Indeterminate Forms And Their Behaviours

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Abstract: - This study shows that there exist solutions to the seven main indeterminate forms that are raised in the world of mathematics . Some limits of functions are said to be indeterminate when merely knowing the limiting behaviour of individual parts of expression is not sufficient to actually determine the overall limit .There will be a study of a certain typical number whereas, if a variable x tends to a certain value α (eventually equal to + ∞ or - ∞) , a certain function Γ(x) does not have an apparent limit from the first view . To eliminate the indetermination (or looking for the right value of Γ (α) , is to find this limit if it exists .

Key-Words: - infinity , indeterminate, equivalent infinitely large principal , equivalent infinitely small principal , limit , function .

1 Introduction

The term " indeterminate " is sometimes used as a synonym for unknown or variable[1] . A mathematical expression which is not definitively or precisely determined may be considered as indeterminate form . Recognizing that the function Γ could be already defined by α : however , what is interesting is then not the initial value of Γ (α) but the limit of Γ (x) . If this limit L exists , then the function is equal to Γ (x) for x ≠ α and equals to L for x = α . We have already faced such problems when we were studying the description form of a rational fraction on the neighbourhood of a pole or at infinity, and when we were studying the limit of $\frac{\sin x}{x}$ as x → 0 . Similarly , when we were searching the calculation of the derivation , we found the limit of : $\frac{\xi(x+h)-\xi(x)}{h}$ if h → 0 ,this lead to study an indeterminate form with respect to a variable h . Calculate a derivation is then eliminate the indetermination .

[1] Becker and Weispfenning 1993 , p.188 .

So it is not sure that the knowledge of the theory of derivations and their recognize that, by changing the variables β = x – α if α is not finite or $\beta = \frac{1}{x}$ if α is infinite , we may always depend on the case where $\alpha = 0$. For instance , $\lim_{x\to 0}\frac{\xi(x)}{\psi(x)} = \frac{0}{0}$ is indeterminate since the value of the overall limit actually depends on the limiting behaviour of the combination of the two functions (e.g. $\lim_{x\to 0}\frac{x}{x} = 1$, while $\lim_{x\to 0}\frac{x^2}{x} = 0$) The indeterminate forms involving 0 , 1 and ∞ are :($\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\times\infty$, $\infty-\infty$, ∞^0, 0^0 , 1^∞)[2]

[2] Thomas and Finney 1996 ; Gellert et al. 1989. Note , however , that there is a certain ambiguity in this enumeration in the sense that symbolic expressions of the form $\frac{\infty}{\infty}$ can perhaps be written as $\infty\times\infty^{-1}$, etc…

Although it is clear that the addition of two large numbers is a large number ($\infty + \infty = \infty$) , it is absolutely vague what would happen if we subtract two large numbers from each others ($\infty - \infty$) or divide them ($\frac{\infty}{\infty}$) . Knowing that the number on the neighbourhood of zero is considered as a small number while a number close to $\pm\infty$ is considered as a large number ; Many are confused about the number - ∞ by considering it as the smallest number among the real numbers while in fact it is a large number which happens to be negative and it is not real .Large and small here are to be understood in terms of quantities while the set of real numbers has a natural order which is not of concerns to us here [3].

2. Indeterminate forms

The indeterminate forms involving 0 , 1 and ∞ are $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, ∞^0, 0^0 , 1^∞

2.1 Form $\frac{0}{0}$

To the persistent but misguided reader who insists on asking "What happens if I do divide by zero,"[4] It is very well known indeterminate form .Γ (x) is presented then in the form of the quotient

$\Gamma(x) = \frac{\xi(x)}{\psi(x)}$,the functions ξ and ψ both tends to zero as x $\to \alpha$.

As we mention before , it is then directed to the case where $\alpha = 0$, which means the case where the functions ξ and ψ are infinitely small for x is small (This last variable is known as infinitely small principal) .

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[3] www.sosmath.com/calculus/indforms/intro/intro.html . We say that we have an indeterminate form. May be one of the most important indeterminate form is the quotient of two small (or large) numbers.

