The number “6” in Planar Tilings

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Abstract: This paper obtains some important properties of planar normal tiling and proves purely combinatorially Grünbaum’s Theorem. Moreover, we give the six-neighbor-theorem and definite “relative density” to describe the increase of tiles with some special properties. Finally, $\Gamma_{pm}-$tilings are classified by their adjacent-types.

Key–Words: Normal tiling, Adjacent, Neighbor, Adjacent-graph, Relative density, Adjacent-type,

1 Introduction

Tilings appears in many fields such as art design (architecture), crystal structure in thin sheet of materials (metallurgy and geology), combination of molecule (chemistry), cell arrangements in skin and membranes of animals and plants (biology), image enhancement and coding (communication theory) and so on. Thus it looks so necessary to study mathematically tiling which relates to combinatorics([17]), number theory([1]), topology([10]) and wavelet([15]) and so on. A planar tiling is a kind of covering of the plane without any overlap and gap. To say mathematically, if a compact sets family $\mathscr{A} := \{T_i,\ i \in \mathbb{Z}^+\}$ satisfies that $\bigcup T_i = \mathbb{R}^2$ and $int(T_i) \cap int(T_j) = \phi,\ \forall i \neq j$, then $\mathscr{A}$ is called a *planar tiling* and each member of $\mathscr{T}$ is called *planar tile*. The complexity of tiling mainly come from two hands: strange structures of tiles and their various ways to fit each other. For instance, polygons can tile a plane through a group (honeycomb), or in random (rimous riverbed). Moreover, some tiling can be formed by translating one tile which may be a square or a fractal even. Reinhardt points out that stringent restrictions must be imposed on tilings if any meaningfully general results are to be obtained, which also indicates that the classification of tiling is necessary. There is a class of tiling called “well-behaved” tiling, with better local topological structure and global combinatorial properties, which plays a crucial role in researches of various tilings.

Normal tiling is an important kind of “well-behaved” tiling with some restrictions on the structure of the tiles and the way to fit together. A planar tiling $\mathscr{T}$ is called *normal*, if the following three conditions hold:

C.1 *every tile of $\mathscr{T}$ is a topological disk (disk-like);*

C.2 *The intersection of every two tiles of $\mathscr{T}$ is a connected set, that is, it does not consist of two (or more) distinct and disjoint parts;*

C.3 *the tiles of $\mathscr{T}$ are uniformly bounded.*

C.1 implies that each tile is bounded by a simple closed curve. In the tiling which satisfies **C.1** and **C.2**, the Schönflies theorem ([13]) tells us that the boundary of each tile is simple closed curve and the intersection of two tiles is either empty, or a single point, or an arc. And **C.3** means that there exist two fixed positive numbers R and r for the tiling in question, called the *circle-parameters* of the normal tiling, so that every tile can contain some circular disk of radius r and be contained in some circular disk of radius R, which implies that all tiles have some balance properties.

In normal tiling, the *vertex (n-covered-points)* of tile (tiling) are those points which are covered by n($\geq$3) tiles. The point covered only by two tiles are called *2-covered-points*. Two tiles T_1 and T_2 are *neighbors*, if $T_1 \cap T_2 \neq \phi$. And they are *adjacent (edge-neighbor)*, if $int(T \cup T') \cap \partial T = \phi$, that is to say, their intersection is an arc called the *edge* of tile (tiling). If $T_1 \cap T_2$ is one point, they are *vertex-neighbor*. The interest of people in normal tiling mostly focus on the topological structure of tiles and combinatorial properties of tiling. However, neighbor and adjacent rightly reflect the details of these two hands of normal tiling. Actually, the neighbors of tile determine its boundary and the neighbor-relation decides the arranging mode. Therefore, two tilings with the same neighbor-relation are regarded as the same one in some sense, notwithstanding they are possibly with various structures. Laves ([8], [9]) shows that if all tileS of monohedral tiling (generated by one tile) have neighbors with the uniform number n, then $n \leq 21$. Fejes Tóth ([16]) points out that for normal tiling (not always Monohedral), n may be infinite. Moreover, Fejes Tóth ([17]) has a conjecture that in every monohedral tiling with convex tiles, "21" is the maximum of neighbor-number of tile, so do even for non-convex tiles. Bezdek ([3]) asserts that in normal tiling with convex tiles, if every tile has the same numbers of vertices and neighbors, the neighbor-number just should be 6, 7, 8, 9, 10, 12, 14, 16 or 21.

The definition of normal tiling looks so simply, but it is usually difficult to be applied directly. For example, is the aperiodic tiling, showed in Fig.1, normal? It is bewildering that to make a decision just by the definition of normal tiling. Nevertheless, we can decidedly say "no" by Grünbaum's theorem (called six-adjacent-theorem)([6]) which is an essential and universal result about normal tiling. But his topological proof looks so obscure and lengthy.

In this paper, we give a purely combinatorial and legible proof for the six-adjacent-theorem in virtue of adjacent-graph. Moreover, the six-neighbor-theorem, describing the global properties of normal tiling from other angles, is given. The two theorems indicate that the number "6" is a special constant for normal tiling. We also raise rationally the conception of "relative density" to describe the increase of tiles with some special properties and estimate the lower boundary of relative density of six-adjacent-tile in normal tiling. Finally, as an example to apply the six-neighbor-theorem, the adjacent-type and the combinatorial classification of Γ_{pm}−tiling is obtained.

2 Some properties of Normal Tiling

A tiling is called *locally finite* if any circular disk, centered any point, meets only a finite number of tiles. Locally finite tiling can avoid some singular geometric structures. The following two lemmas offer a good many conveniences to our work.

Lemma 1 *([6]) For any tile T of locally finite tiling $\mathscr{T}$,*

(i) T has only a finite number of neighbors, and the intersection of any two tiles consists of a finite number of connected components.

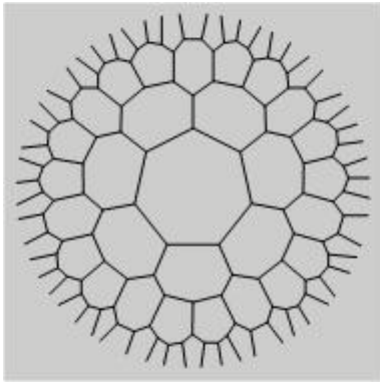

Figure 1: Is normal the tiling with convex heptagons?

