Some Preconditioning Techniques for Linear Systems

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Abstract: New convergence intervals of parameters α_i are derived and applied for solving the modified linear systems, which enables a better understanding of how parameters should be chosen. The convergence theorem for H-matrix is given. Meanwhile, we discuss the convergence results for M-matrices linear systems and give some new preconditioners. Numerical examples are used to illustrate our results.

Key–Words: Convergence, H-matrix, M-matrix, Preconditioner, Gauss-Seidel method

1 Introduction

In numerical linear algebra, the theory of M-and H-matrices is very important for the solution of linear systems of algebra equations by iterative methods (see, e.g., [1], [4-9],[12-14]). For example, (a) in linear complementarity problem (LCP) (see section 10.1 of [10] for specific applications), where we are interested in finding a $z \in R^n$ such that $z \geq 0$, $Mz + q \geq 0$, $z^T(Mz + q) = 0$, with $M \in R^{n\times n}$ and $q \in R^n$ given, a sufficient condition for a solution to exist, and to be found by a modification of an iterative method, especially of SOR, is that M is an H-matrix, with $m_{i,i} > 0, i = 1, \cdots, n$[15]; (b) in fluid analysis, in the car modeling design [16], [17], it was observed that large linear systems with an H-matrix coefficient A are solved iteratively much faster if A is postmultiplied by a suitable diagonal matrix D, with $d_{i,i} > 0, i = 1 \cdots, n$, so that AD is strictly diagonally dominant. We consider the following linear system

$$Ax = b, \tag{1}$$

where A is an $n \times n$ square matrix, x and b are two $n-$ dimensional vectors. For any splitting, $A = M - N$ with the nonsingular matrix M, the basic iterative method for solving the linear system (1) is as follows:

$$x^{i+1} = M^{-1}Nx^i + M^{-1}b \qquad i = 0, 1, 2, \cdots, \tag{2}$$

Direct methods, based on the factorization of the coefficient matrix A into easily invertible matrices, are widely used and are the solver of choice in many industrial codes, especially where reliability is the primary concern. Indeed, direct solvers are very robust, and they tend to require a predictable amount of resources in terms of time and storage [19, 20]. With a state-of-the-art sparse direct solver (see, e.g., [21]) it is possible to efficiently solve in a reasonable amount of time linear systems of fairly large size, particularly when the underlying problem is two dimensional. Direct solvers are also the method of choice in certain areas not governed by PDEs, such as circuits, power system networks, and chemical plant modeling.

To be fair, the traditional classification of solution methods as being either direct or iterative is an oversimplification and is not a satisfactory description of the present state of affairs. First, the boundaries between the two classes of methods have become increasingly blurred, with a number of ideas and techniques from the area of sparse direct solvers being transferred (in the form of preconditioners) to the iterative camp, with the result that iterative methods are becoming more and more reliable. Second, while direct solvers are almost invariably based on some version of Gaussian elimination, the field of iterative methods comprises a bewildering variety of techniques, ranging from truly iterative methods, like the classical Jacobi, Gauss-Seidel, and SOR iterations, to Krylov subspace methods, which theoretically converge in a finite number of steps in exact arithmetic, to multilevel methods. To lump all these techniques under a single heading is somewhat misleading, especially when preconditioners are added to the picture.

The focus of this paper is on preconditioning techniques for improving the performance and reliability of Krylov subspace methods. It is widely recognized that preconditioning is the most critical ingredient in the development of efficient solvers for challenging problems in scientific computation, and that the importance of preconditioning is destined to increase even further. Indeed, much effort has been put in the development of effective preconditioners, and preconditioning has been a more active research area than either direct solution methods or Krylov subspace methods for the past few years. Because an optimal general-purpose preconditioner is unlikely to exist, this situation is probably not going to change in the foreseeable future.

In general, a good preconditioner P should meet the following requirements:
(1)The preconditioned system should be easy to solve.
(2)The preconditioner should be cheap to construct and apply.

The first property means that the preconditioned iteration should converge rapidly, while the second ensures that each iteration is not too expensive. Notice that these two requirements are in competition with each other. It is necessary to strike a balance between the two needs. With a good preconditioner, the computing time for the preconditioned iteration should be significantly less than that for the unpreconditioned one.

We now transform the original system (1) into the preconditioned form

$$PAx = Pb, \tag{3}$$

where P is a nonsingular matrix. The corresponding basic iterative method is given in general by

$$x^{i+1} = M_P^{-1}N_P x^i + M_P^{-1}Pb \qquad i = 0, 1, 2, \cdots,$$

where $PA = M_P - N_P$ is a splitting of PA.

The preconditioners for solving the modified linear systems were considered by Milaszewicz[1] who based his idea on previous ones(see, e.g.,[2-4]), by Gunawardena et al.[5], by Kohno et al.[6] who extended the main idea in [5], by Li and Sun[7] who extended the class of matrices considered in [6], and recently by A.Hadjidimos et al.[9] who generalize the most common preconditioners. Many results were obtained for further preconditioners (see, e.g.,[13-14]).

In a simpler form, Milaszewicz [1] considered the preconditioner

$$P_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ -a_{2,1} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1,1} & 0 & \cdots & 1 & 0 \\ -a_{n,1} & 0 & \cdots & 0 & 1 \end{bmatrix},$$

which eliminates the elements of the first column of A below the diagonal. Gunawardena et al. [5] considered as a preconditioner the matrix

$$S_1 = \begin{bmatrix} 0 & -a_{1,2} & 0 & \cdots & 0 \\ 0 & 0 & -a_{2,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

whose effect on A is to eliminate the elements of the first upper diagonal.