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[4] Derbyshire (2004, p. 36) provides the slightly flippant but firm and concise response, "You can't. It's against the rules." Even in fields other than the real numbers, division by zero is never allowed (Derbyshire 2004, p. 266) .

To facilitate this exposition we put it in this case .First, assume that the ratio

$$\Gamma(x) = \frac{\xi(x)}{\psi(x)} \to 1 \quad \text{as} \quad x \to 0 .$$

It gives that ξ and ψ are related with a relation of the form $\xi(x) = \psi(x)(1 + \mu(x))$ where $\mu(x)$ is a function that tends to zero as x tends to zero . If we may write such relation, then we would say that ξ and ψ are equivalent infinitely small and denoted by : $\xi \sim \psi$. The writer will verify this relation is an equivalent relation .we say that ξ is infinitely small of order n with respect to x if $\xi(x) \sim a x^n$ with $n > 0$ and $a \neq 0$. If this happens then we can also say that $a x^n$ is the principal part of the infinitely small function ξ . It is easy to obtain this principal part when ξ has a limited expansion with respect to x. If :

$$\xi(x) = a_p x^p + a_{p+1} x^{p+1} + \ldots + a_m x^m + x^m \mu(x)$$

is a such expansion , a_p is distinct from zero , the principal part of ξ is $a_p x^p$.

This is posed , when $\Gamma(x) = \frac{\xi(x)}{\psi(x)}$ presents the indeterminate form $\frac{0}{0}$, we can , in order to study the description form of ξ , replace ξ and ψ by equivalents infinitely small ξ_1 and ψ_1 . So we can write :

$$\xi = \xi_1 (1 + \mu(x)) \text{ and } \psi = \psi_1 (1 + \mu(x))$$

Hence $\Gamma = \frac{\xi}{\psi} = \frac{\xi_1}{\psi_1}(1 + \mu(x))$ is described as $\frac{\xi_1}{\psi_1}$ and then have the same limit eventually .

In particular , if ξ and ψ are respectively infinitely very small of orders n and m with respect to x of principal parts $a x^n$ and $b x^m$, then the function Γ will be compatible to the quotient $\frac{a x^n}{b x^m}$, that is :

for n > m , ξ (x) tends to zero with x .

for n < m , ξ (x) tends to ∞ as x tends to zero .

for n = m , ξ (x) tends to $\frac{a}{b}$ as x tends to zero .

Thus , to eliminate the indetermination of ξ search , if it is possible , for limited expansions with respect to x of ξ and ψ .

Practice :

Let us search for the limit of the function

y = $\frac{\sinh x - \sin x}{\tan x - x}$ as x $\to$ 0 .

It is clear that the indeterminate form is $\frac{0}{0}$. Using the limited expansion of sinh x and sin x we have :

$$\sinh x - \sin x = \left(x + \frac{x^3}{6} + x^3 \mu(x) \right) - \left(x - \frac{x^3}{6} + x^3 \mu(x) \right) = \left(\frac{x^3}{3} + x^3 \mu(x) \right) .$$

Similarly , by using the limited expansion of tan x :

$$\tan x - x = \left(x + \frac{x^3}{3} + x^3 \mu(x) \right) - x$$

$$= \left(\frac{x^3}{3} + x^3 \mu(x) \right)$$

Hence , y has the same limit as the ratio

$\frac{x^3/3}{x^3/3}$ =1 .Thus, y $\to$ 1 as x $\to$ 0

Remark : if there are limited expansions of ξ (x) and ψ (x) , such as :

$$\xi(x) = a_n x^n + a_{n+1} x^{n+1} + \ldots + a_q x^q + x^q \mu(x)$$

$$\psi(x) = b_m x^m + b_{m+1} x^{m+1} + \ldots + b_r x^r + x^r \mu(x)$$

with $a_n \neq 0$ and $b_m \neq 0$, then we can write the limited expansion :

