The Optimal Stopping Times of American Call Options with Dividend-paying and Placing Stocks

GUANGQIN LI
Zhejiang Wanli University
Institute of Mathematics
8 Qianhu South Road, Ningbo
CHINA

Abstract: American options can be exercised at any time during their lifetime. This paper addresses the optimal stopping time of several kinds of American call options.

Key–Words: stopping time, American call option, martingale, equivalent martingale measure, dividend-paying and placing rate

1 Introduction

Option is a kind of financial derivative which came into being in the middle 1970s in America [1],[3]. As a efficient way to reduce risks, it has developed quickly since its emergence. According to transaction time, options can be divided into two sections: European options and American ones [4]. American options give the holder the right to exercise them at or before the expiry date, so the payoff of American options is determined by not only the price of underlying assets at maturity but also the price path. This property of American options makes it difficult to value them and determine optimal exercise moment [2]. Pricing and hedging on options is one of the most important problems in mathematical finance, and this problem was abroad discussed in a complete market [9] [11]. In recent years many scholars seek to study such a question in an incomplete market [12], for example, a market with transaction cost.

In this paper, we consider the optimal stopping time and the price of American call options with the underlying stock paying dividend. We can show that the optimal stopping time of standard American option is their maturity in the model of continuous time, and the optimal stopping time of perpetual one does not exist. In the end we can also give the optimal stopping time of perpetual American call options when stock prices follow a jump-diffusion process and the initial price of them.

The paper is organised as follows. In section 1,we Introduction the concept of the option. In section 2, we give the necessarily preliminary.In section 3,4 the optimal stopping times is presented.

2 The Preliminary

Assumed that the financial market is composed of two kinds of assets. One of them, is called stock, whose price of per share $S(t)$ satisfies the equation

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t) \qquad (1)$$

$$S(0) = s_0 > 0$$

Additionally, there is a risk-free asset, called the bond, whose price is given by

$$\begin{cases} dB(t) = B(t)r(t)dt \\ B(0) = 1 \end{cases} \qquad (2)$$

Here $W(t)$ is a standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, P)$, endowed with a filtration $\mathcal{F} = \{\mathcal{F}_t\}_{0\le t\le T}$, which is the P-augmentation of the natural filtration $\mathcal{F}^W(t) := \sigma(W(s) : 0 \le s \le t), 0 \le t \le T$, generated by $W(\cdot)$. The process $r(\cdot)$(interest rate for lending), the process $\mu(\cdot)$ (return rate of the stock) and the process $\sigma(\cdot)$ (volatility of the stock) are all assumed to be progressively measurable with respect to $\mathcal{F}$. Moreover, the $\sigma(\cdot)$ is assumed to be positive, and all processes $r(\cdot), \mu(\cdot), \sigma(\cdot), \sigma^{-1}(\cdot)$ are assumed to be bounded, uniformly in $(t, \omega) \in [0, T] \times \Omega$.

Let

$$\theta(t) := \sigma^{-1}(t)(\mu(t) - r(t))$$

It is called the relative risk process of the market, and $\theta(t)$is bounded and $\mathcal{F}$ progressively measurable, so by Itô formula, for any $0 \le t \le T$,

$$Z_0(t) := \exp\left\{-\int_0^t \theta(s)dW(s) - \frac{1}{2}\int_0^t \theta^2(s)ds\right\},$$

is a martingale, and the process

$$W^0(t) := W(t) + \int_0^t \theta(s)ds, \quad 0 \le t \le T$$

is a Brownian motion under the probability measure

$$P^0(A) := E(Z_0(t); A), \quad A \in \mathcal{F}$$

Definition 1 1) An $\mathcal{F}$ progressively measurable process $\phi : [0,T] \times \Omega \to R$ is called a portfolio process if $\int_0^T \phi^2(s)ds < \infty \ a.s.$; 2) An $\mathcal{F}$ adapted process $C : [0,T] \times \Omega \to [0,\infty)$ is called a culmulative consumption process if C is increasing, right continuous with $C(0) = 0, C(T) < \infty \ a.s.$.

Definition 2 1) A portfolio $\phi(t) = \{\phi_0(t), \phi_1(t)\}$ is called a self-financing portfolio if the wealth process satisfies $dV(t) = \phi_0(t)dB(t) + \phi_1(t)dS(t)$; 2) We say that a portfolio $\phi(t)$ is admissible if the wealth process satisfies almost surely $V^{\phi(t)}(t) \ge 0$, for any $0 \le t \le T$.

Definition 3 1) A stopping time τ^* is called the optimal exercised moment of a American option, if for any admissible self-financing portfolio $\phi(t)$, $V^{\phi(t)}(\tau^*) \ge f(\tau^*) \ P^0 - a.s.$, we always have,

$$V^{\phi(t)}(\tau^*) = f(\tau^*).$$

Here $f(t)$ is a non-negative adaptive process with respect to the American contingent claim. If $V^{\phi(t)}(t) \ge f(t), 0 \le t \le T, P^0 - a.s.$, we call the portfolio $\phi(t)$ the duplicative portfolio of the contingent claim. Thus the meaning of the above definition is apparent. If there exists a set of positive probability such that $V^{\phi(t)}(t,\omega) \ge f(t,\omega)$ when ω is in the set, then the holder won't exercise his(her) right but choose to continue to hold it.

Lemma 1 *Assumed that the stochastic process $\phi(t)$ is a portfolio process and that a non-negative supermartingale $Z(t)$ satisfies $Z(t) = 1 + \int_0^t \phi(t)dW(t)$, if $E(Z(t)) = 1$, then*

$$W^*(t) = W(t) - \int_0^t Z^+(s)\phi(s)ds$$

is a Brownian motion under Q probability measure. Here Q measure satisfies $dQ = Z(T)dP$, $Z^+(t) = Z(t) \vee 0$[5].