(ii) Any boundary point of T must belong to at least one other tile of $\mathscr{T}$.

(iii) The boundary of T consists of a finite number of edges of $\mathscr{T}$, and each vertex is a connected component of the intersection of some three tiles of $\mathscr{T}$.

Lemma 2 *Every normal tiling $\mathscr{T}$ is locally finite.*

Proof. We consider any one disk $D(x, P)$ of radius x, centered point P on plane. Then it is clear that any tile which meets $D(x, P)$ must lie entirely inside the disk $D(x + 2R, P)$ whose area equal to $\pi(x + 2R)^2$, where R is a circle-parameter of $\mathscr{T}$. However, every tile contains a disk with radius r so that its area is πr^2 at least. Thus not more than $M(x) = \lfloor \pi(x + 2R)^2/\pi r^2 \rfloor + 1$ tiles can meet $D(x, P)$. Since D is arbitrary and $M(x)$ is finitely fixed for x, $\mathscr{T}$ is locally finite. $\square$

Figure 2: A normal tiling traced by black curves and its adjacent-graph formed by tint points and lines.

After being drawn on plane, the tiling become its *topological graph*. The *adjacent-graph* G_A of tiling is an abstract graph whose vertices are all tiles of the tiling and edge are all adjacent-relation in the tiling. Similarly, the *neighbor-graph* G_N of a tiling is with vertex-set all tiles of the tiling and edge-set all neighbor-relation in the tiling. Obviously, the adjacent-graph and neighbor-graph can exhibit mainly the tiling way, but avoid the complexity produced by the topology and shape of single tile.

Lemma 3 *The adjacent-graph of normal tiling is always planar.*

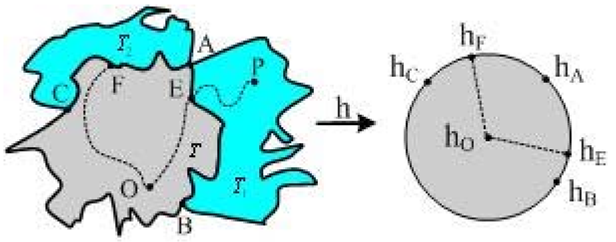

Figure 3: The planar adjacent-graph of normal tiling.

Proof. Let any one tile T in normal tiling $\mathscr{T}$ meet its two adjacents T_1 and T_2 to two edges AB and AC, which is shown in Fig.3. We can take arbitrarily two inner-points E and F of AB and AC. Let the homeomorphism h map T to the circular disk D. Then by Lemma 1, the vertices A, B and C, the 2-covered-points E and F, the edges AB and AC are mapped to the points on the circle h_A, h_B, h_C, h_E, h_F and the arcs $\widetilde{h_Ah_B}$ and $\widetilde{h_Ah_C}$, where h_E and h_F still are the inner-points of $\widetilde{h_Ah_B}$ and $\widetilde{h_Ah_C}$ respectively. Moreover, the inverse-image O of the center h_O of D is an inner-point of T. Two line-segments $\overline{h_Oh_E}$ and $\overline{h_Oh_F}$ in D meet only at the point h_O, and then their inverse-images $\widetilde{OE}(= h^{-1}(\overline{h_Oh_E}))$ and $\widetilde{OF}(= h^{-1}(\overline{h_Oh_F}))$ are two arcs in T meeting only at the point O. Similarly, we can obtain an arc $\widetilde{PE}$ in T_1, where P is an inner-point of T_1. $\widetilde{OP}$ forms an arc from an inner-point of T, through a common boundary-point E of T and T_1, to an inner-point of T_1. All such arcs as $\widetilde{OP}$ for every adjacent of T meet at O. Doing the same thing for all tile of $\mathscr{T}$, we can obtain a graph embedded on plane which is dual to the topological graph of $\mathscr{T}$. As be seen easily, the graph is rightly an planar embedding of the adjacent-graph of T. □

Lemma 4 *In normal tiling, two tiles are neighbors if and only if the vertices corresponding to them in adjacent-graph lie on one common face.*

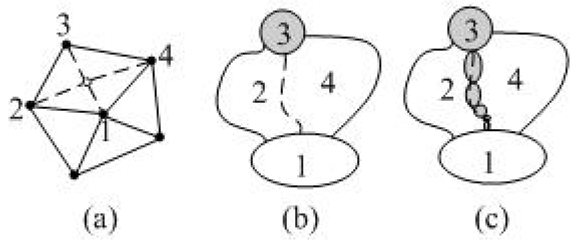

Figure 4: The adjacent-graph decides the neighbor-graph.

Proof. We need only to consider the simple case that four vertices 1, 2, 3 and 4 in the adjacent-graph corresponding to tiles T_1, T_2, T_3, and T_4 of normal tiling $\mathscr{T}$ lie on the common face shown in Fig.3, where T_2 and T_4 are adjacents of T_1.

Suppose that T_3 is not a neighbor of T. Then either T_2 and T_4 are adjacents (Fig.4 (b)), or there are a string of finite adjoining tiles between T_1 and T_3 (Fig.4 (c)). In Fig.4 (b), vertices 1 and 3 should be blocked off by the circuit generated the vertices corresponding to all adjacents of T_1, which contradicts to 1 and 3 lying on a common face. In Fig.4 (c), if regarded all tiles surrounded by T_1, T_2, T_3 and T_4 as a whole tile, 1 and 3 should also be blocked off by some one circuit (Fig.4 (a)). Hence, T_1 and T_3 should be neighbor. It is easy to obtain inductively the similar result for the case of more vertices on one common face.

To prove the necessary part of the lemma is only an inverse thought. □

Remark 5 *Lemma 3 and 4 imply that the adjacent-graph can decide neighbor-graph, and thus it is really essential, which enlightens us to discuss the global properties of normal tiling by purely combinatorial method.*

3 Six-adjacent-theorem and Six-neighbor-theorem

We introduce the conception of *patch* of Grünbaum. Patch consists of a finite number of tiles with a property that their union is a topological disk. There is a standard procedure for constructing patches. We take a connected set $\mathcal{S}$ (for example a square) in tiling $\mathscr{T}$. The set $\mathscr{M}$ of all tiles of $\mathscr{T}$ that meet $\mathcal{S}$ is considered firstly. The union of the tiles of $\mathscr{M}$, the white area in Fig.5, will clearly to be connected but it may fail to be simply connected. Adjoining to $\mathscr{M}$ just enough tiles, the black area $\mathscr{N}$ in Fig.5, to fill up the "holes" (see to Fig.5), we obtain the patch $\mathscr{P}(\mathcal{S})$ (=$\mathscr{M} \cup \mathscr{N}$) generated by $\mathcal{S}$.