In 1997, Kohno et al.[6] proposed a general method for improving the modified Gauss-Seidel method with the modified matrix $P = I + S_\alpha$, if A is a nonsingular diagonally dominant $Z-$ matrix with some conditions, where

$$S_\alpha = \begin{bmatrix} 0 & -\alpha_1 a_{1,2} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\alpha_{n-1}a_{n-1,n} \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

they showed numerically that the modified Gauss-Seidel method is superior to the other methods if the parameters $\alpha_i (i = 1, 2, \cdots, n-1)$ are chosen appropriately.

In 2003, A.Hadjidimos et al.[9] considered The generalized (parametrized) preconditioner used in this case is of the form

$$P_\alpha = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ -\alpha_2 a_{2,1} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\alpha_{n-1}a_{n-1,1} & 0 & \cdots & 1 & 0 \\ -\alpha_n a_{n,1} & 0 & \cdots & 0 & 1 \end{bmatrix},$$

they parametrized Milaszewicz's preconditioner by using the idea in [6] and base our Jacobi and GaussC-Seidel iterative schemes on the ones in [1] (see also [4]), sufficient conditions on the α's are given that guarantee convergence and the best, in some sense, set of the α's is found.

In this paper, we consider the preconditioned linear system of the form

$$\tilde{A}x = \tilde{b}, \tag{4}$$

where $\tilde{A} = (I + S_\alpha)A$ and $\tilde{b} = (I + S_\alpha)b$. From the equality

$$\tilde{A} = (I + S_\alpha)A = (I + S_\alpha)(I - L - U)$$

$$= I - L - S_\alpha L - U + S_\alpha - S_\alpha U,$$

If we apply the Gauss-Seidel iterative method to the preconditioned linear system (4), then we get the preconditioned Gauss-Seidel iterative method whose iteration matrix is

$$\tilde{T} = (\tilde{D} - \tilde{L})^{-1}\tilde{U}. \quad (5)$$

This paper is organized as follows. In Section 2, we present some definitions and preliminary results. In Section 3, we consider the convergence of the preconditioned Gauss-Seidel method for H-matrix and derive new convergence intervals of parameters α_i which are compared with that in Theorem 3.4[12]. In Section 4, we discuss the convergence results for M-matrix. In Section 5, we will give some new preconditioner. Meanwhile, we give a few numerical examples to illustrate our results.

2 Preliminaries

We first recall the following: A matrix $A = (a_{i,j}) \in R^{n\times n}$ is called a Z-matrix if $a_{i,j} \leq 0$ for $i \neq j$. A real vector $x = (x_1, x_2, \cdots, x_n)^T$ is called nonnegative(positive) and denoted by $x \geq 0 (x > 0)$ if $x_i \geq 0 (x_i > 0)$ for all i. Similarly, a matrix $A = (a_{i,j})$ is called nonnegative and denoted by $A \geq 0$ if $a_{i,j} \geq 0$ for all i, j. Let $B = (b_{i,j}) \in R^{n\times n}$, then we denote $A \geq B (A > B)$ if $a_{i,j} \geq b_{i,j} (a_{i,j} > b_{i,j})$ for any i, j.

Definition 1 *A matrix A is called an $M-$ matrix if $A = sI - B, B \geq 0$ and $s > \rho(B)$, where $\rho(B)$ denotes the spectral radius of B.*

Definition 2 *A matrix A is an $H-$ matrix, if its comparision matrix $\langle A\rangle = (\bar{a}_{i,j})$ is an $M-$ matrix, where $\bar{a}_{i,j}$ is*

$$\bar{a}_{i,i} = |a_{i,i}|, \qquad \bar{a}_{i,j} = -|a_{i,j}|, \quad i \neq j.$$

Definition 3 *([8]). The splitting $A = M - N$ is called $H-$ splitting if $\langle M\rangle - |N|$ is an $M-$ matrix.*

Lemma 4 *([10]). Let A be a Z -matrix. Then the following statements are equivalent:*
(a)A is a nonsingular M-matrix.
(b)All principle submatrices of A are nonsingular M-matrices.
(c)All principle minors are positive.

Lemma 5 *([11]). Let A be an H-matrix, then $|A^{-1}| \leq \langle A\rangle^{-1}$.*

Lemma 6 *([12]). Let*

$$\beta_i = 1 + \frac{|a_{i,i+1}| + 1}{|a_{i,i+1}|(2\|\langle A\rangle^{-1}\|_\infty - 1)},$$

then $\beta_i > 1, i = 1, 2, \cdots, n-1$.

Remark 7 *In Remark 3.3 [12], we know that the question of whether taking $\alpha_i \in [0, \beta_i)$ is advantageous is not taken into account. After all, to compute the values of β_i, one has to compute $\|\langle A\rangle^{-1}\|_\infty$, which implies additional work.*

Theorem 8 *([12]). Let A be an H-matrix. Then for any $\alpha_i \in [0, \beta_i), i = 1, 2, \cdots, n-1$($\beta_i$ defined as in Lemma 2.6), $\tilde{A}$ is an H-matrix and $\rho(\tilde{T}) < 1$.*