Lemma 2 *Assumed that $W(t)$ is a Brownian motion under P measure, for any $\alpha \in R$, let*

$$\tau_\alpha = \inf\{t \ge 0 : W(t) = \alpha\} = \inf\{t \ge 0 : W(t) \ge \alpha\},$$

then $P\{\tau_\alpha < \infty\} = 1$. In addition, for any $\alpha, \beta \in R$, let

$$\tau_{\alpha,\beta} = \inf\{t \ge 0 : W(t) = \alpha t + \beta\},$$

then $P\{\tau_{\alpha,\beta} < \infty\} = 1$.

Proof Let $M(t) = \sup_{s \le t}\{W(t)\}$, we have

$$\begin{aligned} & F(t) = P\{\tau_\alpha \le t\} = P\{M(t) \ge \alpha\} \\ = & \ 2P\{W(t) \ge \alpha\} = \frac{2}{\sqrt{2\pi t}} \int_\alpha^\infty e^{-x^2/2t}dx. \end{aligned}$$

Thus,

$$\begin{aligned} & P\{\tau_\alpha \le \infty\} = \lim_{t\to\infty} F(t) \\ = & \ 1 - \lim_{t\to\infty} \frac{2}{\sqrt{2\pi t}} \int_{-\infty}^\alpha e^{-x^2/2t}dx = 1. \end{aligned}$$

So let

$$\frac{dQ}{dP} = Z^\beta_\infty, \quad Z^b_t = \exp\{\mu W(t) - \frac{1}{2\mu^2 t}\},$$

then $W^*(t) = W(t) - \alpha t$ is a Brownian motion under Q measure,and $\tau_{\alpha,\beta} = \tau^*_\alpha = \inf\{t \ge 0 : W^*(t) = \beta\}$, therefore,

$$P\{\tau_{\alpha,\beta} < \infty\} = Q\{\tau^*_\alpha < \infty\} = 1.$$

In the following, $\mu(t), r(t), \sigma(t)$ are abbreviated to μ, r, σ respectively.

Lemma 3 *There exists an equivalent martingale measure P^* of P measure such that the discount process $\widetilde{S}(t) = e^{-rt}S(t)$ $(0 \le t \le T)$ is a martingale under P^*.*

Proof Let

$$Z(t) = \exp\left\{-\frac{\mu - r}{\sigma}Z(t) - \frac{1}{2}(\frac{\mu - r}{\sigma})^2 t\right\},$$

then $Z(t) > 0, P - a.s.$ and $E(Z(t)) = 1$, so according to Lemma 1,

$$dP^* := Z(t)dP$$

defines a probability measure on $(\Omega, \mathcal{F}_T)$. If we let

$$W(t) := B(t) + \frac{\mu - r}{\sigma}t,$$

$W(t)$ is a standard Brownian motion under P^*. So,

$$\begin{aligned} & d\widetilde{S}(t) = -re^{-rt}S(t)dt + e^{-rt}dS(t) \\ = & \ \widetilde{S}(t)[(\mu - r)dt + \sigma dB(t)] \\ = & \ \widetilde{S}(t)\sigma dW(t). \end{aligned} \tag{3}$$

Thus $\widetilde{S}(t) = \widetilde{S}(0)\exp\left\{\sigma W(t) - \frac{\sigma^2}{2}t\right\}$ is a martingale under P^*

3 The Optimal Stopping times without Dividend-paying and Placing

The valuation function of the American call option is $U(X_t) = (X_t - k)^+$, the holder always want to get the maximal profit, that is to say, he(she) hopes to find the optimal exercise moment τ^* such that

$$\begin{aligned} C^*(0) &= \sup_\tau E^*[e^{-rt}(S(\tau) - k)^+] \\ &= E^*[e^{-r\tau^*}(S(\tau^*) - k)^+]. \end{aligned}$$

Theorem 1 *i)The optimal stopping time of the American call option is T;*
ii)The optimal stopping time of the permanent American call option does not exist. Let

$$X(t) := \sigma W(t) + (r - \sigma/2)t,$$

$$T_\epsilon := \frac{1}{r}\ln\left(\frac{k}{\epsilon S(0)}\right), 0 < \epsilon < k,$$

then T_ϵ is the ϵ optimal stopping time, and

$$C^*(0)(1-\epsilon) \le E^*[e^{-rT_\epsilon}(S(T_\epsilon) - k)^+] \le C^*(0).$$

Proof i)For $S(t) = \exp\left\{\sigma W(t) + \left(r - \frac{\sigma^2}{2}t\right)\right\}$, then

$$e^{-rt}U(S(t)) = \left(\exp\left\{\sigma W(t) - \frac{\sigma^2}{2}t\right\}\right)^+,$$

here $\exp\left(\sigma W(t) - \frac{\sigma^2}{2}t\right)$ is a martingale, so by Jessen's inequality, $\left(\exp\left\{\sigma W(t) - \frac{\sigma^2}{2}t\right\} - ke^{-rt}\right)^+$ is a submartingale. According to Doob's stopping time theorem, for any $\tau < T$, we have,

$$e^{-r\tau}U(S(\tau)) \le e^{-rT}U(S(T))$$

That's, $\tau^* = T$ is the optimal stopping time.
ii) The price process satisfies

$$S(t) = \exp\left\{\sigma W(t) + \left(r - \frac{\sigma^2}{2}t\right)\right\},$$