$c_0 + c_1 x + \ldots + c_s x^s + x^s \mu(x)$ of quotient :

$$\frac{\xi / x^n}{\psi / x^m} = \frac{a_n + a_{n+1} x + \ldots a_q x^{q-n} + x^{q-n} \mu(x)}{b_m + b_{m+1} x + \ldots + b_r x^{r-m} + x^{r-m} \mu(x)}$$

and then we have the following expansion of Γ(x) :

$$\Gamma(x) = c_0 x^{n-m} + c_1 x^{n-m+1} + \ldots + c_s x^{n-m+s} + x^{n-m+s} \mu(x) .$$

For n $\geq$ m , it is an expansion limit of Γ and for n < m it is an expansion generalized with respect to infinitely large X = $\frac{1}{x}$ of the form

$$\phi(x) = \frac{c_0}{X^k} + \frac{c_1}{X^{k-1}} + \ldots + \frac{c_{k-1}}{X} + c_k + c_{k+1} X + \ldots + c_{k+n} X^n + X^n \mu(x)$$

and then get , referring back to the variable x , the expansion :

$$\xi(x) = c_0 x^k + c_1 x^{k-1} + \ldots + c_{k-1} x + c_k + \frac{c_{k+1}}{x} + \ldots + \frac{c_{k+n}}{x^n} + \frac{1}{x^n} \mu(x) .$$

2.2 Form $\frac{\infty}{\infty}$

What is infinity divided by infinity? Most people would say it equals one . (infinity) x (infinity) = (infinity) , so (infinity) = (infinity) / (infinity) . Depending on how you figure it out, you can get infinity divided by infinity to equal just about anything[5].

It is the case where Γ is represented in the form Γ (x) = $\frac{\xi(x)}{\psi(x)}$, where the functions ξ and ψ increase indefinitely and absolutely when x tends to α .We write :

$$\Gamma = \frac{1/\psi(x)}{1/\xi(x)},$$

we notice that it has the indeterminate form $\frac{0}{0}$. Some particular remarks are concerned with case .

It is better , by taking a new variable $\beta = \frac{1}{x-\alpha}$, to consider the limit under study when x $\rightarrow \infty$. Then we say that x is infinitely large principal , when ξ and ψ are related by a relation of the form ξ (x) = ψ (x) (1 + $\mu(\frac{1}{x})$) such that $\mu(x)$ tends to 0 with $\frac{1}{x}$, then the two functions ξ and ψ are two equivalent infinitely large and we write : $\xi \sim \psi$.

This relation is an equivalent relation and permits the ratio $\frac{\xi(x)}{\psi(x)} \rightarrow 1$.

We say that ξ is infinitely of order n if we have :

ξ (x) ~ a x^n with a $\neq 0$, n > 0 . We say then that a x^n is the principal part of infinitely large ξ .

This is said , when Γ (x) = $\frac{\xi(x)}{\psi(x)}$ has the indeterminate form $\frac{\infty}{\infty}$ as x $\rightarrow \infty$, to study the replacement limit Γ, we can replace ξ and ψ with equivalents infinitely large . In particular if ξ and ψ are infinitely large functions of orders n and m with respect to x whereas the principal parts are respectively a x^n and b x^m , we notice that F has the same limit as the quotient $\frac{a x^n}{b x^m}$ and then :

For n > m , Γ tends to ∞ with x .

For n < m , Γ tends to 0 as x tends to ∞

For n = m , Γ tends to $\frac{a}{b}$ as x tends to ∞

In the following , we are going to deal with two cases of indeterminate forms of the form $\frac{\infty}{\infty}$.

y = $\frac{e^x}{x^m}$ as x $\rightarrow +\infty$, m is a strictly positive number (the case where m is null or negative is evident) .

We notice that if x is positive , then the function $e^x - x$ is always increasing since its derivation $e^x - 1$ is always positive or null for x > 0 .

Because $e^x - x = 1$ for x = 0 , then the ratio

$$\frac{e^x}{\sqrt{x}} \rightarrow \infty \text{ as } x \rightarrow +\infty .$$

Suppose that $z = \frac{x}{2m}$, the equality :

$$\frac{e^x}{x^m} = \frac{1}{(2m)^m}\left(\frac{e^z}{\sqrt{z}}\right)^{2m}$$

proves that $\frac{e^x}{x^m} \rightarrow +\infty$ at the same time as

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[5]Ronak S. Dr. Ken Mellendorf

Physics Instructor Illinois Central College z , which means that at the same time as x and then when m is strictly positive ,the ratio $\frac{e^x}{x^m} \to +\infty$ at the same time as x .