Theorem 9 *([12]). Let $A = I - L_h - U_h$ be an H-matrix, where L_h is a strictly lower triangular matrix and U_h is a general matrix. Suppose that $A = (I - L_h) - U_h$ is an H-compatible splitting of A. If for any $\alpha_i \in [0, 1], i = 1, 2, \cdots, n-1$, the iteration matrix of the IMGS method corresponding to A is $T^h_\alpha = (I - L_h - S_\alpha L_h)^{-1}(U_h - S_\alpha + S_\alpha U_h)$, then $\rho(T^h_\alpha) < 1$. Moreover, if $\hat{T^h_\alpha}$ is the iteration matrix of the IMGS method corresponding to $\langle A\rangle$, then*

$$\rho(T^h_\alpha) \leq \rho(\hat{T^h_\alpha}) < 1.$$

Remark 10 *In Remark 3.5 [12], the author are not sure whether the result of Theorem 2.8 can be extended to the case $\alpha_i \in [0, \beta_i), i = 1, 2, \cdots, n-1(\beta_i > 1)$ theoretically and how β_i should be computed. Moreover, he gave a example to show that it is not general the case.*

Example 11 *Let*

$$A = \begin{bmatrix} 1 & 0.2 & 0 \\ 0.2 & 1 & 0.2 \\ 0.1 & 0 & 1 \end{bmatrix},$$

then A is an H-matrix. If we randomly choose $\alpha_1 = 0.5$, $\alpha_2 = 1.5 > 1$, then

$$1 > \rho(T^h_\alpha) = 0.04050 > \rho(\hat{T}^h_\alpha) = 0.032$$

and the comparison result of Theorem 9 does not hold.

Remark 12 *In the proof of Theorem 2.9 [12], the author use the inequality $|T^h_\alpha| \leq \hat{T^h_\alpha}$, and $\hat{T^h_\alpha} = (\hat{E^h_\alpha})^{-1}\hat{F}^h_\alpha$, where $(\hat{E^h_\alpha})^{-1}, \hat{F}^h_\alpha$ are nonnegative. but if $\alpha_i > 1$, this can not ensure that $\hat{F}^h_\alpha$ is nonnegative,* *hence,* $|T^h_\alpha| \le \hat{T^h_\alpha}$ *can not hold. Example 4.2 [12] shows* $\rho(T^h_\alpha) > \rho(\hat{T^h_\alpha})$*. For example, let*

$$A = \begin{bmatrix} 1 & 0.5 & 0.6 \\ 0.1 & 1 & 0.1 \\ 0.2 & 0.2 & 1 \end{bmatrix},$$

it is easy to show that A *is an* H*-matrix, by (7), we have* $\alpha_1 \in [0, 1.6368), \alpha_2 \in [0, 3.3350)$*, we let* $\alpha_1 = 1.5, \alpha_2 = 3$*,*

$$L = \begin{bmatrix} 0 & 0 & 0 \\ -0.1 & 0 & 0 \\ -0.1 & -0.2 & 0 \end{bmatrix}, U = \begin{bmatrix} 0 & -0.5 & -0.6 \\ 0 & 0 & -0.1 \\ 0 & 0 & 0 \end{bmatrix}.$$

by direct computation, we have $\rho(T^h_\alpha) = 0.2949 < 0.3120 = \rho(\hat{T^h_\alpha})$*. From example 4.2 [12], the above example and the process of the proof of the Theorem 2.9 [12], we conclude that Theorem 2.9 [12] can not be extended to the case* $\alpha_i \in [0, \beta_i)$*.*

In Section 3, we present new convergence intervals of parameters α_i, which need not compute $\|\langle A\rangle^{-1}\|_\infty$. The further result is obtained which is much extensive than Theorem 2.8.

3 Convergence theorem for H-matrix

In this Section, we will consider the preconditioned Gauss-Seidel method for H-matrices. For convenience, we still use some notions and definitions in Section 2. We first give two well-known results:

Lemma 13 *([3]) Let* A *have nonpositive off-diagonal entries. Then a real matrix* A *is* $M-$ *matrix if and only if there exists some vector* $u = (u_1, \cdots, u_n)^T > 0$ *such that* $Au > 0$*.*

Lemma 14 *([8]) Let* $A = M - N$ *be a splitting. If it is an* $H-$ *splitting,then* A *and* M *are* $H-$ *matrices and* $\rho(M^{-1}N) \le \rho(\langle M\rangle^{-1}|N|) < 1$*.*

Lemma 15 *Let* A *be an* $H-$ *matrix with unit diagonal elements,* $a_{i,i+1} \ne 0 (i = 1, \cdots, n-1)$*. Let* $u = (u_1, \cdots, u_n)^T$ *be a positive vector such that* $\langle A\rangle u > 0$*. Define*

$$\alpha'_i = \frac{u_i - \sum\limits_{j=1}^{i-1} |a_{i,j}|u_j - \sum\limits_{j=i+2}^{n} |a_{i,j}|u_j + |a_{i,i+1}|u_{i+1}}{|a_{i,i+1}| \sum\limits_{j=1}^{n} |a_{i+1,j}|u_j},$$

then $\alpha_i > 1$*,* $i = 1, \cdots, n-1$*.*

Proof: Let $u = (u_1, \cdots, u_n)^T$ be a positive vector such that $\langle A\rangle u > 0$. From the definition of $\langle A\rangle$, we have

$$u_i - \sum_{\substack{j=1\\ j\ne i}}^{n} |a_{i,j}|u_j > 0, \qquad i = 1, 2, \cdots, n-1.$$