so,

$$e^{-rt}U(S(t)) = e^{-rt}e^{\gamma X(t)}e^{-\gamma X(t)}U(S(t)).$$

Let $M(t) = e^{-rt}e^{\gamma X(t)}$, if $M(t)$ is a martingale, γ must satisfies

$$-\frac{1}{2}\gamma^2\sigma^2 = \gamma(r - \frac{\sigma^2}{2}) - r,$$

that is,

$$\gamma_1 = 1, \quad \gamma_2 = -\frac{2r}{\sigma^2}.$$

Therefore, by Girsanov theorem,

$$M(t) := \exp\{\gamma\sigma W(t) - \frac{1}{2}\gamma^2\sigma^2 t\}$$

is a martingale. Let $M(0) = 1$ and

$$f(x) = e^{\gamma x}(S(0)e^x - k), \quad x \in (\ln\frac{k}{S(0)}, \infty)$$

If $f'(x) = 0$, $x^* = \ln\frac{k}{S(0)} + \ln\frac{\gamma}{\gamma - 1}$.So $f(x)$ attain its maximum $\left(\frac{\gamma-1}{k}\right)^{\gamma-1}\gamma^{-\gamma}S(0)^\gamma$ at $x = x^*$. Denote

$$C^* = \left(\frac{\gamma-1}{k}\right)^{\gamma-1}\gamma^{-\gamma}S(0)^\gamma$$

Thus, for any $t \ge 0$,

$$e^{-rt}U(S(t)) = e^{-\gamma X(t)}U(S(t))M(t) \le C^*M(t).$$

Therefore, by Doob's theorem of stopping time,

$$E\left[e^{-rt}U(S(t))I_{\{\tau<\infty\}}\right] \le C^*$$

Especially,

$$E\left[e^{-r\tau^*}U(S(\tau^*))I_{\{\tau^*<\infty\}}\right] \le C^*E(M(\tau^* I_{\{\tau^*<\infty\}})).$$

While $\gamma_1 = 1$, $\gamma_2 = -\frac{2r}{\sigma^2}$, $f'(x) > 0$, so $f(x)$ is a monotonically increasing function at interval $(\ln\frac{k}{S(0)}, \infty)$, and can't attain its maximum, thus there doesn't exist the optimal stopping time. As to T_ϵ,

$$\begin{aligned} & E[e^{-rT_\epsilon}U(S(T_\epsilon))] \\ = & S(0)E\left[e^{\sigma W(T_\epsilon) - \frac{1}{2}\sigma^2 T_\epsilon} - ke^{-rT_\epsilon}\right]^+ \\ = & S(0)E\left[e^{\sigma W(T_\epsilon) - \frac{1}{2}\sigma^2 T_\epsilon} - ke^{-rT_\epsilon}\right] \\ & \cdot I_{\left\{\exp\{\sigma W(T_\epsilon) - \frac{1}{2}\sigma^2 T_\epsilon\} - ke^{-rT_\epsilon} > 0\right\}} \\ = & S(0)E\left[e^{\sigma W(T_\epsilon) - \frac{1}{2}\sigma^2 T_\epsilon} - ke^{-rT_\epsilon}\right] \end{aligned}$$

$$
\begin{aligned}
- \quad & S(0)E\left[e^{\sigma W(T_\epsilon)-\frac{1}{2}\sigma^2 T_\epsilon} - ke^{-rT_\epsilon}\right] \\
& \cdot I_{\left\{\exp\{\sigma W(T_\epsilon)-\frac{1}{2}\sigma^2 T_\epsilon\} - ke^{-rT_\epsilon} < 0\right\}} \\
\geq \quad & S(0)\left[1-\frac{k}{S(0)}e^{-rT_\epsilon}\right] \\
= \quad & S(0)(1-\epsilon).
\end{aligned}
$$

The desired conclusion is got.

In the following we consider a kind of American call options whose underlying asset's prices follow a jump-diffusion process.

Assumed that the stock's price process satisfies the following stochastic differential equation:

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t) + US(t)dN(t) \quad (4)$$

where $N(t)$ is a Poisson process with parameter being λ, U is a square integrable random variable, $U > -1, P-a.s.$, additionally, $W(t), N(t), U$ are independent.

The solution of equation (5) is,

$$S(t) = S(0)\exp\{e^{(\mu-\frac{\sigma^2}{2})t} + \sigma W(t)\}\prod_{n=1}^{N(t)}(1+U_n),$$

where $U_1, U_2, \cdots$ are random variable with independent identical distribution function, U_n is the jumping height of the underlying stock's price at time τ_n.

Lemma 4 $\{\phi_0(t), \phi_1(t)\}$ *is self-financing if and only if*

$$d\widetilde{V}(t) = \phi_1(t)\widetilde{S}(t)(\sigma dW^*(t) + UdN(t)).$$

where $W^*(t) = W(t) + \int_0^t \theta(s)ds,\ 0 \leq t \leq T.$

By It$\hat{o}$'s Lemma and Girsanov Theorem, we can easily get it [7].

Lemma 5 [8] *If* $E(|U_1|) < \infty$, *the stock's price process* $\widetilde{S}(t)$ *with jump-diffusion is a martingale if and only if*

$$\mu = r - \lambda E(U_1).$$

Theorem 2 *If the underlying stock's price process is a one with Poisson jump, the optimal stopping time of the corresponding permanent American call option* $f(X(t)) = \left\{e^{X(t)}\prod_{n=1}^{N(t)}(1+U_n) - k\right\}^+$ *is*

$$\tau^* = \inf\left\{t \geq 0 : \sigma W(t) = x^* - \left[r - \frac{\sigma^2}{2} - \lambda E(U_1)\right]t\right\},$$

and the price of the option should be

$$C^* = e^{-\gamma_1 x^*}\left\{e^{x^*}\prod_{n=1}^{N(t)}(1+U_n) - k\right\}^+, \quad (5)$$

where $x^* = \ln\frac{\gamma_1 k}{\gamma_1 - 1} - \sum_{n=1}^{N(t)}\ln(1+U_n).$

Proof $C^* = \sup_{x\in R}\{e^{\gamma x}f(x)\}$ can attain its extremum at

$$x^* = \ln k + \ln\frac{\gamma}{\gamma-1} - \sum_{n=1}^{N(t)}\ln(1+U_n).$$

Thus it should have

$$X(t) > \ln k - \sum_{i=1}^{N(t)}\ln(1+U_i),$$

that's, $\gamma > 1$, so it must have $\gamma = \gamma_1$. Therefore the optimal stopping time is

$$\tau^* = \inf\{t \geq 0 : X(t) = x^*\},$$

Noted that

$$X(t) = \sigma W(t) + \left[r - \lambda E(U_1) - \frac{\sigma^2}{2}\right]t$$

is a martingale under an equivalent martingale measure, so

$$\tau^* = \inf\left\{t : \sigma W(t) = x^* - \left[r - \lambda E(U_1) - \frac{\sigma^2}{2}\right]t\right\}.$$