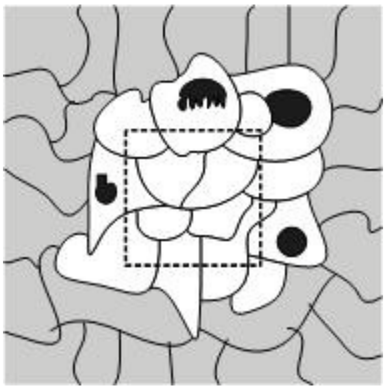

Figure 5: To generate a patch by a square.

The following lemma points out that the low boundaryof the number of adjacents of tile in normal tiling.

Lemma 6 *Every tile of normal tiling should be with at least three adjacents, that is to say, the degree of every vertex of adjacent-graph is at least 3.*

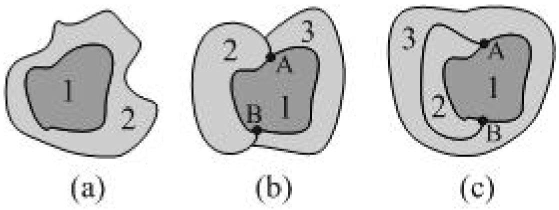

Figure 6: Every tile should be with at least three adjacents.

Proof. Let T_1, T_2, and T_3 be three tiles of normal tiling $\mathscr{T}$. Suppose that T_1 has uniquely one adjacent T_2, then $\partial T_1 = T_1 \cap T_2$ being homeomorphic to a circle $\mathbb{C}$, which is a contradiction to **C.2** in the definition of normal tiling. And suppose that T_1 has only two adjacents T_2 and T_3 (see to Fig.6), then $\partial T_1 = (T_1 \cap T_2) \cup (T_1 \cap T_3)$. Let the map $\tau : \partial T_1 \to \mathbb{C}$ be a homeomorphism. Since both $T_1 \cap T_2$ and $T_1 \cap T_3$ are connected and $\tau((T_1 \cap T_2) \cup (T_1 \cap T_3)) = \mathbb{C}$, $T_2 \cap T_3$ should contain the vertices A and B . And because of the connectedness of $T_2 \cap T_3$, we have that $T_2 \cap T_3 = \overline{\partial T_2 - (T_1 \cap T_2)}$, i.e., T_2 is surrounded by T_1 and T_3. Hence, it will hold that T_3 surrounds $T_1 \cup T_2$ shown in Fig.6. But by Jordan Curve Theorem ([13]), T_3 will divide the plane into two disjoint connected parts, which conflicts to T_3 being disk-like. □

By the local finiteness of normal tiling, adjacent-graph of a patch should be a connected finite graph which satisfies the following Euler's Theorem.

Lemma 7 *(Euler's Theorem)([4]) Let $v(\mathcal{L})$, $e(\mathcal{L})$ and $f(\mathcal{L})$ be respectively the numbers of vertices, edges, and faces of adjacent-graph of the patch generated by a curve $\mathcal{L}$. Then they will satisfy the following equation*

$$v(\mathcal{L}) - e(\mathcal{L}) + f(\mathcal{L}) = 1.$$

Grünbaum shows topologically the following called six-adjacent-theorem. Here, we give a more compendious purely combinational proof.

Theorem 8 *Every normal tiling contains a infinite numbers of tiles each of which has at most six adjacents.*

Proof. Suppose that some one normal tiling $\mathscr{T}$ contains only finitely many tiles each of which has at most six adjacents. We can take arbitrarily a circuit $\mathcal{L}$ which includes all vertice with degrees at most 6 in the adjacent-graph G_A of $\mathscr{T}$ shown in Fig.7. By Jordan Curve Theorem ([13]), $\mathcal{L}$ should divide G_A into three parts: $\mathcal{L}$, outside $\mathcal{L}$ (unbounded connected component), and inside $\mathcal{L}$ (bounded connected component). We use respectively $v_1(\mathcal{L})$, $v_2(\mathcal{L})$, $v_3(\mathcal{L})$, $e_1(\mathcal{L})$, $e_2(\mathcal{L})$, and $f(\mathcal{L})$ to denote the numbers of the vertices on $\mathcal{L}$, the vertices with degrees at least seven inside $\mathcal{L}$, the vertices with degree at most six inside $\mathcal{L}$, the edges on $\mathcal{L}$, the edges inside $\mathcal{L}$ and the faces inside $\mathcal{L}$. Then by Lemma 7, it holds that

$$[v_1(\mathcal{L})+v_2(\mathcal{L})+v_3(\mathcal{L})]-[e_1(\mathcal{L})+e_2(\mathcal{L})]+f(\mathcal{L}) = 1. \tag{1}$$

It is clear that

$$v_1(\mathcal{L}) = e_1(\mathcal{L}). \tag{2}$$

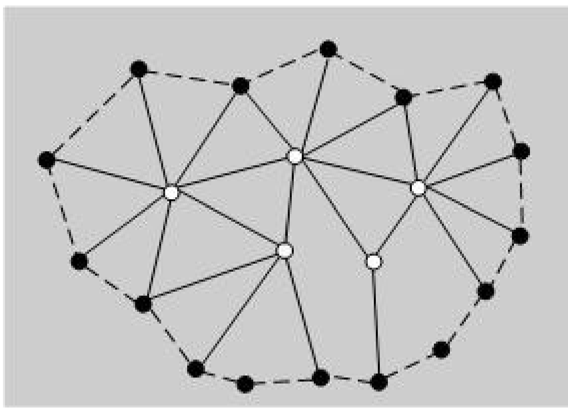

Figure 7: A circuit in adjacent-graph.