From the equality

$$u_i - \sum_{j=1}^{i-1} |a_{i,j}|u_j - \sum_{j=i+2}^{n} |a_{i,j}|u_j$$
$$+|a_{i,i+1}|u_{i+1} - |a_{i,i+1}| \sum_{j=1}^{n} |a_{i+1,j}|u_j$$
$$= u_i - \sum_{\substack{j=1\\ j\ne i}}^{n} |a_{i,j}|u_j + |a_{i,i+1}|(u_{i+1} - \sum_{\substack{j=1\\ j\ne i+1}}^{n} |a_{i+1,j}|u_j).$$

Observe that

$$u_i - \sum_{\substack{j=1\\ j\ne i}}^{n} |a_{i,j}|u_j > 0, u_{i+1} - \sum_{\substack{j=1\\ j\ne i+1}}^{n} |a_{i+1,j}|u_j > 0.$$

Then we have

$$u_i - \sum_{j=1}^{i-1} |a_{i,j}|u_j - \sum_{j=i+2}^{n} |a_{i,j}|u_j + |a_{i,i+1}|u_{i+1}$$
$$-|a_{i,i+1}| \sum_{j=1}^{n} |a_{i+1,j}|u_j > 0.$$

Which is equivalent to

$$u_i - \sum_{j=1}^{i-1} |a_{i,j}|u_j - \sum_{j=i+2}^{n} |a_{i,j}|u_j + |a_{i,i+1}|u_{i+1} > |a_{i,i+1}| \sum_{j=1}^{n} |a_{i+1,j}|u_j > 0, \qquad i = 1, 2, \cdots, n-1.$$

Hence we have

$$\alpha'_i = \frac{u_i - \sum\limits_{j=1}^{i-1} |a_{i,j}|u_j - \sum\limits_{j=i+2}^{n} |a_{i,j}|u_j + |a_{i,i+1}|u_{i+1}}{|a_{i,i+1}| \sum\limits_{j=1}^{n} |a_{i+1,j}|u_j} > 1 \qquad i = 1, 2, \cdots, n-1.$$

□

Theorem 16 *Let* A *be an* $H-$ *matrix with unit diagonal elements,* $A_\alpha = (I + S_\alpha)A = M_\alpha - N_\alpha$*,* $M_\alpha = I - L - S_\alpha L$ *and* $N_\alpha = U - S_\alpha + S_\alpha U$*. Let* $u = (u_1, \cdots, u_n)^T$ *be a positive vector such that* $\langle A\rangle u > 0$*, assume that* $a_{i,i+1} \ne 0$ *for* $i =$

$1, 2, \cdots, n-1$, and α_i' are defined as Lemma 3.3. Then $0 \le \alpha_i < \alpha_i'$, the splitting $A_\alpha = M_\alpha - N_\alpha$ is an $H-$ splitting and $\rho(M_\alpha^{-1}N_\alpha) < 1$.

Proof: Let $M_\alpha = I - L - S_\alpha L$ and $N_\alpha = U - S_\alpha + S_\alpha U$. Then $\tilde{T} = M_\alpha^{-1}N_\alpha$.

Let $[(\langle M_\alpha\rangle - |N_\alpha|)u]_i$ be the ith element in the vector $(\langle M_\alpha\rangle - |N_\alpha|)u$ for $i = 1, 2, \cdots, n-1$. Then we obtain

$$[(\langle M_\alpha\rangle - |N_\alpha|)u]_i$$

$$= |1-\alpha_i a_{i,i+1}a_{i+1,i}|u_i - \sum_{j=1}^{i-1}|a_{i,j} - \alpha_i a_{i,i+1}a_{i+1,j}|u_j$$

$$- \sum_{j=i+1}^{n}|a_{i,j} - \alpha_i a_{i,i+1}a_{i+1,j}|u_j$$

$$\ge u_i - \alpha_i|a_{i,i+1}a_{i+1,i}|u_i - \sum_{j=1}^{i-1}|a_{i,j}|u_j$$

$$-\alpha_i \sum_{j=1}^{i-1}|a_{i,i+1}a_{i+1,j}|u_j - \sum_{j=i+2}^{n}|a_{i,j}|u_j$$

$$-\alpha_i \sum_{j=i+2}^{n}|a_{i,i+1}a_{i+1,j}|u_j - |1-\alpha_i||a_{i,i+1}|u_{i+1},$$

and

$$[(\langle M_\alpha\rangle - |N_\alpha|)u]_n = u_n - \sum_{\substack{j=1\\ j\ne n}}^{n}|a_{nj}|u_j > 0.$$

If $0 \le \alpha_i \le 1$ for $i = 1, 2, \cdots, n-1$, then we have

$$[(\langle M_\alpha\rangle - |N_\alpha|)u]_i \ge u_i - \sum_{\substack{j=1\\ j\ne i}}^{n}|a_{i,j}|u_j + \alpha_i|a_{i,i+1}|u_{i+1}$$

$$-\alpha_i|a_{i,i+1}|\sum_{\substack{j=1\\ j\ne i+1}}^{n}|a_{i+1,j}|u_j$$

$$= (u_i - \sum_{\substack{j=1\\ j\ne i}}^{n}|a_{i,j}|u_j) + \alpha_i|a_{i,i+1}|(u_{i+1} - \sum_{\substack{j=1\\ j\ne i+1}}^{n}|a_{i+1,j}|u_j)$$

> 0.