By lemma 1, $P\{\tau^* < \infty\} = 1$, in addition, $C^* = \sup_{x\in R}\{e^{\gamma x}f(x)\}$, thus,

$$C^* = e^{-\gamma_1 x^*}\left\{e^{x^*}\prod_{n=1}^{N(t)}(1+U_n) - k\right\}^+.$$

4 The Optimal Stopping Times with Dividend-paying and Placing

4.1 The Model

In this model, we still assume that the financial market is composed of two kinds of assets. The risky asset, is called stock, whose price of per share $S(t)$ satisfies the following equation, for any $0 \leq t \leq T$,

$$
\begin{aligned}
dS(t) &= S(t-)[\mu(t)dt + \sigma(Y(t))dW(t) \\
&\quad + \int_a^\infty x\tilde{N}(dt, dx)]
\end{aligned}
$$

$$S(0) = s_0 > 0$$

where $\tilde{N}(t,\cdot) = N(t,\cdot) - tv(\cdot)$ is a Poisson-valued martingale measure, $N(t,\cdot)$ is a Poisson random measure corresponding to $X(t)$, which is a Lévy process on a complete probability space, $v(\cdot) = E[N(1,\cdot)]$ is the Lévy measure of $X(t)$. And $\mu, \sigma \geq 0$ are given functions, $W(t)$ and $\tilde{N}(t,\cdot)$ are independent. Furthermore, for any $0 \leq t \leq T$,$Y(t)$ satisfies

$$dY(t) = \alpha(m - Y(t))dt + \beta d\hat{W}(t)$$

$$d\hat{W}(t) = \rho dW(t) + \sqrt{1-\rho^2}dB(t)$$

where $Y(t)$ is another stochastic process, $B(t)$ is another standard Brownian motion which is independent to $W(t)$, α, β, ρ, m are some constants, and ρ is the correlation coefficient between $W(t)$ and $\hat{W}(t)$. Additionally, there is a risk-free asset, called the bond, whose price is still given by equation(2).

Furthermore, we can have

$$\begin{aligned} dS(t) &= S(t-)[\mu(t)dt + \sigma(Y(t))dW(t) \\ &\quad + \int_a^\infty x\tilde{N}(dt,dx)] \qquad (6) \\ dY(t) &= \alpha(m - Y(t))dt + \beta\rho dW(t) \\ &\quad + \sqrt{1-\rho^2}dB(t) \qquad (7) \end{aligned}$$

4.2 The Equivalent Martingale Measure

Let

$$\beta(t) = \exp\left\{-\int_0^t r(s)ds\right\},\ \ 0 \leq t \leq T.$$

$\beta(t)$ is the so-called discount factor. Then we denote the discount of $S(t)$ by $\tilde{S}(t)$, that's,

$$\tilde{S}(t) = \beta(t)S(t) = \exp\left(-\int_0^t r(s)ds\right)S(t)$$

By Itô's formula, we can have,

$$\begin{aligned} d\tilde{S}(t) &= d\left\{\exp\left(-\int_0^t r(s)ds\right)S(t)\right\} \\ &= \exp\left(-\int_0^t r(s)ds\right)dS(t) \\ &\quad -r(t)\exp\left(-\int_0^t r(s)ds\right)S(t-)dt \\ &= \exp\left(-\int_0^t r(s)ds\right)S(t-)[\mu(t)dt \\ &\quad +\sigma(Y(t))dW(t) + \int_a^\infty x\tilde{N}(dt,dx) - r(t)dt] \\ &= \tilde{S}(t-)[(\mu(t) - r(t))dt + \sigma(Y(t))dW(t) \\ &\quad + \int_a^\infty x\tilde{N}(dt,dx) \qquad (8) \end{aligned}$$

Let X be a Lévy process, whose Lévy measure is v, and f be a mapping from $[0,T] \times A \times \Omega$ ($A \in \mathcal{B}(R_0), R_0 = R - \{0\}$) to R. Denote the set of f which satisfies the followings by $\mathrm{H}(T,A)$,

(i)f is predictable;

(ii)$\|f\|^2_{T,A} := E\left(\int_0^T \int_A |f(t,x)|^2 v(dx)dt\right) < \infty$.

Especially, if f is the mapping from $[0,T] \times \Omega$ to R, then $\mathrm{H}(T)$.

Lemma 6 *(i)If $f(t,x) \in \mathrm{H}(T,A)$, $\int_0^t \int_A f(s,x)\tilde{N}(ds,dx)$ is a square integral martingale;*

(ii)If $f(t) \in \mathrm{H}(T)$, $\int_0^t f(s)dW(s)$ is a continuous square integral martingale.

If $f(t) \in \mathrm{H}(T)$ and

$$E\left(\exp\left(\int_0^T f^2(s)ds\right)\right) < \infty, \qquad (9)$$

let

$$X_1(t) := \int_0^t f(s)dW(s) - \frac{1}{2}\int_0^t f^2(s)ds$$

thus $e^{X_1(t)}$ is a exponential martingale. And if $g(x)$ satisfies

$$\int_{R_0} g^2(x)v(dx) < \infty, \int_{R_0} e^{g(x)}v(dx) < \infty \qquad (10)$$

let

$$\begin{aligned} X_2(t) &:= \int_A g(x)\tilde{N}(t,dx) \\ &\quad -t\int_A \left[e^{g(x)} - 1 - g(x)\right]v(dx), \end{aligned}$$

then for any $A \in \mathcal{B}(R_0)$, $e^{X_2(t)}$ is a exponential martingale. So $e^{Y(t)} := e^{X_1(t)+X_2(t)}, 0 \leq t \leq T$ is also a exponential martingale.