Hence, we can deduce that

$$v_2(\mathcal{L}) + v_3(\mathcal{L}) - e_2(\mathcal{L}) + f(\mathcal{L}) = 1. \tag{3}$$

Since each edge links two vertices rightly and each face contains at least three edges, we have the following two inequalities by Lemma 6 that

$$\begin{array}{rcl} 2e_2(\mathcal{L}) - e_1(\mathcal{L}) & \geq & 7v_2(\mathcal{L}) + 3v_3(\mathcal{L}) \\ \Rightarrow e_2(\mathcal{L}) & \geq & \frac{7v_2(\mathcal{L})+3v_3(\mathcal{L})+v_1(\mathcal{L})}{2}; \end{array} \quad (4)$$

$$\begin{array}{rcl} 3f(\mathcal{L}) & \leq & e_1(\mathcal{L}) + 2e_2(\mathcal{L}) \\ \Rightarrow f(\mathcal{L}) & \leq & \frac{e_1(\mathcal{L})+2e_2(\mathcal{L})}{3}. \end{array} \quad (5)$$

Therefore, by Eq.(1)-(4), we have

$$\begin{array}{rcl} 1 & = & v_2(\mathcal{L}) + v_3(\mathcal{L}) - e_2(\mathcal{L}) + f(\mathcal{L}) \\ & \leq & v_2(\mathcal{L}) + v_3(\mathcal{L}) - e_2(\mathcal{L}) + \frac{e_1(\mathcal{L})+2e_2(\mathcal{L})}{3} \\ & = & v_2(\mathcal{L}) + v_3(\mathcal{L}) - \frac{1}{3}e_2(\mathcal{L}) + \frac{1}{3}v_1(\mathcal{L}) \\ & \leq & v_2(\mathcal{L}) + v_3(\mathcal{L}) \\ & & -\frac{1}{3} \cdot \frac{7v_2(\mathcal{L})+3v_3(\mathcal{L})+v_1(\mathcal{L})}{2} + \frac{1}{3}v_1(\mathcal{L}) \\ & = & -\frac{v_2(\mathcal{L})}{6} + \frac{v_3(\mathcal{L})}{2} + \frac{v_1(\mathcal{L})}{6} \end{array} \quad (6)$$

That is to say that

$$v_1(\mathcal{L}) \geq v_2(\mathcal{L}) - 3v_3(\mathcal{L}) + 6 \quad (7)$$

In the following development, we will obtain a contradiction to Eq.(7).

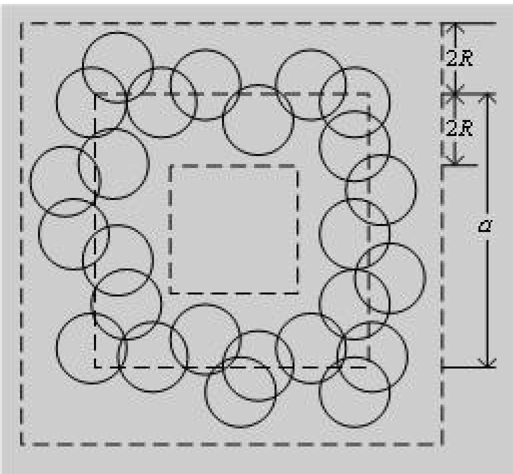

Figure 8: The boundary tiles of square patch.

We also take a square $\mathcal{S}$ with edge a to include all tiles each of which has at most six adjacents on plane and use $\mathscr{P}(\mathcal{S})$ to denote the patch generated by $\mathcal{S}$. Clearly, the adjancent graph of $\mathscr{P}(\mathcal{S})$ will produce a circuit $\mathcal{L}$. We still use the denotations in what mentioned above. Since there exist two constants R and r such that each of tile is contained in a disk with radius R and contains a disk with radius r, the square $\mathcal{S}_1$ with edge length $a + 4R$ can contain the entire patch $\mathscr{P}(\mathcal{S})$ and all tiles to meet the boundary of $\mathcal{S}$ are outside of the square $\mathcal{S}_2$ with edge length $a - 4R$, which is shown in Fig.8. Hence, we have that

$$\begin{array}{rcl} v_1(\mathcal{L}) \cdot \pi r^2 & \leq & (a + 4R)^2 - (a - 4R)^2 \\ \Rightarrow v_1(\mathcal{L}) & \leq & \frac{16Ra}{\pi r^2}, \end{array} \quad (8)$$

and

$$\begin{array}{rcl} & & [v_1(\mathcal{L}) + v_2(\mathcal{L}) + v_3(\mathcal{L})] \cdot \pi R^2 \\ & \geq & area(\mathcal{S}) = a^2 \\ \Rightarrow v_2(\mathcal{L}) & \geq & \frac{a^2}{\pi R^2} - \frac{16Ra}{\pi r^2} - v_3(\mathcal{L}). \end{array} \quad (9)$$

Since $v_3(\mathcal{L})$ is a fixed finite constant, $v_2(\mathcal{L})$ will exceed seriously $v_1(\mathcal{L})$ as a becomes enough large, which contradicts to Eq.(7). Therefore, the theorem holds. □

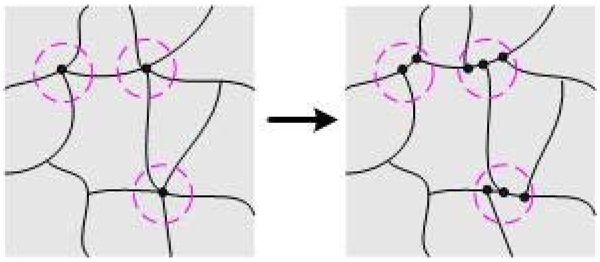

Figure 9: The vertices uniformization in circle.

As shown in Fig.9, when the valence of some one vertex A in a tile T of $\mathscr{T}$ is more three, those edges which meet at A but not be in T are slipped along edges except one keeping being fixed. In this way, a new tiling $\mathscr{T}^*$, called *the vertices uniformization* of $\mathscr{T}$, is constructed. A notable characteristic of such vertices uniformization tiling is that the valence of each vertex equals to 3. Honeycomb just is such an excellent example.

