On the ratio processes induced from the mean-field Bouchaud-Mezard model

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Abstract: In this article, we develop the ratio processes $\{Y_i(t)\}_{t\geq 0}$, $i = 1, 2, \cdots, n$ induced from the mean-field Bouchaud-Mezard model. The limit m_i of the long-time average of the ratio process $\{Y_i(t)\}_{t\geq 0}$ is studied and compared with all others. We shows that a strictly increasing sequence $\{\sigma_k\}_{k=1}^{n}$ of the investment volatilities implies a strictly decreasing sequence $\{m_k\}_{k=1}^{n}$ of the limits, given appropriate J, based on both theoretical and numerical analyses. It reveals a negative correlation between the investment volatilities and the ratio processes. As an empirical application, this negative correlation can be employed to characterize the mean-field Bouchaud-Mezard model. Our main result also indicates that an agent whose spontaneous growth or decrease in wealth due to investment in stock markets is always small will eventually become rich in the mean-field Bouchaud-Mezard model.

Key-Words: Mean-field Bouchaud-Mezard model, Wealth distribution, Ratio process, Volatility, Ergodic, Long-time average

1 Introduction

In the nineteenth century, Pareto studied the distribution of personal wealth to characterize the economic status of a country. He found that the distribution of the personal wealth follows a power-law distribution [8]. Hereafter, Gibrat clarified that the Pareto law is applicable only in the range of high wealth [4]. He concluded that the personal wealth distribution in the middle wealth range follows a log-normal distribution. Recently, Drvagulescu and Yakovenko showed that the wealth distribution at very low wealth is essentially exponential [3]. Although, no consensus has been reached on the distributions in the middle and low-wealth ranges, the distribution in the high-wealth range is today generally believed to follow a power-law tail [10].

To model the high wealth range, Bouchaud and Mezard introduced a linear mean-field model [1], borrowed from the physics of directed polymers, to describe the dynamics of the individual wealth $X_i(t)$ in a given society of n agents [9]. With reference to an earlier work [5], this study considers the following stochastic differential equation;

$$dX_i(t) = \sigma_i X_i(t) dB_i(t) + \frac{J}{n}\sum_{k=1}^{n}\{X_k(t) - X_i(t)\}dt,$$

$$X_i(0) > 0, \quad i = 1, 2, \cdots, n,$$

where $B_1(t), B_2(t), \cdots$ are mutually independent standard Brownian motions, and the investment volatility σ_i describes the spontaneous growth or decrease of wealth due to investment in stock markets as well as the positive constant $\frac{J}{n}$, the amount of wealth that all agents exchange with all others.

Since investment volatility σ_i depends on the *i*th agent, the model considered herein incorporates fluctuations of personal wealth among agents. The earlier cited work neglected this consideration [5], and assumed the investment volatilities to be identically equal.

Now, let

$$Y_i(t) = \frac{X_i(t)}{\sum_{k=1}^{n} X_k(t)}, \qquad i = 1, 2, \cdots, n.$$

The wealth ratio $Y_i(t)$ is considerable because $Y_i(t)$ is the value of personal wealth normalized to overall wealth and has a value of between zero and one. A study of the arrangement in sizes between $\{X_1(t), X_2(t), \cdots, X_{n-1}(t), X_n(t)\}$ clearly shows that $Y_i(t) > Y_j(t)$ if and only if $X_i(t) > X_j(t)$. With the aid of being bounded and order-preserving, we take $Y_i(t)$ as the object of this study. And we call $\{Y_i(t)\}_{t\geq 0}$ "the ratio process induced from the mean-field Bouchaud-Mezard model" in this article.

According to the proof of Theorem 2.2 elsewhere [5], $\left(Y_1(t), Y_2(t), \cdots, Y_{n-1}(t)\right)$ is ergodic for $n \geq 2$. That is, there exists the unique invariant probability measure $\pi(\cdot)$ on $\mathbb{S}$ such that for any $f \in L(\pi)$,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f\left(Y_1(s), \cdots, Y_{n-1}(s)\right) ds = \int_{\mathbb{S}} f\left(y_1, \cdots, y_{n-1}\right) d\pi, \tag{1}$$

where

$$\mathbb{S} = \left\{ \left(y_1, y_2, \cdots, y_{n-1}\right) \in (0,1)^{n-1} : \sum_{k=1}^{n-1} y_k < 1 \right\}.$$

Because $Y_n(t) = 1 - \sum_{k=1}^{n-1} Y_k(t)$ and $y_n = 1 - \sum_{k=1}^{n-1} y_k$, plainly, (1) admits

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f\left(Y_1(s), \cdots, Y_{n-1}(s), Y_n(s)\right) ds = \int_{\mathbb{S}} f\left(y_1, \cdots, y_{n-1}, y_n\right) d\pi, \tag{2}$$

for any bounded continuous function f that is defined on $\mathbb{S} \times (0,1)$. The subtle distinction between Eqs. (1) and (2) should be noted.