If $1 < \alpha_i < \alpha_i'$ for $i = 1, 2, \cdots, n-1$, then we have

$$[(\langle M_\alpha\rangle - |N_\alpha|)u]_i \ge u_i - \sum_{j=1}^{i-1}|a_{i,j}|u_j - \sum_{j=i+2}^{n}|a_{i,j}|u_j$$

$$+|a_{i,i+1}|u_{i+1} - \alpha_i|a_{i,i+1}|\sum_{j=1}^{n}|a_{i+1,j}|u_j > 0.$$

Therefore, by Lemma 3.1, $\langle M_\alpha\rangle - |N_\alpha|$ is an $M-$matrix for $0 \le \alpha_i < \alpha_i'(i = 1, 2, \cdots, n-1)$. Namely, $A_\alpha = M_\alpha - N_\alpha$ is an $H-$ splitting for $0 \le \alpha_i < \alpha_i'(i = 1, 2, \cdots, n-1)$. Hence, from Lemma 3.2, we know $\rho(\tilde{T}) = \rho(M_\alpha^{-1}N_\alpha) < 1$ for $0 \le \alpha_i < \alpha_i'(i = 1, 2, \cdots, n-1)$. □

Remark 17 *In Theorem 2.8 [12], for a given matrix A, the value of β_i is determine. We need compute $\|\langle A\rangle^{-1}\|_\infty$, then we compute the β_i by (7). But by Lemma 3.3, we see that α_i is computed which is concerned with the vector u.*

Example 18 *Let*

$$A = \begin{bmatrix} 1 & \frac{1}{2} \\ -1 & 1 \end{bmatrix}.$$

It is clear that A is an H-matrix. By Lemma 6, we have $\beta_1 = \frac{10}{7}$, so the convergence intervals of α_i are $[0, \frac{10}{7})$. Since $u = (\frac{4}{5}, 1)^T$ such that $\langle A\rangle^{-1}u > 0$, from Lemma 14, we have $\alpha_i' = \frac{13}{9}$, then we have the convergence intervals of α_i are $[0, \frac{13}{9})$. So the convergence interval of parameters α_i in Theorem 3.3 is much wider than in Theorem 2.7 [12].

Example 19 *Let*

$$B = \begin{bmatrix} 1 & -0.2 & -0.1 \\ 0 & 1 & -0.1 \\ -0.1 & -0.4 & 1 \end{bmatrix}.$$

It is clear that A is an M-matrix, so it is an H-matrix. By Lemma 6, we have $\beta_1 = 3.7188$ and $\beta_2 = 5.9846$, so the convergence intervals of $\alpha_i, i = 1, 2$ are $[0, 3.7188)$ and $[0, 5.9846)$, respectively. Since $u = (1, 1, 1)^T$ such that $\langle B\rangle^{-1}u > 0$, from Lemma 14, we have $\alpha_1' = 5$ and $\alpha_2' = 7.3333$, then the convergence intervals of $\alpha_i, i = 1, 2$ are $[0, 5)$ and $[0, 7.3333)$, respectively. So the convergence interval of parameters α_i in Theorem 3.3 is much wider than in Theorem 2.7 [12].

Remark 20 *By the proof of Theorem 2.8 [12], Example 17 and Example 18, we observe that the vector $r = \langle A\rangle^{-1}e, e = (1, 1, \cdots, 1)^T$, that is, $\langle A\rangle^{-1}r = e > 0$. So we see that the vector r is the special vector such that condition of Theorem 3.1. So the results of Theorem 2.8 [12] is the special case of Theorem 3.1.*

4 Convergence results for M-matrix

Without loss of generality, we still use the notes in Section 2 and Section 3, in order to prove our results, we first give some Definitions and Lemmas.

Let $\mathcal{E}$ be a real Banach space, $\mathcal{E}'$ its dual and $\pounds(\mathcal{E})$ the space of all bounded linear operators mapping $\mathcal{E}$ into itself. We do not distinguish between the norms of these spaces, writing simply $\|\cdot\|$ in each case. When $\mathcal{E}$ is the n-dimensional real space $\Re^n$, $\pounds(\mathcal{E})$ is the space of $n \times n$ matrices.

We assume that $\mathcal{E}$ is generated by a normal cone K, i.e., $\mathcal{E} = K - K$ where K has the following properties: (i) $K + K \subset K$, (ii) $\alpha K \subset K$ for $\alpha \geq 0$, (iii) $K \bigcap (-K) = 0$, (iv) $\bar{K} = K$ where $\bar{K}$ denotes the norm-closure of K, and (v) for $x, y \in K$ there exists $\sigma > 0$ such that $\|x + y\| \geq \sigma\|x\|$. When $\mathcal{E}$=$\Re^n$, a generating cone is $K = \Re^n_+$, the set of nonnegative vectors, i.e. of vectors with nonnegative entries.

Let $K' = \{x' \in \mathcal{E}' : \langle x, x'\rangle = x'(x) \geq 0$ for all $x \in K\}$. It can be shown that K' is also a closed normal cone generating $\mathcal{E}'$; see Ivo Marek and D.B.Szyld [7].

We say that operator $A \in \pounds(\mathcal{E})$ has property "d" if its dual A' possesses a Frobenius eigenvector in the dual cone, i.e. if there exists $x' \in K'$ such that $A'x' = \rho(A)x'$, where $\rho(A)$ denotes the spectral radius of A.

When $\mathcal{E}$=$\Re^n$ and $K = \Re^n_+$, all operators in $\pounds(\mathcal{E})$, i.e. all matrices, have property "d".