Lemma 9 *The uniformization process of normal tiling can not increase the number of neighbors of any tile.*

Proof. Every vertex are only with finite valence due to local finiteness of normal tiling. The neighbors of tile T are decided completely by the edges to intersect T, that to say, the neighbors of T are rightly those tiles whose some edges meet T. Let the valence of vertex A of tile T be $n(\geq 3)$ (Fig.10 (a)). If $n - 3$ edges are moved along only two edges of T in uniformization processing of A (Fig.10 (b)), new neighbors of T will not appear and just all vertex-neighbors at A become adjacents of T. Otherwise, the edges meeting T decrease (Fig.10 (c)), and then the number of neighbors of T get less. Hence, the lemma holds. □

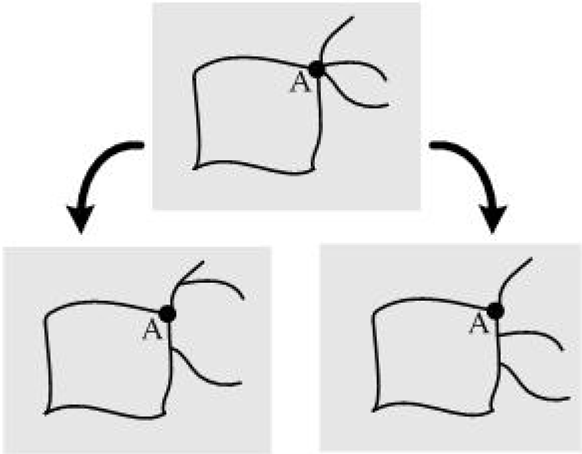

Figure 10: The uniformization process can not increase the number of neighbors.

Let $\mathscr{L}$ be any one circuit in the adjacent-graph of normal tiling $\mathscr{T}^*$. Denote the numbers of the vertices on $\mathscr{L}$, the vertices with at most 5-degree inside $\mathscr{L}$, the vertices with at least 6-degree inside $\mathscr{L}$, the edges on $\mathscr{L}$, the edges inside $\mathscr{L}$ and the faces (tiles) inside $\mathscr{L}$ by $\mathscr{V}_1(\mathscr{L})$, $\mathscr{V}_2(\mathscr{L})$, $\mathscr{V}_3(\mathscr{L})$, $\mathscr{E}_1(\mathscr{L})$, $\mathscr{E}_2(\mathscr{L})$, and $\mathscr{F}(\mathscr{L})$.

Then we have the following lemma to list some obvious properties of $\mathscr{T}^*$.

Lemma 10 *For normal tiling $\mathscr{T}^*$ and an arbitrary circuit $\mathscr{L}$ in its, the following facts hold*
(i) $\mathscr{V}_1(\mathscr{L}) = \mathscr{E}_1(\mathscr{L})$;
(ii) All neighbors of each tile inside $\mathscr{L}$ must be its adjacents.
(iii) The adjacent-graph of $\mathscr{T}^*$ is a triangulation of the plane.

Theorem 11 *Every normal tiling $\mathscr{T}$ contains infinitely many tiles each of which has at least six neighbors.*

Proof. According to Lemma 9, we only need to prove the vertex uniformization tiling $\mathscr{T}^*$ satisfying the theorem. Suppose that there is $\mathscr{T}^*$ only containing finitely many tiles each of which has at least six neighbors. Any one circuit $\mathscr{L}$, containing all vertices with at least 6-degree, can be taken arbitrarily in G_A^*, the adjacent graph of $\mathscr{T}^*$. Euler's Formula holds inside $\mathscr{L}$ and Lemma 10 (ii) implies that $G_A^* = G_N^*$. Then we have

$$\begin{array}{rcl} [\mathscr{V}_1(\mathscr{L}) + \mathscr{V}_2(\mathscr{L}) + \mathscr{V}_3(\mathscr{L})] & & \\ -[\mathscr{E}_1(\mathscr{L}) + \mathscr{E}_2(\mathscr{L})] + \mathscr{F}(\mathscr{L}) & = & 1. \end{array} \quad (10)$$

By Lemma 10 (i), we deduce that

$$\mathscr{V}_2(\mathscr{L}) + \mathscr{V}_3(\mathscr{L}) - \mathscr{E}_2(\mathscr{L}) + \mathscr{F}(\mathscr{L}) = 1. \quad (11)$$

Let the number of neighbors of the tile with the most neighbors in $\mathscr{T}^*$ be $M(\geq 6)$. Then, by Lemma 11 (ii) and (iii), we have the formulas

$$\begin{array}{rcl} 3\mathscr{F}(\mathscr{L}) & = & 2\mathscr{E}_2(\mathscr{L}) + \mathscr{E}_1(\mathscr{L}) \\ \Rightarrow \mathscr{F}(\mathscr{L}) & = & \frac{\mathscr{E}_1(\mathscr{L})+2\mathscr{E}_2(\mathscr{L})}{3}. \end{array} \quad (12)$$

and

$$\begin{array}{rcl} & & 5[\mathscr{V}_1(\mathscr{L}) + \mathscr{V}_2(\mathscr{L})] + M\mathscr{V}_3(\mathscr{L})] \\ & \geq & 2[\mathscr{E}_1(\mathscr{L}) + \mathscr{E}_2(\mathscr{L})] \\ \Rightarrow \mathscr{E}_2(\mathscr{L}) & \leq & \frac{3\mathscr{V}_1(\mathscr{L})+5\mathscr{V}_2(\mathscr{L})+M\mathscr{V}_3(\mathscr{L})]}{2}. \end{array} \quad (13)$$

Hence, by Eq.(11)-(13), we have

$$\begin{array}{rcl} 1 & = & \mathscr{V}_2(\mathscr{L}) + \mathscr{V}_3(\mathscr{L}) - \mathscr{E}_2(\mathscr{L}) + \mathscr{F}(\mathscr{L}) \\ & = & \mathscr{V}_2(\mathscr{L}) + \mathscr{V}_3(\mathscr{L}) - \mathscr{E}_2(\mathscr{L}) + \frac{\mathscr{E}_1(\mathscr{L})+2\mathscr{E}_2(\mathscr{L})}{3} \\ & = & \mathscr{V}_2(\mathscr{L}) + \mathscr{V}_3(\mathscr{L}) - \frac{\mathscr{E}_2(\mathscr{L})}{3} + \frac{\mathscr{V}_1(\mathscr{L})}{3} \\ & \geq & \mathscr{V}_2(\mathscr{L}) + \mathscr{V}_3(\mathscr{L}) \\ & & -\frac{1}{3} \cdot \frac{3\mathscr{V}_1(\mathscr{L})+5\mathscr{V}_2(\mathscr{L})+M\mathscr{V}_3(\mathscr{L})}{2} + \frac{\mathscr{V}_1(\mathscr{L})}{3} \\ & = & \frac{\mathscr{V}_2(\mathscr{L})}{6} - \frac{\mathscr{V}_1(\mathscr{L})}{6} - (\frac{M}{6} - 1)\mathscr{V}_3(\mathscr{L}) \end{array} \quad (14)$$