Suppose that f_1, f_2 satisfy (9) and that h satisfies (10), let

$$\begin{aligned} Z(t) &= \int_0^t f_1(s)dW(s) - \frac{1}{2}\int_0^t f_1^2(s)ds \\ &+ \int_0^t f_2(s)dW(s) - \frac{1}{2}\int_0^t f_2^2(s)ds \\ &+ \int_0^t \int_a^\infty g(x)\tilde{N}(ds,dx) \\ &- \int_0^t \int_a^\infty \left[e^{g(x)} - 1 - g(x)\right]v(dx)ds \end{aligned}$$

thus $e^{Z(t)}$ is a exponential martingale, and $E(e^{Z(t)}) = 1, 0 \leq t \leq T$.

Let $dP^* := e^{Y(T)}dP$, then P^* is equivalent to P. And also let

$$W^*(t) := W(t) - \int_0^t f_1(s)ds,$$

$$B^*(t) := B(t) - \int_0^t f_2(s)ds,$$

$$\begin{aligned}\tilde{N}^*(t,A) &:= \tilde{N}(t,A) - \int_0^t \int_A \left[e^{g(x)} - 1\right] v(dx)ds \\ &= N(t,A) - \int_0^t \int_A e^{g(x)} v(dx)ds \\ &:= N(t,A) - tv^*(A),\end{aligned}$$

where $v^*(A) = \int_A e^{g(x)}v(dx), A \in \mathcal{B}(R_0)$. By Girsanov theorem, $W^*(t), B^*(t)$ both are standard Brownian motions under measure P^*, and $N(t,A)$ is a Poisson process whose intensity is $v^*(A)$, $\tilde{N}^*(t,A)$ is a martingale(A is given), and $W^*(t), B^*(t), \tilde{N}^*(t,A)$ are independent to each other.

Then we have,

$$\begin{aligned}& d\tilde{S}(t) = \tilde{S}(t-)\left[(\mu(t) - r(t))dt\right. \\ & +\sigma(Y(t))(dW^*(t) + f_1(t)dt) \\ & \left. + \int_a^\infty x\left(\tilde{N}^*(\cdot,dx) + (e^{g(x)} - 1)v(dx)\right)dt\right] \\ = \ & \tilde{S}(t-)[(\mu(t) - r(t) + \sigma(Y(t)f(t) \\ & + \int_a^\infty (e^{g(x)} - 1)v(dx))dt + \sigma(Y(t))dW^*(t) \\ & + \int_a^\infty x\tilde{N}^*(t)(dt,dx)]. \qquad (11)\end{aligned}$$

Furthermore, if

$$\mu(t) - r(t) + \sigma(Y(t)f(t) + \int_a^\infty (e^{g(x)} - 1)v(dx) = 0,$$

then,

$$\begin{aligned}& d\tilde{S}(t) \\ = \ & \tilde{S}(t-)\left[\sigma(Y(t))dW^*(t) + \int_a^\infty x\tilde{N}^*(t)(dt,dx)\right].\end{aligned}$$

Thus, by Itô formula, we can have,

$$\begin{aligned}& dS(t) = d\left[e^{\int_0^t r(s)ds}\tilde{S}(t)\right] \\ = \ & e^{\int_0^t r(s)ds}d\tilde{S}(t) + r(s)e^{\int_0^t r(s)ds}\tilde{S}(t-)dt \\ = \ & e^{\int_0^t r(s)ds}\tilde{S}(t-)\cdot \\ & \cdot\left[\sigma(Y(t))dW^*(t) + \int_a^\infty x\tilde{N}^*(dt,dx)\right] \\ & +r(t)e^{\int_0^t r(s)ds}\tilde{S}(t-)dt \\ = \ & e^{\int_0^t r(s)ds}\tilde{S}(t-)[r(t)dt + \sigma(Y(t))dW^*(t) \\ & + \int_a^\infty x\tilde{N}^*(dt,dx)] \\ = \ & S(t-)\left[r(t)dt + \sigma(Y(t))dW^*(t)\right. \\ & \left. + \int_a^\infty x\tilde{N}^*(dt,dx)\right].\end{aligned}$$

And (7) can be changed as,

$$\begin{aligned}& dY(t) = \alpha(m - Y(t))dt + \beta\rho\left[dW^*(t) + f_1(t)dt\right] \\ & +\beta\sqrt{1-\rho^2}\left[dW^*(t) + f_2(t)dt\right] \\ = \ & \left[\alpha(m - Y(t)) + \beta\rho f_1(t) + \beta\sqrt{1-\rho^2}f_2(t)\right]dt \\ & +\beta\rho dW^*(t) + \beta\sqrt{1-\rho^2}dW^*(t).\end{aligned}$$

If let

$$\delta(t) := \rho f_1(t) + \sqrt{1-\rho^2}f_2(t),$$

then,

$$\begin{aligned}dY(t) &= \left[\alpha(m - Y(t)) + \beta\delta(t)\right]dt \\ &+ \beta\rho dW^*(t) + \beta\sqrt{1-\rho^2}dW^*(t).\end{aligned} \qquad (12)$$

4.3 The Stopping Times

Denote the yield function of the European(American) option by $y(x)$, so the value $V(t)$ of the option at time t should be,

$$V\left(t, S(t), Y(t)\right) = E^*\left[e^{-\int_t^T r(s)ds}y(S(T))|\mathcal{F}_t\right],$$

The discounted value is,

$$\begin{aligned}\tilde{V}\left(t, S(t), Y(t)\right) &= e^{-\int_0^t r(s)ds}V(t, S(t), Y(t)) \\ &= E^*\left[e^{-\int_0^t r(s)ds}y(S(T))|\mathcal{F}_t\right]\end{aligned}$$

Let V_1, V_2 be the value of the American call option and the European one respectively at time t.

