. Let

$$F = \{x \in L \mid x \ge ab,\ a \in F_1,\ b \in F_2\}.$$

Clearly, $F \subseteq F_1 \vee_{\mathcal{F}_v} F_2$. If $x \in F_1 \vee_{\mathcal{F}_v} F_2 =< F_1 \cup F_2 >_{F_v}$, then it follows from Theorem 3.2 that there are $a_1, ...,\ a_s \in F_1$, $b_1, ...,\ b_t \in F_2$ and $m_1, \dots, m_s, n_1, \dots, n_t \in \mathcal{N}^+$ such that

$$x \ge (a_1^{(m_1)} \cdots a_s^{(m_s)})(b_1^{(n_1)} \cdots b_t^{(n_t)}),$$

where s, t are nonnegative integers. Let $a = a_1^{(m_1)} \cdots a_s^{(m_s)} \in F_1$ and $b = b_1^{(n_1)} \cdots b_t^{(n_t)} \in F_2$. Then, $x \ge ab$, i.e, $x \in F$. Thus, $F_1 \vee_{\mathcal{F}_v} F_2 \subseteq F$.

Therefore, $F =< F_1 \cup F_2 >_{F_v} = F_1 \vee_{\mathcal{F}_v} F_2$.□

Let A, B be two subset of L. By Theorem 14 and its proof, we can see that

$$< A \cup B >_{F_v} =< A >_{F_v} \vee_{\mathcal{F}_v} < B >_{F_v}.$$

In particular, for any $a, a_1, ..., a_n \in L$, we have that

$$\begin{aligned} &F(a) = F \vee_{\mathcal{F}_v} < a >_{F_v} \quad \forall F \in \mathcal{F}_v, \\ &< a_1, ..., a_n >_{F_v} \\ &=< a_1 >_{F_v} \vee_{\mathcal{F}_v} \cdots \vee_{\mathcal{F}_v} < a_n >_{F_v}. \end{aligned}$$

Theorem 33 *Let v be a vt-operator on L. If $A \in \mathcal{F}_v$ and B is an upper subset of L, i.e., $b \in B, c \in L, b \le c \Rightarrow c \in B$, then $A \cap < B >_{F_v} =< A \cap B >_{F_v}$.*

Proof: Clearly, $< A \cap B >_{F_v} \subseteq A \cap < B >_{F_v}$.

If $x \in A \cap < B >_{F_v}$, then $x \in A$ and $x \in < B >_{F_v}$. Thus, it follows from Theorem 11 that there are $b_1, ..., b_t \in B$ and $n_1, \dots, n_t \in \mathcal{N}^+$ such that $x \ge b_1^{(n_1)} \cdots b_t^{(n_t)}$. Noting that A and B are two upper subsets of L, we see that $x \vee b_1, ..., x \vee b_t \in A \cap B$. It follows from Theorem 7 that

$$\begin{aligned} &x \ge (x \vee b_1^{(n_1)}) \cdots (x \vee b_t^{(n_t)}) \\ &\ge (x^{(n_1)} \vee b_1^{(n_1)}) \cdots (x^{(n_t)} \vee b_t^{(n_t)}) \\ &= (x \vee b_1)^{(n_1)} \cdots (x \vee b_t)^{(n_t)} \end{aligned}$$

and hence $x \in < A \cap B >_{F_v}$. Thus,

$A \cap < B >_{F_v} \subseteq < A \cap B >_{F_v}$.

Therefore, $A \cap < B >_{F_v} =< A \cap B >_{F_v}$. □

Theorem 34 *Let v be a vt-operator on L. Then $< \mathcal{F}_v;\ \subseteq,\ \cap,\ \vee_{\mathcal{F}},\ \{1\},\ L >$ is a complete Brouwerian lattice.*

Proof: For any $A, B_j \in \mathcal{F}_v (j \in J)$, where J is any nonempty index set, noting that $\cup_{j\in J} B_j$ is a upper subset of L, we see that

$$
\begin{aligned}
& A \cap (\vee_{\mathcal{F}_v\ j\in J} B_j) = A\cap < \cup_{j\in J} B_j >_{F_v} \\
= \ & < A \cap (\cup_{j\in J} B_j) >_{F_v} = < \cup_{j\in J} (A \cap B_j) >_{F_v} \\
= \ & \vee_{\mathcal{F}_v\ j\in J} < A \cap B_j >_{F_v} = \vee_{\mathcal{F}_v\ j\in J} (A \cap B_j).
\end{aligned}
$$

Therefore, it follows from Chapter V, Theorem 24 in [3] that $< \mathcal{F}_v;\ \{1\},\ L,\ \subseteq,\ \cap,\ \vee_{\mathcal{F}_v} >$ is a complete Brouwerian lattice. □

6 Conclusion and future work

In this paper, based on Hájek [8, 9], Vychodil [17], Rachůnek and Šalounová [14], we study the concept of v-filters of residuated lattices with weak vt-operators, axiomatize very true operators and discuss filters and v-filters of residuated lattices with weak vt-operator. We also give the formulas for calculating the v-filters generated by subsets, and show that the lattice of v-filters of a residuated lattice with a vt-operation is a complete Brouwerian lattice.

In a forthcoming paper, we will study prime v-filters of a residuated lattices with weak vt-operators.

Acknowledgements: This material is based upon work funded by Zhejiang Provincial Natural Science Foundation of China under Grant No.Y1080239.

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