We will notice that this does not mean that e^x is always greater than x^m .

Let us propose now the behaviour of the

ratio $\frac{\ln x}{x^m}$ as $x \to +\infty$, m is strictly positive.

Assume that $\ln x = t$, we have :

$$\frac{\ln x}{x^m} = \frac{t}{e^{mt}} = \frac{1}{m}\frac{mt}{e^{mt}} ,$$

referring to what has preceded , this ratio tends to 0 as $mt \to +\infty$, which also means that as $x \to +\infty$. Hence , the number m is strictly positive , and the ratio $\frac{\ln x}{x^m} \to 0$ as $x \to +\infty$.

These two important results lead to treat many other indeterminate forms . For example , in

finding the limit of $y = \frac{e^x}{1/x}$ as $x \to -\infty$.

The function y will take then the indeterminate form $\frac{0}{0}$ .We let $x = -\beta$, the equality $y = \frac{-\beta}{e^{\beta}}$ shows that $y \to 0$ as $\beta \to +\infty$. The limit to be found is then 0 .

2.3 Form $0 \times \infty$

It is the case of the function $\Gamma(x) = \xi(x)\,\psi(x)$ such that , as $x \to \alpha$, the function $\xi \to 0$ and the function ψ augments indefinitely . By writing $\Gamma = \frac{\xi}{1/\psi}$, we notice that it is then transformed to the form $\frac{0}{0}$.

Let $t = \frac{1}{x}$ we have : for $m > 0$,

$$\lim_{x \to 0} x^m \ln x = \lim_{t \to +\infty} \frac{-\ln t}{t^m} = 0 .$$

The number m is strictly positive , the product $x^m \ln x \to 0$ as $x \to 0$ with positive values .

Also here we can , in order to study the limit of Γ , replace ξ by equivalent infinitely small and ψ

by equivalent infinitely large .In particular , if

$\alpha = 0$, when ξ is infinitely small whose principal

part with respect to x is $a x^n$ and when ψ is

infinitely large whose principal part with respect to

$\frac{1}{x}$ is $\frac{b}{x^m}$, we recognize that Γ is behaved as the function $a b x^{n-m}$.

Example :

Let us study the limit of $y = \frac{x^2 - 1}{x} e^{\frac{1}{x}}$ as $x \to 0$

with positive values ,

$\frac{x^2 - 1}{x} \to -\infty$ and $e^{\frac{1}{x}} \to +\infty$, then

$y \to -\infty$ and there is no indeterminate form .

As $x \to 0$ with negative values , we notice that

$\frac{x^2 - 1}{x} \to +\infty$ and $e^{\frac{1}{x}} \to 0$; so we are in front of an indeterminate form . To eliminate

the indetermination we can let $x = \frac{-1}{\beta}$. Then we

have :

$$y = \frac{1-1/\beta^2}{1/\beta} e^{-\beta} = \frac{\beta}{e^{\beta}} \left(1 - \frac{1}{\beta^2}\right).$$

As $x \to 0$ with positive values , $\beta \to +\infty$ and

the quotient $\frac{\beta}{e^{\beta}} \to 0$ and $\left(1 - \frac{1}{\beta^2}\right) \to 1$.

Hence $y \to 0$.

2.4 Form $\infty - \infty$

In intuitive terms, the difference between two very large positive numbers can be small, even zero, thus indeterminate, but the sum of two very large positive numbers must be a very large positive number. Obviously (-inf)+(+inf) is also indeterminate[6].

This is the case where the function Γ (x) is represented in the form $\xi(x) - \psi(x)$, both functions ξ and ψ tend to $+\infty$ (or both tend to $-\infty$) as x tends to α .In this case we can not substitute ξ and ψ by equivalents infinitely large .So let $\xi(x) = x^2 + x$ and $\psi(x) = x^2$;

as $x \to \infty$, ξ (x) and ψ (x) are equivalents to

x^2 and then $\xi(x) - \psi(x) = x$ is not compatible as $x^2 - x^2 = 0$.To treat the indeterminate form $\infty - \infty$, we can try to put Γ in a product form by writing $\Gamma = \xi(x)\left(1 - \frac{\psi(x)}{\xi(x)}\right)$.