Definition 21 *Let $A = (a_{ij}) \in R^{n\times n}$, $A = M - N$ is called a splitting of A if M is a nonsingular matrix. The splitting is called:*
(a)convergent if $\rho(M^{-1}N) < 1$.
(b)regular if $M^{-1} \geq 0$ and $N \geq 0$.
(c)weak regular if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$.
(d)nonnegative if $M^{-1}N \geq 0$.
(e)M-splitting if M is a nonsingular M-matrix and $N \geq 0$.

Definition 22 *We call $A = M - N$ the Gauss-Seidel splitting of A, if $M = D - E$ and $N = F$, where D is the diagonal part and $-E$ and $-F$ are strictly lower and upper triangular parts of A, respectively. In addition, the splitting is called*
(a)Gauss-Seidel convergent if $\rho(M^{-1}N) < 1$.
(b)Gauss-Seidel regular if $M^{-1} = (D - E)^{-1} \geq 0$ and $N = F \geq 0$.
(c)Gauss-Seidel weak regular if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$.

Lemma 23 *([7]) Let A be irreducible, $A = M - N$ be an M-splitting. Then there is a positive vector x such that $M^{-1}Nx = \rho(M^{-1}N)x$.*

Lemma 24 *([10]) Let $A \geq 0$ be a nonnegative matrix. Then the following hold:*
(a)If $Ax \geq \beta x$ for a vector $x \geq 0$ and $x \neq 0$, then $\rho(A) \geq \beta$.
(b)If $Ax \leq \gamma x$ for a vector $x > 0$, then $\rho(A) \leq \gamma$; Moreover, if A is irreducible and if $\beta x \leq Ax \leq \gamma x$, equality excluded, for a vector $x \geq 0$ and $x \neq 0$, then $\beta < \rho(A) < \gamma$ and $x > 0$.

Lemma 25 *([10]) Let A be a Z-matrix. Then the following statements are equivalent:*
(a)A is a nonsingular M-matrix.
(b)There is a positive vector x such that $Ax > 0$.
(c)All principal submatrices of A are nonsingular M-matrices.
(d)All principal minors are positive.

Lemma 26 *([18]) Let A be a nonsingular M-matrix and let $A_\alpha = (I + S_\alpha)A = M_\alpha - N_\alpha$ be the Gauss-Seidel splitting of A_α, where $\alpha = (\alpha_1, \cdots, \alpha_{n-1}, 1)$, $0 \leq \alpha_i \leq 1$. If $\rho(T_\alpha) > 0$, then for any nonnegative Perron vector x of T_α we have $Ax \geq 0$.*

Lemma 27 *([28]) Let $A_1 = M_1 - N_1$ and $A_2 = M_2 - N_2$ be weak regular splittings with $T_1 = M_1^{-1}N_1$, $T_2 = M_2^{-1}N_2$ having property "d". Let $x \geq 0$, $z \geq 0$ be such that $T_1x = \rho(T_1)x$, $T_2z = \rho(T_2)z$. If*

$$M_1^{-1} \geq M_2^{-1},$$

and either $(A_1 - A_2)x \geq 0$, $A_1x \geq 0$, or $(A_1 - A_2)z \geq 0$, $A_1z \geq 0$ with $z > 0$, then

$$\rho(T_1) \leq \rho(T_2).$$

Moreover, if $M_1^{-1} > M_2^{-1}$ and if $N_1 \neq N_2$, then

$$\rho(T_1) < \rho(T_2).$$

Theorem 28 *Let A be an irreducible nonsingular M-matrix, $A_\alpha = (I + S_\alpha)A = M_\alpha - N_\alpha$, $M_\alpha = I - L - S_\alpha L$ and $N_\alpha = U - S_\alpha + S_\alpha U$. Then $\tilde{T} = (I - L - S_\alpha L)^{-1}(U - S_\alpha + S_\alpha U)$, for $\alpha_i \in [0, 1]$, we have*

$$\rho(\tilde{T}) \leq \rho(T) < 1.$$

Proof: Let $A = I - L - U$ be an irreducible nonsingular M-matrix. It is easy to show that A_α is also an M-matrix for $\alpha_i \in [0, 1]$. Observe that $I - L$ is an M-matrix and U is nonnegative. So $(I - L) - U$ is an M-splitting of A. By Lemma 23, there exists a positive vector x such that

$$Tx = (I - L)^{-1}Ux = \lambda x, \qquad (6)$$

where λ denotes the spectral radius of T.

We can write (6) equivalently as

$$Ux = \lambda(I - L)x, \qquad (7)$$

or

$$\lambda x - \lambda Lx - Ux = 0. \tag{8}$$

We next consider
$\tilde{T}x - \lambda x$
$= (I - L - S_\alpha L)^{-1}(U - S_\alpha + S_\alpha U)x - \lambda x$

$= (I-L-S_\alpha L)^{-1}[(U-S_\alpha+S_\alpha U)-\lambda(I-L-S_\alpha L)]x.$

By (8), we have
$\tilde{T}x - \lambda x$
$= (I - L - S_\alpha L)^{-1}(-S_\alpha + S_\alpha U + \lambda S_\alpha L$
$= (I - L - S_\alpha L)^{-1}[-S_\alpha + \lambda S_\alpha(I - L) + \lambda S_\alpha L]x$
$= (\lambda - 1)(I - L - S_\alpha L)^{-1}S_\alpha x.$
It is clear that $I - L - S_\alpha L$ is an M-matrix, so $(I - L - S_\alpha L)^{-1}$ is nonnegative. Since $0 < \lambda < 1$ and $S_\alpha \geq 0$, thus, we have that $\tilde{T}x - \lambda x \leq 0$. From Lemma 24, we have