Theorem 3 *If the yield function $y(x)$ is convex and $y(0) = 0$, and the underlying stock doesn't distribute dividend, then the value of the American call option should be equivalent to the one of the European, that is, for any $0 \le t \le T$,*

$$V_1\left(t, S(t), Y(t)\right) = V_2\left(t, S(t), Y(t)\right).$$

Proof Easily we can show that for any $t \in [0,T]$,

$$V_2(t, S(t), Y(t)) \le V_1(t, S(t), Y(t)).$$

On the other hand,because the yield function $y(x)$ is convex and $y(0)=0$, for any $0\le\theta\le 1$,

$$y(\theta x)\le \theta y(x).$$

Since

$$0\le e^{-\int_t^T r(s)ds}\le 1,$$

then for any $\tau\in\mathcal{T}_{t,T}$, we have,

$$y(e^{-\int_t^T r(s)ds}S(T))\le e^{-\int_t^T r(s)ds}y(S(T)),$$

where $\mathcal{T}_{t,T}$ is the set of all stopping times during $[t,T]$.

In addition,

$$\tilde{S}(t)=e^{-\int_o^t r(s)ds}S(t)$$

is a martingale under measure P, so

$$\begin{aligned}
V_2(t) &= E\left[e^{-\int_t^T r(s)ds}y(S(T))|\mathcal{F}_t\right]\\
&= E\left[E\left(e^{-\int_t^T r(s)ds}y(S(T))|\mathcal{F}_\tau\right)\mid\mathcal{F}_t\right]\\
&= E\left[e^{-\int_t^\tau r(s)ds}E\left(e^{-\int_\tau^T r(s)ds}y(S(T))|\mathcal{F}_\tau\right)\mid\mathcal{F}_t\right]\\
&\ge E\left[e^{-\int_t^\tau r(s)ds}E\left(y(e^{-\int_\tau^T r(s)ds}S(T))|\mathcal{F}_\tau\right)\mid\mathcal{F}_t\right]\\
&= E\left[e^{-\int_t^\tau r(s)ds}E\left(y(e^{\int_0^\tau r(s)ds}\tilde{S}(T))|\mathcal{F}_\tau\right)\mid\mathcal{F}_t\right]\\
&\ge E\left[e^{-\int_t^\tau r(s)ds}y\left(E(e^{\int_0^\tau r(s)ds}\tilde{S}(T)|\mathcal{F}_\tau)\right)\mid\mathcal{F}_t\right]\\
&= E\left[e^{-\int_t^\tau r(s)ds}y\left(e^{\int_0^\tau r(s)ds}E(\tilde{S}(T)|\mathcal{F}_\tau)\right)\mid\mathcal{F}_t\right]\\
&= E\left[e^{-\int_t^\tau r(s)ds}y\left(e^{\int_0^\tau r(s)ds}\tilde{S}(\tau)\right)\mid\mathcal{F}_t\right]\\
&= E\left[e^{-\int_t^\tau r(s)ds}y\left(S(\tau)\right)\mid\mathcal{F}_t\right],
\end{aligned}$$

that's,

$$\begin{aligned}
V_2(t) &\ge \sup_{\tau\in\mathcal{T}_{t,T}} E\left[e^{-\int_t^\tau r(s)ds}y\left(S(\tau)\right)\mid\mathcal{F}_t\right]\\
&= V_1(t).
\end{aligned}$$

Thus the desired result is got.

Remark (i) From theorem 3, T is the optimal stopping time of the American call option. That is to say, it need not be exercised before the expire date. This conclusion coincide with the result we get before. (ii) The yield function of the call option is convex, so the price of the American call option should be equal to the one of the European.

Suppose that $V(t,S(t),Y(t))\in C^{1,2,2}$, by Itô formula,

$$\begin{aligned}
&dV(t,S(t),Y(t))\\
=\;& \frac{\partial}{\partial t}V(t,S(t-),Y(t))dt\\
&+\frac{\partial}{\partial s}V(t,S(t-),Y(t))r(t)S(t-)dt\\
&+\frac{\partial}{\partial s}V(t,S(t-),Y(t))\sigma(Y(t))S(t-)dW^*(t)\\
&+\frac{\partial}{\partial y}V(t,S(t-),Y(t))[\alpha(m-Y(t))+\beta\delta(t)dt\\
&+\beta\rho dW^*(t)+\beta\sqrt{1-\rho^2}dB^*(t)]\\
&+\frac{\partial^2}{\partial s\partial y}V(t,S(t-),Y(t))\beta\rho\sigma(Y(t))S(t-)dt\\
&+\frac{1}{2}\frac{\partial^2}{\partial s^2}V(t,S(t-),Y(t))\sigma^2(Y(t))S(t-)dt\\
&+\frac{\beta^2}{2}\frac{\partial^2}{\partial y^2}V(t,S(t-),Y(t))S(t-)dt\\
&+\int_a^\infty[V(t,(1+x)S(t-),Y(t))\\
&-V(t,S(t-),Y(t))]\tilde{N}^*(dt,dx)\\
&+\int_a^\infty[V(t,(1+x)S(t-),Y(t))\\
&-V(t,S(t-),Y(t))\\
&-xS(t-)\frac{\partial}{\partial s}V(t,S(t-),Y(t))]v^*(dx)dt\\
=\;& [\frac{\partial}{\partial t}V(t,S(t-),Y(t))\\
&+\frac{\partial}{\partial s}V(t,S(t-),Y(t))r(t)S(t-)\\
&+\frac{\partial}{\partial y}V(t,S(t-),Y(t))(\alpha(m-Y(t))+\beta\delta(t))\\
&+\frac{\partial^2}{\partial s\partial y}V(t,S(t-),Y(t))\beta\rho\sigma(Y(t))S(t-)\\
&+\frac{1}{2}\frac{\partial^2}{\partial s^2}V(t,S(t-),Y(t))\sigma^2(Y(t))S(t-)\\
&+\frac{\beta^2}{2}\frac{\partial^2}{\partial y^2}V(t,S(t-),Y(t))S(t-)\\
&+\int_a^\infty[V(t,(1+x)S(t-),Y(t))\\
&-V(t,S(t-),Y(t))\\
&-xS(t-)\frac{\partial}{\partial s}V(t,S(t-),Y(t))]v^*(dx)]dt\\
&+[\frac{\partial}{\partial s}V(t,S(t-),Y(t))\sigma(Y(t))S(t-)\\
&+\frac{\partial}{\partial y}V(t,S(t-),Y(t))\beta\rho]dW^*(t)
\end{aligned}$$