That is to say that

$$\mathscr{V}_1(\mathscr{L}) \geq \mathscr{V}_2(\mathscr{L}) - (M-6)\mathscr{V}_3(\mathscr{L}) - 6 \quad (15)$$

To be similar to the latter part of the proof of Theorem 8, we also obtain two equalities

$$\mathscr{V}_1(\mathscr{L}) \leq \frac{16Ra}{\pi r^2} \quad (16)$$

and

$$\mathscr{V}_2(\mathscr{L}) \geq \frac{a^2}{\pi R^2} - \frac{16Ra}{\pi r^2} - \mathscr{V}_3(\mathscr{L}) \quad (17)$$

Since both $\mathscr{V}_3(\mathscr{L})$ and M are fixed finite numbers for one given $\mathscr{T}^*$, Eq.(15) implies that $\mathscr{V}_1(\mathscr{L})$ can dominate linearly $\mathscr{V}_2(\mathscr{L})$. But Eq.(16) and Eq.(17) tell us that $\mathscr{V}_1(\mathscr{L})$ increases linearly at most and $\mathscr{V}_1(\mathscr{L})$ does in square at least with the largening of a. This is a contradiction. □

Remark 12 *Six-adjacent-theorem and six-neighbor-theorem reflect the global properties of normal tiling in two different hands. In particular, there are always an infinite numbers of tiles with at most six neighbors and ones with at least six neighbors in the uniformized tiling.*

4 The Relative Density in Normal Tiling

Ostensibly, for general normal tiling, Six-adjacent-Theorem points out the number of tiles with six adjacents at most. And in fact, suppose that we watch

$\mathscr{T}$ through a "window", Six-adjacent-Theorem also reflects those tiles increase infinitely as the "window" expands. To characterize the increasing, we define "the relative density" in this section.

We continue to denote the patch generated by the square $\mathcal{S}$ by $\mathscr{P}(\mathcal{S})$, and the boundary of adjacent-graph of $\mathscr{P}(\mathcal{S})$ by $\mathcal{B}_\mathcal{S}$. The relative density of $p-$tile (the tile with some property "p") is defined by

$$RD_p(\mathscr{T}) := \varliminf_{|\mathcal{S}|\to+\infty} \frac{v_p(\mathcal{B}_\mathcal{S})}{v(\mathcal{B}_\mathcal{S})},$$

where $v_p(\mathcal{B}_\mathcal{S})$ is the number of vertices of adjacent-graph corresponding to $p-$tiles inside $\mathcal{B}_\mathcal{S}$, and $v(\mathcal{B}_\mathcal{S})$ is the number of all vertices inside $\mathcal{B}_\mathcal{S}$, and $|\mathcal{S}|$ is the edge-length of $\mathcal{S}$.

If the denotations in the proof of Theorem 8 still are used, then

$$\varliminf_{|\mathcal{S}|\to+\infty} \frac{v_3(\mathcal{B}_\mathcal{S})}{v_1(\mathcal{B}_\mathcal{S}) + v_2(\mathcal{B}_\mathcal{S}) + v_3(\mathcal{B}_\mathcal{S})}$$

is the relative density of 6-adjacent-tile (the tile described in Theorem 8).

The following lemma is a foundational result to character the balance of normal tiling.

Lemma 13 *(Normality Lemma)([14]) If $\mathscr{T}$ is a normal tiling, then for every $x > 0$,*

$$\lim_{a\to\infty} \frac{v(a+x,P)}{v(a,P)} = 1$$

where $v(a,P)$ is the number of vertices in adjacent-graph of the patch generated by the square with edge-length a and center P.

Lemma 14 *For a given square $\mathcal{S}$, $RD_p(\mathscr{T})$ will be irrelative to the position and size of the initial square.*

Proof. Let the relative density of $p-$tile for a given square $\mathcal{S}$ with edge-length a and center P be

$$\varliminf_{a\to\infty} \frac{v_p(\mathcal{S})}{v(\mathcal{S})}.$$

It is obvious that the relative densities corresponding to all squares with center-point P has a same value by the definition of relative density. Thus we can take arbitrarily another square $\mathcal{S}_1$ with center P_1 and enough edge-length such that

$$v_p(\mathcal{B}_\mathcal{S}) \le v_p(\mathcal{B}_{\mathcal{S}_1}), \quad v(\mathcal{B}_\mathcal{S}) \le v(\mathcal{B}_{\mathcal{S}_1}).$$

We also take a square S' with center P and enough edge-length such that

$$v_p(\mathcal{B}_{\mathcal{S}_1}) \le v_p(\mathcal{B}_{\mathcal{S}'}), \quad v(\mathcal{B}_{\mathcal{S}_1}) \le v(\mathcal{B}_{\mathcal{S}'}).$$

Hence,

$$\frac{v_p(\mathcal{B}_\mathcal{S})}{v(\mathcal{B}_\mathcal{S})} \cdot \frac{v(\mathcal{B}_\mathcal{S})}{v(\mathcal{B}_{\mathcal{S}'})} \le \frac{v_p(\mathcal{B}_{\mathcal{S}_1})}{v(\mathcal{B}_{\mathcal{S}_1})} \le \frac{v_p(\mathcal{B}_{\mathcal{S}'})}{v(\mathcal{B}_{\mathcal{S}'})} \cdot \frac{v(\mathcal{B}_{\mathcal{S}'})}{v(\mathcal{B}_\mathcal{S})}.$$