$$\rho(\tilde{T}) \leq \rho(T) < 1.$$

□

Example 29 *Consider a $n\times n$ matrix of A of the form*

$$A = \begin{bmatrix} 1 & c_1 & c_2 & c_3 & c_1 & \cdots \\ c_3 & 1 & c_1 & c_2 & \ddots & c_1 \\ c_2 & c_3 & \ddots & \ddots & \ddots & c_3 \\ c_1 & \ddots & \ddots & 1 & c_1 & c_2 \\ c_3 & \ddots & c_2 & c_3 & 1 & c_1 \\ \vdots & c_3 & c_1 & c_2 & c_3 & 1 \end{bmatrix},$$

where $c_1 = -2/n, c_2 = 0, c_3 = -1/n + 2$. It is clear that the matrix A satisfies the assumptions of Theorem 28. Numerical results for this matrix A are given in Table 1 and Table 2.

We consider Example 29, it is clear to show that A is an irreducible nonsingular M-matrix. The initial approximation of x^0 is taken as a zero vector, and that b is chosen so that $x = (1, 2, \cdots, n)^T$ is the solution of the linear system (1). Here $\|x^{k+1}-x^k\|/\|x^{k+1}\| \leq 10^{-6}$ is used as the stopping criterion.

All experiments were executed on a PC using MATLAB programming package.

In order to show that the preconditioned Gsuss-Seidel method is superior to the basic Gauss-Seidel method. In Table 2, we report the CPU time (T) and the number of iterations (IT) for the basic and the preconditioned Gauss-Seidel method. Here GS represents the restarted Gauss-Seidel method, the preconditioned restarted Gauss-Seidel method is noted by PGS.

Table 1
Spectral radius of the iteration matrices $\rho(T), \rho(\tilde{T})$ for Example 29

n	$\rho(T)$	$\rho(\tilde{T})$
60	0.9471	0.9436
90	0.9642	0.9626
120	0.9729	0.9720
150	0.9782	0.9776
180	0.9818	0.9814
210	0.9844	0.9841

Table 2
CPU time and the number of the basic and the preconditioned Gauss-Seidel method for Example 29

n	$IT(GS)$	$CPU(GS)$	$IT(PGS)$	$CPU(PGS)$
60	232	0.2500	219	0.0620
90	340	0.1720	327	0.1410
120	446	0.3280	433	0.3280
150	551	3.6870	538	3.2030
180	655	8.5630	642	8.3280
210	758	16.8590	746	16.7030

5 Other preconditioners

In 2002, Kotakemori et al.[22] proposed to use the preconditioner

$$P = I + S_m,$$

where

$$(S_m)_{i,j} = \begin{cases} -a_{i,k_i}, & j = k_i, \\ 0, & j \neq k_i, \end{cases}$$

and $k_i = \min\{j|\max_{j>i}|a_{i,j}|, i < n\}$.

Li Wen [18] gave a counterexample to show that the result of Theorem 1.1 is not true. For example

$$A = \begin{bmatrix} 1 & -0.1 & -0.1 & -0.1 & -0.2 \\ -0.1 & 1 & -0.1 & -0.1 & -0.2 \\ -0.1 & -0.1 & 1 & -0.1 & -0.2 \\ -0.1 & -0.1 & -0.1 & 1 & -0.2 \\ -0.1 & -0.1 & -0.1 & -0.1 & 1 \end{bmatrix}, \tag{9}$$

by computation, A satisfies the conditions of Theorem 1.1, but $\rho(M_m^{-1}N_m) = 0.1555 > 0.1497 = \rho(M_s^{-1}N_s)$. Hence, the convergence rate of the preconditioned iterative method with the preconditioner $I + S_m$ is not in general faster than that of the preconditioner $I + S$.

In 2005, Li Jicheng [24] considered a new preconditioner

$$S_{\alpha,\beta} = \begin{bmatrix} 0 & -\alpha_1 a_{1,2} & \cdots \\ -\beta_2 a_{2,1} & 0 & \ddots \\ \vdots & \vdots & \ddots \\ 0 & 0 & \ddots \\ 0 & 0 & \cdots \end{bmatrix}$$

The author discussed the preconditioned Gauss-Seidel iterative method for Z-matrices and gave the comparison theorem between the preconditioned Gauss-Seidel iterative method and the classical iterative method. Recently, Wang Xuezhong et al. [25] applied this preconditioner to the H-matrices linear system and obtained the sufficient condition for the convergence of the preconditioned Gauss-Seidel iterative method.

To improve the convergence of preconditioned Gauss-Seidel iterative method, Liu Qingbing and Chen Guoliang [26] give the two new preconditioners

$$\hat{S} = \begin{bmatrix} 0 & -a_{1,2} & \cdots & -a_{1,k_1} & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & -a_{n-1,n} \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix},$$

where $k_i = \min j \in \{j | \max_{j>i+1} |a_{i,j}|, i < n\}$. and

$$\hat{S}_1 = \begin{bmatrix} 0 & -a_{1,2} & \cdots & \cdots & -a_{1,n} \\ 0 & 0 & \cdots & -a_{2,k_2} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & -a_{n-1,n} \\ -a_{n,1} & -a_{n,2} & \cdots & \cdots & 0 \end{bmatrix}.$$

The authors discussed the convergence of the two preconditioned Gauss-Seidel iterative methods and gave some comparison results between the new preconditioned Gauss-Seidel iterative methods and the preconditioned Gauss-Seidel iterative method in [6].