$$+\frac{\partial}{\partial y}V(t,S(t-),Y(t))\beta\sqrt{1-\rho^2}dB^*(t)$$
$$+\int_a^\infty [V(t,(1+x)S(t-),Y(t))$$
$$-V(t,S(t-),Y(t))]\tilde{N}^*(dt,dx).$$

Similarly, we can have,

$$\begin{aligned} & d\tilde{V}(t,S(t),Y(t) \\ = & d\left(e^{-\int_0^t r(s)ds}V(t,S(t),Y(t))\right) \\ = & e^{-\int_0^t r(s)ds}dV(t,S(t),Y(t)) \\ & -r(t)e^{-\int_0^t r(s)ds}V(t,S(t),Y(t))dt \\ = & e^{-\int_0^t r(s)ds}[dV(t,S(t),Y(t)) \\ & -r(t)V(t,S(t),Y(t))dt]. \end{aligned}$$

Therefore, we can get the following theorem,

Theorem 4 *If the call options' value function $V(t,s,y) \in C^{1,2,2}$, it must satisfies the following equation*

$$rs = \frac{\partial V}{\partial t} + rs\frac{\partial V}{\partial s} + [\alpha(m-y)+\beta\delta]\frac{\partial V}{\partial y}$$
$$+\frac{1}{2}\sigma^2(y)s^2\frac{\partial^2 V}{\partial s^2} + \beta\rho\sigma(y)s\frac{\partial^2 V}{\partial s\partial y}$$
$$+\frac{1}{2}\beta^2\frac{\partial^2 V}{\partial y^2} + \int_a^\infty (V^* - V - xs\frac{\partial V}{\partial s})v^*(dx)$$
$$V(T,s,y) = y(s), \forall s \geq 0$$

where

$$V^* = V(t,(1+x)s,y), v^*(dx) = e^{H(x)}v(dx).$$

As to $V(t,s,y) \in C^{1,2,2}$, we can define such a operator C_t as,

$$C_tV(t,s,y) := r(t)s\frac{\partial V}{\partial s} + [\alpha(m-y)+\beta\delta(t)]\frac{\partial V}{\partial y}$$
$$+\frac{1}{2}\sigma^2(y)s^2\frac{\partial^2 V}{\partial s^2} + \beta\rho\sigma(y)s\frac{\partial^2 V}{\partial s\partial y}$$
$$+\frac{1}{2}\beta^2\frac{\partial^2 V}{\partial y^2} + \int_a^\infty (V^* - V - xs\frac{\partial V}{\partial s})v^*(dx)$$

Thus we can have,

$$\frac{\partial V}{\partial t} + C_tV - r(t)V = 0. \quad (13)$$

Theorem 5 *Assumed that the underlying stock of the American call option will distribute rights and dividend at time $t_i (i = 1,2,\cdots,n)$, and the proportion is $\theta_i (i = 1,2,\cdots,n)$ correspondingly.*

(i) $t \in [t_{i-1}, t_i), i = 1,2,\cdots,n$, *the option's value function $V(t,s,y)$ satisfies*

$$\begin{cases} \frac{\partial}{\partial t}V(t,s,y) + C_tV(t,s,y) - r(t)V(t,s,y) = 0 \\ V(t_i-,s,y) = \max\{y(s), V(t_i-,\theta_i s,y)\} \end{cases}$$

(ii) $t \in [t_n, T]$, *the option's value function*

$$\begin{cases} \frac{\partial}{\partial t}V(t,s,y) + C_tV(t,s,y) - r(t)V(t,s,y) = 0 \\ V(T,s,y) = y(s) \end{cases}$$

The optimal time to exercise the option is $t_1-, t_2-, \cdots, t_n-$ or T, and the condition that the option is exercised at time t_i- is

$$V(t_i, \theta_i s, y) < y(s), \forall s \geq 0.$$

Proof 1) Firstly, we assume that the yield function of the American call options is $y(x) = (x-k)^+$, and that the underlying stock will pay some dividend and place some shares only at the same time $t_1 \in [0,T]$. Denote t_1- by the instantaneousness before the dividend and rights are distributed, and t_1+ by the instantaneousness after the dividend and rights are distributed. We also assume that the price before the dividend and rights are granted is $S(t_1-)$, $0 \leq \beta < 1$ is the so-called cash dividend ration (that's the ratio of cash dividend to $S(t_1)$); $\delta_1(\delta_1 \geq 0)$ bonus shares (tax included) will be allotted to each share to the whole shareholders; furthermore, $\delta_2(\delta_2 \geq 0)$ shares will be placed to each share to the whole shareholders, denote $\gamma(0 < \gamma < 1)$ by the ratio of the price of placed shares to $S(t_1-)$. According to the principle that the value of the stock should be equivalent before and after the dividend and rights are distributed, the price of the underlying stock after the dividend and rights being distributed should be