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[6] www.physicsforums.com/showthread.php?t=155248 - 43k

We can also refer to the case where α is infinity , that is the case where $x \to \infty$ and search to expand ξ and ψ in the form :

$$\xi = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 + \mu\left(\frac{1}{x}\right)$$

$$= P(x) + \mu\left(\frac{1}{x}\right)$$

$$\psi = b_m x^m + b_{m-1} x^{m-1} + \ldots + b_0 + \mu\left(\frac{1}{x}\right)$$

$$= Q(x) + \mu\left(\frac{1}{x}\right)$$

Hence $\xi(x) - \psi(x)$ will be compatible as the polynomial $P(x) - Q(x)$.

Example :

Study the limit of $y = \sqrt[6]{x^6 + x^2 + 1} - \sqrt[4]{x^4 + 1}$ as $x \to +\infty$ .We can write :

$$\sqrt[6]{x^6 + x^2 + 1} = x\left(1 + \frac{1}{x^4} + \frac{1}{x^6}\right)^{1/6}$$

$$= x\left[1 + \frac{1}{6}\left(\frac{1}{x^4} + \frac{1}{x^6}\right) + \frac{1}{x^4}\mu\left(\frac{1}{x}\right)\right]$$

$$= x + \frac{1}{6x^3} + \frac{1}{x^3}\mu\left(\frac{1}{x}\right).$$

Similarly , $\sqrt[4]{x^4 + 1} = x\left(1 + \frac{1}{x^4}\right)^{1/4}$

$$= x\left[1 + \frac{1}{4x^4} + \frac{1}{x^4}\mu\left(\frac{1}{x}\right)\right]$$

$$\sqrt[4]{x^4 + 1} = x + \frac{1}{4x^3} + \frac{1}{x^3}\mu\left(\frac{1}{x}\right).$$

Then , $y = [x + \frac{1}{6x^3} + \frac{1}{x^3}\mu\left(\frac{1}{x}\right)] -$

$$[x + \frac{1}{4x^3} + \frac{1}{x^3}\mu\left(\frac{1}{x}\right)]$$

$$= -\frac{1}{12}\times\frac{1}{x^3} + \frac{1}{x^3}\mu\left(\frac{1}{x}\right).$$

Hence $y \to 0$ as x augmented indefinitely .

The same thing is happened to y which is

infinitely small along which its principal part

with respect to $\frac{1}{x}$ is $-\frac{1}{12x^3}$ .Pushing further the preceding limited expansions , we can have a limited expansion of y with respect to $\frac{1}{x}$ of arbitrary raising order .

2.5 Forms ∞^0 , 0^0 , 1^∞

You may be surprised to see the second item in the list, since 0^0 at first sight looks like a fairly tame object. In fact, however, if you take a look on any given day at an Internet newsgroup devoted to discussion of mathematics, you are likely to find a lively debate going on about whether 0^0 is equal to 1, or is undefined. (Sometimes these arguments are started by a "crank" who charges that the mathematics profession is covering up some embarrassing scandal associated with the meaning of 0^0.)[7]

Let $y = [u(x)]^{v(x)}$.

The relation $y = e^{v(x)\ln u(x)}$ shows that y takes

an indeterminate form because of v(x) ln u(x) , that is if v(x) ln u(x) has the form $0 \times \infty$. This is possible in *three different ways :*

a) $v(x) \to 0$ and $\ln u(x) \to +\infty$, e.g.

$u(x) \to +\infty$. It is the indeterminate form ∞^0 .

b) $v(x) \to 0$ and $\ln u(x) \to -\infty$, e.g.

$u(x) \to 0$. It is the indeterminate form 0^0 .

c) v(x) augmented indefinitely and $\ln u(x) \to 0$, e.g. $u(x) \to 1$. It is the indeterminate form 1^∞ .The elimination of the indetermination of y depends on that of v(x) ln u(x) , e.g. that of ln y , too .