In fact, by the properties of M-matrices, we know that if A is an M-matrix, then there exists a positive vector x such that $Ax > 0$. If we define a nonnegative matrix H as a preconditioner, where each row of H is nonzero, then we have that $HAx > 0$. Thus, if we guarantee that H is a Z-matrix, by Lemma 13, we know that HA is also an M-matrix. We can applied the Gauss-Seidel iterative method to HA, furthermore, by matrix splitting theorem, we can conclude that the convergence rate of the preconditioned Gauss-Seidel iterative method for HA is faster than that of the classical iterative method. By these knowledge, we next give other preconditioners.

Other preconditioners derived from S_1 and P_1 were considered.

Upper triangular preconditioner. We consider the upper triangular part of the coefficient matrix A to be the preconditioner. Namely

$$H_1 = \begin{bmatrix} 1 & -a_{1,2} & -a_{1,3} & \cdots & \cdots & -a_{1,n} \\ 0 & 1 & -a_{2,3} & \cdots & \cdots & -a_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & -a_{n-1,n} \\ 0 & 0 & \cdots & \cdots & \cdots & 1 \end{bmatrix}.$$

Lower triangular preconditioner. We consider the lower triangular part of the coefficient matrix A to be the preconditioner. Namely

$$H_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ -a_{2,1} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_{n-1,1} & -a_{n-1,2} & \cdots & 1 & 0 \\ -a_{n,1} & -a_{n,2} & \cdots & -a_{n,n-1} & 1 \end{bmatrix}.$$

Combination preconditioner. We tried using a combination of the upper triangular preconditioner and the lower triangular precondtioner together, leading to a new preconditioner:

$$H_3 = \alpha H_1 + \beta H_2 - I,$$

where $\alpha = (\alpha_1, \cdots, \alpha_{n-1}, 1)^T$ and $\beta = (1, \beta_2, \cdots, \beta_n)^T$, $\alpha_i \in [0, 1], i = 1, \cdots, n-1$, $\beta_j \in [0, 1], j = 2, \cdots, n$.

In order to validate the above preconditioner, we consider Example 29, we let $c_1 = -2/n, c_2 = -1/n + 1, c_3 = -1/n + 2$, The initial approximation of x^0 is taken as a zero vector, and that b is chosen so that $x = (1, 2, \cdots, n)^T$ is the solution of the linear system (1). Here $\|x^{k+1} - x^k\|/\|x^{k+1}\| \leq 10^{-6}$ is used as the stopping criterion. In Table 3 and Table 4, we report the CPU time (T) and the number of iterations (IT) for the basic and the preconditioned Gauss-Seidel method, respectively. Here GS represents the restarted Gauss-Seidel method, the preconditioned restarted Gauss-Seidel method is noted by H_1GS and H_2GS.

Table 3

CPU time and the number of the basic and the preconditioned Gauss-Seidel method for Example 29

n	$IT(GS)$	$CPU(GS)$	$IT(H_1GS)$	$CPU(H_1GS)$
60	1318	0.4210	664	0.1870
90	1263	0.6870	638	0.3130
120	1237	1.0470	627	0.5000
150	1221	7.8900	620	3.9690
180	1211	17.0470	615	8.5470
210	1204	29.5930	612	15.1720

Table 4

CPU time and the number of the basic and the preconditioned Gauss-Seidel method for Example 29

n	$IT(GS)$	$CPU(GS)$	$IT(H_2GS)$	$CPU(H_2GS)$
60	1318	0.4210	912	0.3750
90	1263	0.6870	878	0.6410
120	1237	1.0470	862	1.0780
150	1221	7.8900	852	5.8910
180	1211	17.0470	846	12.7660
210	1204	29.5930	841	20.7970

6 Conclusions

The development of efficient and reliable preconditioned iterative methods is the key for the successful application of scientific computation to the solution of many large-scale problems. Therefore it is not surprising that this research area continues to see a vigorous level of activity. In this paper, we have attempted to highlight some of the developments that have taken place in recent years. Among the most important such developments, we emphasize the following:
(1)Improved robustness, frequently achieved by transferring techniques from sparse direct solvers (such as reorderings and scalings) to preconditioners.
(2)Improved performance through the use of blocking.
(3)The emergence of a new class of general-purpose, parallel preconditioners based on sparse approximate inverses.
(4)The increasingly frequent use of iterative solvers in industrial applications.

There are many further important problems and ideas that we have not been able to address in this article, even within the relatively narrow context of purely algebraic methods. For example, nothing has been said about preconditioning techniques for the special linear systems arising in the solution of eigenvalue problems [28, 29, 30]. Also, we have not been able to include a discussion of certain graph theoretically motivated techniques that show promise and have been given increased attention, known as Vaidya-type and support graph preconditioners [31, 32, 33]. Furthermore, we have hardly touched on the very important topic of software for preconditioned iterative methods.

In concluding this paper we stress the fact that in spite of recent progress, there are still important areas where much work remains to be done. While efficient preconditioners have been developed for certain classes of problems, such as self-adjoint, second-order scalar elliptic PDEs with positive definite operator, much work remains to be done for more complicated problems. For example, there is a need for reliable and efficient preconditioners for symmetric indefinite problems [34].

Ideally, an optimal preconditioner for problem (1) would result in an O(n) solution algorithm, would be perfectly scalable when implemented on a parallel computer, and would behave robustly over large problem classes. However, incremental progress leading to increasingly more powerful and reliable preconditioners for specific types of problems is within reach and can be expected to continue for the foreseeable future.

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