If $a \to \infty$, then $|\mathcal{S}|$, $|\mathcal{S}_1|$ and $|\mathcal{S}'|$ go to the infinity synchronously. By the Normality Lemma,

$$\lim_{a\to\infty} \frac{v(\mathcal{B}_\mathcal{S})}{v(\mathcal{B}_{\mathcal{S}'})} = \lim_{a\to\infty} \frac{v(\mathcal{B}_{\mathcal{S}'})}{v(\mathcal{B}_\mathcal{S})} = 1.$$

We have already known that

$$\varliminf_{a\to\infty} \frac{v_p(\mathcal{B}_\mathcal{S})}{v(\mathcal{B}_\mathcal{S})} = \varliminf_{a\to\infty} \frac{v_p(\mathcal{B}_{\mathcal{S}'})}{v(\mathcal{B}_{\mathcal{S}'})}.$$

Finally,

$$\varliminf_{a\to\infty} \frac{v_p(\mathcal{B}_\mathcal{S})}{v(\mathcal{B}_\mathcal{S})} = \varliminf_{a\to\infty} \frac{v_p(\mathcal{B}_{\mathcal{S}_1})}{v(\mathcal{B}_{\mathcal{S}_1})} = \varliminf_{a\to\infty} \frac{v_p(\mathcal{B}_{\mathcal{S}'})}{v(\mathcal{B}_{\mathcal{S}'})}.$$

□

The following theorem implies that the increasing infinitely of 6-adjacent-tile is rather rapidly.

Theorem 15 *In every normal tiling, 6-adjacent-tile have always a positive relative density. And it has a lower boundary $\frac{r^2}{4R^2}$, where R is the minimum and r is the maximum satisfying* ***C.3****.*

Proof. We take arbitrarily a square $\mathcal{S}$ with length $a(= |\mathcal{S}|)$. For $v_1(\mathcal{B}_\mathcal{S})$, $v_2(\mathcal{B}_\mathcal{S})$ and $v_3(\mathcal{B}_\mathcal{S})$, Eq.(7), (8) and (9) keep holding. From Eq.(7) and (8), we can obtain

$$3v_3(\mathcal{B}_\mathcal{S}) \ge v_2(\mathcal{B}_\mathcal{S}) + 6 - \frac{16Ra}{\pi r^2} \tag{18}$$

According to Eq.(9) and (19), we have

$$3v_3(\mathcal{B}_\mathcal{S}) \ge \frac{a^2}{\pi R^2} - \frac{32Ra}{\pi r^2} + 6 - v_3(\mathcal{B}_\mathcal{S}) \tag{19}$$

Moreover,

$$v_3(\mathcal{B}_\mathcal{S}) \ge \frac{a^2}{4\pi R^2} - \frac{8Ra}{\pi r^2} + \frac{3}{2} \tag{20}$$

It holds which is similar to Eq.(8) and (9) that

$$(v_1(\mathcal{B}_\mathcal{S}) + v_2(\mathcal{B}_\mathcal{S}) + v_3(\mathcal{B}_\mathcal{S}))\pi r^2 \le (a+4R)^2 \tag{21}$$

Hence, we have

$$\begin{array}{ll} & \frac{v_3(\mathcal{B}_\mathcal{S})}{v_1(\mathcal{B}_\mathcal{S})+v_2(\mathcal{B}_\mathcal{S})+v_3(\mathcal{B}_\mathcal{S})} \\ \ge & \left(\frac{a^2}{4\pi R^2} - \frac{8Ra}{\pi r^2} + \frac{3}{2}\right) \cdot \frac{\pi r^2}{(a+4R)^2} \end{array} \tag{22}$$

Letting a go to the infinity, we can immediately deduce

$$\varliminf_{a\to+\infty} \frac{v_3(\mathcal{B}_\mathcal{S})}{v_1(\mathcal{B}_\mathcal{S}) + v_2(\mathcal{B}_\mathcal{S}) + v_3(\mathcal{B}_\mathcal{S})} \ge \frac{r^2}{4R^2}.$$

□

Remark 16 *We believe that tiles with at least six neighbors also has a relative density with positive lower boundary. But it is a pity that we have not found it.*

Example 17 *Fig.11 is a kind of familiar design of ceramic floor tiling which is normal, where both the relative densities of squares (with 4 adjacents) and octagons (with 8 adjacents) equal to* $\frac{1}{2}$.

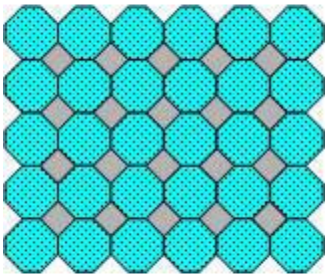

Figure 11: Positive relative density.

5 Adjacent-graph of $\Gamma_{pm}-$tiling

A *planar crystallographic group* is a discrete cocompact subgroup $\Gamma \subset \mathbf{Isom}(\mathbb{R}^2)$, where $\mathbf{Isom}(\mathbb{R}^2)$ is the group of isometries on $\mathbb{R}^2$. The planar tiling $\Gamma(T) := \cup\{\gamma(T) : \gamma \in \Gamma\}$ is called *crystallographic tiling*, if $T \subset \mathbb{R}^2$ is compact and Γ is a crystallographic group. $\Gamma(T)$ should be made of congruent tiles and locally finite. Fedorov ([7]) points out that the planar crystallographic group only have 17 kinds whose detailed derivation can be found in [12]. Mackay ([11]) mentions an ingenious method of displaying 17 crystallographic groups by various jigsaw pieces fitting together. In this section, we will apply the six-adjacent-theorem to consider $\Gamma_{pm}-$tiling generated by crystallographic group Γ_{pm} and to obtain its a kind of combinatorial classification. We recall the following result of Gelbrich ([5]).

Lemma 18 *Suppose T is disk-like, Γ is a planar crystallographic group. Then the intersection of two arbitrary tiles of the crystallographic tiling $\Gamma(T)$ is either empty, or one point, or a topological line segment.*

This lemma implies that every planar crystallographic tiling with disk-like tiles must be normal. Since each tile of crystallographic tiling is with same adjacent structure, the follow theorem is a natural conclusion of the six-adjacent-theorem and Lemma 6 using the terminology of graph theory.

Theorem 19 *The adjacent-graph of planar disk-like crystallographic tiling is at most 6-connected and at least 3-connected.*

Lattice tiling $\Gamma_{p1}-$tiling, where Γ_{p1} is generated only two translation with different directions, is the simplest kind of 17 crystallographic tilings. Bandt and Gelbrich analyze the classification of $\Gamma_{p1}-$tiling and obtain the following result which implies that its adjacent-graph only is 4-connected or 6-connected.