The function (of two variables) x^y is definitely not

continuous as the point *(x,y)* approaches (0,0).

(That's precisely why 0^0 is an indeterminate form; the

limit depends on the direction of approach to (0,0).) But

we know that a function that's discontinuous at a

point doesn't need to be undefined there.

When a math book says that 1^infinity is an

indeterminate form, they mean that if f(x) or g(x) are

unknown functions, except that we know limit(f(x),

x=a) = 1 and limit(g(x), x=a) = infinity, then we still don't have enough information to compute limit(f(x)^g(x), x= a).If both f(x)and g(x) are known, then to say that the answer is indeterminate is simply a failure to give an answer[8].

Example

1) Limit of $y = \left(1+\frac{1}{x}\right)^x$ as $x \to +\infty$. It is of the form 1^∞ . We have $\ln y = x \ln\left(1+\frac{1}{x}\right)$. Since $\ln\left(1+\frac{1}{x}\right)$ is infinitely small equivalent to $\frac{1}{x}$,

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[8] Maple User Group Email List Archive - Adept Scientific

then ln y has a limit $x \cdot \frac{1}{x} = 1$, and y admits e as a limit . Then we have the following result : $\left(1+\frac{1}{x}\right)^{x} \rightarrow$ e as x $\rightarrow + \infty$.

2) Limit of $y = x^{\frac{1}{x}}$ as $x \rightarrow + \infty$ .It is the indeterminate form ∞^{0} . Since ln y = $\frac{1}{x}$ ln x $\rightarrow$ 0 as $x \rightarrow + \infty$, then y $\rightarrow$ 1 .

3) Limit of y = x^{x} as $x \rightarrow 0$. It is the indeterminate form 0^{0} . We have ln y = x ln x = $-\frac{\ln 1/x}{1/x}$. When $x \rightarrow 0$ (with large values to make y defined) ,

$\frac{1}{x} \rightarrow + \infty$ and ln y $\rightarrow 0$, so y $\rightarrow$ 1 .

3. Power of the Indeterminate Forms

As we see the types of the indeterminate forms and their behaviors , one might ask the question " what would happen if these indeterminate forms do not exist ? " Well , from my point of view , I would say that either we did many mistakes in guessing wrong mistakes such that ($\frac{0}{0} = 1$, $0^{0} = 1$, $1^{\infty} = 1$, etc…)

or Many mathematical expressions that have indeterminate forms will be held and blocked .So these forms give rise to many problems in math , in physics , in engineering , in statistics to have an open gate towards plenty of ways to eliminate their indeterminations . For instance , While Professor Siavash H. Sohrab was working in his paper " Implications of a Scale Invariant Model of Statistical Mechanics to Nonstandard Analysis and the Wave Equation"[9] attainted the formula $D_{\beta} = (\ln N_{AS\beta-1}) \;/\; (\ln N_{AE\beta-1})$. Then how can this professor continue the work when $N_{AS\beta-1} = N_{AE\beta-1} = 1$? in this case $D_{\beta} = \frac{0}{0}$ which is an indeterminate form , and it gave him a gate to go on .

So we as scientists should be convinced that there is nothing impossible in this life . And when we face a case that it looks to be impossible , we would rather considered it as an indeterminate form and work on it .

4. Conclusion

It was cleared that the word indetermination is completely different from the word impossible . The actual numerical values of $\frac{0}{0}$, 1^{∞} , ∞^{o} , 0^{0} , $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$ or 0^{∞} are not well defined ; they depend on where the 0's and infinities came from.

Ariel's is correct that this is not precisely accurate ; it would be more accurate to say that the limits of these quantities are indeterminate forms , not the quantities themselves . The quantities are always undefined ; their limits , however , can have values .

Some Scientists state that $\frac{1}{0} = \pm\infty$, where in fact it is impossible to divide one by zero and it is incredible to write a number equal to infinity because infinity is an indefinite number . Therefore, all what can we say in this situation is that when x approaches zero (not zero) , the expression $\frac{1}{x}$ approaches $\pm\infty$ (not $\pm\infty$) .

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