$$S(t_1+) = \frac{1-\beta+\delta_2\gamma}{1+\delta_1+\delta_2}S(t_1).$$

Denote

$$\alpha := \frac{1-\beta+\delta_2\gamma}{1+\delta_1+\delta_2}.$$

α is called the dividend-paying and placing rate, and easily we can tell that $0 < \alpha \leq 1$. By It$\hat{o}$ formula,

$$\begin{aligned} S(t) &= S(0)\exp\left[\int_0^t r(s)ds - \frac{1}{2}\int_0^t \sigma^2(Y(s))ds\right. \\ &+ \int_0^t \sigma(y(s))dW^*(s) + \int_a^\infty \ln(1+x)\tilde{N}^*(t,dx) \\ &+ \left. t\int_a^\infty (\ln(1+x)-x)v^*(dx)\right]. \end{aligned}$$

So the price process of the underlying stock which will distribute some dividend and rights at time t_1 satisfies:
i) When $0 \le t \le t_1$,

$$\begin{aligned} S(t) &= S(0)\exp\left[\int_0^t r(s)ds - \frac{1}{2}\int_0^t \sigma^2(Y(s))ds \right. \\ &+ \int_0^t \sigma(y(s))dW^*(s) + \int_a^\infty \ln(1+x)\tilde{N}^*(t,dx) \\ &+ \left. t\int_a^\infty (\ln(1+x)-x)v^*(dx)\right]; \end{aligned}$$

ii) When $t_1 > t \le T$,

$$\begin{aligned} S(t) &= \alpha S(t_1-)\exp\left[\int_0^t r(s)ds - \frac{1}{2}\int_0^t \sigma^2(Y(s))ds \right. \\ &+ \int_0^t \sigma(y(s))dW^*(s) + \int_a^\infty \ln(1+x)\tilde{N}^*(t,dx) \\ &+ \left. t\int_a^\infty (\ln(1+x)-x)v^*(dx)\right]. \end{aligned}$$

As to an American call option whose underlying asset is the former stock, its value at time t_1- should be

$$\begin{aligned} & V(t-,S(t-),Y(t-)) \\ =\ & \max\left\{y(S(t-)), E[V(t,S(t),Y(t)) \mid \mathcal{F}_{t-}]\right\}, \end{aligned}$$

where $\mathcal{F}_{t-} = \bigcap_{\epsilon>0}\mathcal{F}_{t-\epsilon}$.

From (12) we can know that almost all sample path of the process $Y(t)$ is continuous with respect to t, in addition, $S(t_1) = \alpha S(t_1-)$, so

$$\begin{aligned} & E(V(t_1,S(t_1),Y(t_1) \mid \mathcal{F}_{t_1-}) \\ =\ & E(V(t_1-,\alpha S(t_1-),Y(t_1-) \mid \mathcal{F}_{t_1-}) \\ =\ & V(t_1-,\alpha S(t_1-),Y(t_1-)). \end{aligned}$$

Therefore, the value of the call option at the instantaneousness before the dividend and rights are distributed is

$$\begin{aligned} & V(t_1-,S(t_1-),Y(t_1-)) \\ =\ & \max\left\{y(S(t_1-)), V(t_1-,\alpha S(t_1-),Y(t_1-))\right\}. \end{aligned}$$

Because there are no dividend and bonus shares during time $[0,t_1)$, the value of the American call option should satisfies the equation (13). If the option is not exercised at time t_1, it can be viewed as an option from t_1 to T during the time interval $(t_1,T]$, and there are still no dividend and bonus shares during it, so here the value of the option still satisfies the equation (13). Thus the optimal exercised time of the American call option whose underlying stock will distribute dividend and grant shares at the same time t_1 should be t_1- or T. And the condition that the option should be exercised at time t_1- is

$$V(t_1,\alpha s,y) < y(s).$$

Therefore, if an American call option's underlying asset—stock will distribute some dividend and grant some bonus shares at time t_1, and the dividend-paying and placing rate is α, the value function of the option satisfies the following equation,

(i) when $t \in [0,t_1)$,

$$\begin{cases} \frac{\partial}{\partial t}V(t,s,y) + C_tV(t,s,y) - r(t)V(t,s,y) = 0 \\ V(t_i-,s,y) = \max\{y(s), V(t_1-,\alpha s,y)\} \end{cases}$$

(ii) when $t \in [t_1,T]$,

$$\begin{cases} \frac{\partial}{\partial t}V(t,s,y) + C_tV(t,s,y) - r(t)V(t,s,y) = 0 \\ V(T,s,y) = y(s) \end{cases}$$

2) Secondly, we assume that the underlying stock of the American call option will distribute some rights and dividend at time $t_i (i = 1,2,\cdots,n)$, and the dividend-paying and placing rate is $\theta_i (i = 1,2,\cdots,n)$ correspondingly, the value function of the option should satisfy the following equation,

(i) when $t \in [t_{i-1},t_i), i = 1,2,\cdots,n,$

$$\begin{cases} \frac{\partial}{\partial t}V(t,s,y) + C_tV(t,s,y) - r(t)V(t,s,y) = 0 \\ V(t_i-,s,y) = \max\{y(s), V(t_i-,\theta_i s,y)\} \end{cases}$$

(ii) when $t \in [t_n,T]$,

$$\begin{cases} \frac{\partial}{\partial t}V(t,s,y) + C_tV(t,s,y) - r(t)V(t,s,y) = 0 \\ V(T,s,y) = y(s) \end{cases}$$

That is to say, the optimal exercised time(stopping time) should be $t_1-, t_2-, \cdots, t_n-$ or T. And the condition that the option should be exercised at time t_i- is

$$V(t_i,\alpha_i s,y) < y(s).$$

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