Lemma 20 *([2]) Let Ω be a topological disk which tiles $(\mathbb{R}^2)$ by lattice translates of lattice $\mathcal{L}$. Then in the tiling $\Omega + \mathcal{L}$ one of the following must be true:*

(i) Ω has no vertex-neighbor and six adjacents $\Omega \pm \alpha$, $\Omega \pm \beta$ and $\Omega \pm (\alpha + \beta)$ for some $\alpha, \beta \in \mathcal{L}$, and $\mathbb{Z}\alpha + \mathbb{Z}\beta = \mathcal{L}$;

(ii) Ω has four adjacents $\Omega \pm \alpha$, $\Omega \pm \beta$ and four vertex-neighbors $\Omega \pm \alpha \pm \beta$ for some $\alpha, \beta \in \mathcal{L}$, and $\mathbb{Z}\alpha + \mathbb{Z}\beta = \mathcal{L}$.

Γ_{pm} can be generated by three planar isometries: two translations with different directions and a mirror-reflection whose axis is parallel to some one translation-direction.

Theorem 21 *It is impossible that the adjacent-graph of disk-like $\Gamma_{pm}-$tiling is 6-connected.*

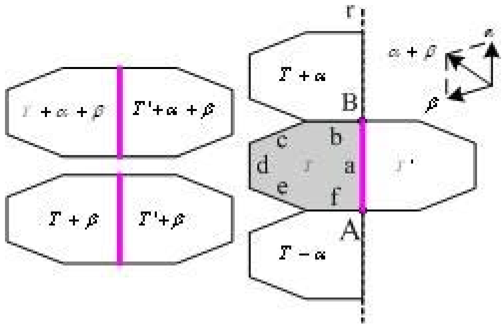

Figure 12: It is impossible that the tile with 6 adjacents appear in $\Gamma_{pm}-$Tiling.

Proof. By the symmetry of $\Gamma_{pm}-$tiling, we just consider arbitrarily one tile T. Suppose T being with 6 adjacents. If the 6 adjacents are all translation-equivalences of T, then the tiling $\Gamma_{pm}(T)$ is a lattice tiling.

Therefore, there is one adjacent T' of T is not a translation-image, but a mirror-image with the axis r, which is shown in Fig.12. If $S(= T \cup T')$ is regarded as a new tile, it is an important fact that S can tile the plane under some one lattice. Hence, by Lemma 20, there is two irrelevant vectors α and β so that one of the following two cases holds:

(i) S has no vertex-neighbor and six adjacents $S \pm \alpha$, $S \pm \beta$ and $S \pm (\alpha + \beta)$;

(ii) S has four adjacents $S \pm \alpha$, $S \pm \beta$.

We can suppose that α is parallel to r by the definition of Γ_{pm}. Case (i) is discussed firstly. For a legible depiction, we denote anticlockwise the edges of T (the intersections of T and its six adjacents) by a, b, c, d, e, f, and draw them with line-segments. As shown in Fig.12, the line-segment $\overline{AB}(= T \cap T')$ is denoted especially by a. We consider six translation-images of a: $\overline{AB} \pm \alpha$, $\overline{AB} \pm \beta$ and $\overline{AB} \pm (\alpha + \beta)$. Since $\alpha(A) = B$ and $B \in b$, then $T \cap (T + \alpha) = b$. Similarly, we have that $T \cap (T - \alpha) = f$. Because of β being not parallel to r, $S + \beta$ should locate wholly on one side (left side) of r and one of $T + \beta$ or $T' + \beta$ should be an adjacent of T. But c, d and e are all impossibly one of $\beta-$translation-images of b, c, d or e. Thus $T' + \beta$ is one adjacent of T, and $T \cap (T' + \beta) = c$, d or e. Similarly, as an adjacent, $T' + \alpha + \beta$ meets T on one of c, d or e. Due to the characteristics of translation and mirror-reflection, there should not be other adjacents of S on left side of r except $S + \beta$ and $S + \alpha + \beta$, which implies that there should not be more adjacents except T', $T + \alpha$, $T - \alpha$, $T' + \beta$ and $T' + \alpha + \beta$. This is a contradiction to our initial supposition. Hence, T has five adjacents at most. □

We give Fig.13 to illustrate the following result in sense of isomorphism.

Theorem 22 *The adjacent-graph of disk-like $\Gamma_{pm}-$tiling can be 3-connected, 4-connected or 5-connected.*

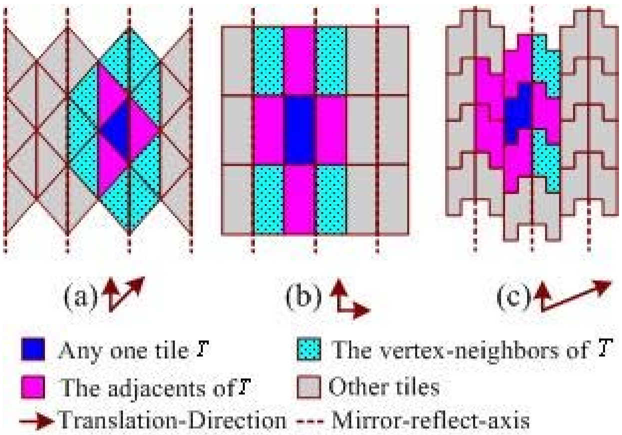

Figure 13: Three possible kinds of pattern in $\Gamma_{pm}-$tiling: (a) 3-adjacents, (b) 4-adjacents,(c) 5-adjacents.

From Theorem 19, 21 and 22, the combinatorial classification of $\Gamma_{pm}-$tiling mentioned in the reference [6] can be obtained directly, which will not be said more.

Conclusion

Six-adjacent-theorem and Six-neighbor-theorem imply that the number "6" play an important role on the combinatorial structure and some intrinsic property of normal tiling, which is perhaps related to the planar kissing-number being 6 rightly. However, we still wonder why it is so. It is worthy exploring the relation between them. Then, a natural question is what happens in high-dimension tilings. A conjecture is that the similar number is "12" for 3-dimension tiling.

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