Since a risk-seeking agent may prefer a high-grade volatility, while a risk-averse agent prefers a low rating, whether $Y_i(t)$ of a risk-seeking agent exceeds that of a risk-averse agent for a long time is of interest. With this motivation,

$$n \geq 2 \text{ and } 0 < \sigma_1 < \sigma_2 < \cdots < \sigma_{n-1} < \sigma_n$$

are assumed. The purpose of this article is to study the arrangement of $\{m_1, m_2, \cdots, m_n\}$ in sizes, where

$$m_i = \int_{\mathbb{S}} y_i \, d\pi, \text{ for } i = 1, 2, \cdots, n-1,$$

$$m_n = 1 - \sum_{k=1}^{n-1} m_k.$$

Our main result is that

$$m_1 > m_2 > \cdots > m_{n-1} > m_n,$$

provided

$$J > \frac{994 n^2 \sigma_n^4}{15 \min\left\{\sigma_j^2 - \sigma_i^2 : 1 \leq i < j \leq n\right\}}. \tag{3}$$

Notably, the interpretation of the main result in economics is as described in Section 5.

This article is outlined as follows. Section 2 presents the main result obtained by theoretical analysis. Section 3 performs a numerical study to verify our main result. Section 4 discusses a conjecture that condition (3) on J may be unnecessary. Section 5 draws a conclusion.

2 Theoretical analysis

For simplicity of showing main result, we introduce the following notations for each fixed i, $1 \leq i \leq n$;

$$u_i = \int_{\mathbb{S}} y_i^2 \, d\pi,$$

$$v_i = \int_{\mathbb{S}} y_i^3 \, d\pi,$$

$$w_i = \int_{\mathbb{S}} y_i^4 \, d\pi,$$

$$f_i = \int_{\mathbb{S}} y_i \left(\sum_{k=1}^{n} \sigma_k^2 y_k^2 \right) d\pi ,$$

$$g_i = \int_{\mathbb{S}} y_i^2 \left(\sum_{k=1}^{n} \sigma_k^2 y_k^2 \right) d\pi ,$$

$$h_i = \int_{\mathbb{S}} y_i^3 \left(\sum_{k=1}^{n} \sigma_k^2 y_k^2 \right) d\pi .$$

In addition, throughout this section, we denote y by $\left(y_1,\cdots,y_{n-1},y_n\right)=\left(Y_1(0),\cdots,Y_{n-1}(0),Y_n(0)\right)$.

2.1 Lemma of showing main result

In order to show our main result, we need the following two crucial lemmas.

Lemma 2.1.1

Assume $n \ge 2$ *and* $J \notin \left\{ \sigma_i^2, \frac{\sigma_i^2}{2}, \frac{\sigma_i^2}{3} : 1 \le i \le n \right\}$.

For each fixed i, $1 \le i \le n$, *we have the following evaluations* (i)-(iv);

(i)

$$m_i = \frac{1}{n} + \frac{1}{J} f_i - \frac{1}{J}\sigma_i^2 u_i , \tag{4}$$

(ii)

$$u_i = \frac{J}{2J-\sigma_i^2}\left\{ \frac{2}{n} m_i + \frac{3}{J} g_i - \frac{4}{J}\sigma_i^2 v_i \right\}, \tag{5}$$

(iii)

$$v_i = \frac{J}{J-\sigma_i^2}\left\{ \frac{1}{n} u_i + \frac{2}{J} h_i - \frac{3}{J}\sigma_i^2 w_i \right\}, \tag{6}$$

(iv)

$$\begin{aligned} & f_i \\ & = \sigma_i^2 v_i + \sum_{k\neq i}^{n} \frac{J\sigma_k^2 u_k}{n\left(3J-\sigma_k^2\right)} \\ & \quad + \sum_{k\neq i}^{n} \frac{2J\sigma_k^2}{n\left(3J-\sigma_k^2\right)} \int_{\mathbb{S}} y_i y_k \, d\pi \\ & \quad + \sum_{k\neq i}^{n} \frac{6\sigma_k^2}{3J-\sigma_k^2} \int_{\mathbb{S}} y_i y_k^2 \left(\sum_{l=1}^{n} \sigma_l^2 y_l^2 \right) d\pi \\ & \quad - \sum_{k\neq i}^{n} \frac{6\sigma_k^4}{3J-\sigma_k^2} \int_{\mathbb{S}} y_i y_k^3 \, d\pi \\ & \quad - \sum_{k\neq i}^{n} \frac{3\sigma_i^2\sigma_k^2}{3J-\sigma_k^2} \int_{\mathbb{S}} y_i^2 y_k^2 \, d\pi . \end{aligned} \tag{7}$$

Proof:

For any fixed i, $1 \le i \le n$, it is clear that $Y_i(t)$ satisfies the following stochastic differential equation;

$$\begin{aligned} & dY_i(t) \\ & = \sigma_i Y_i(t) dB_i(t) - Y_i(t)\left(\sum_{k=1}^{n} \sigma_k Y_k(t) dB_k(t) \right) \\ & \quad + \frac{J}{n} dt - JY_i(t)dt + Y_i(t) \sum_{k=1}^{n} \sigma_k^2 Y_k(t)^2 dt \\ & \quad - \sigma_i^2 Y_i(t)^2 dt. \end{aligned} \tag{8}$$

Now, for any fixed positive integer m, we take $f\left(y_1, y_2, \cdots, y_n\right) = y_i^m$. Applying Ito formula and (8) to evaluate $f\left(Y_1(t), Y_2(t), \cdots, Y_n(t)\right) = Y_i(t)^m$ first, and then taking the expectation on the both sides, finally, differentiating the both sides with respect to t gives

$$\begin{aligned} \frac{d}{dt} E_y Y_i(t)^m = & \frac{mJ}{n} E_y Y_i(t)^{m-1} - mJE_y Y_i(t)^m \\ & + \frac{m(m-1)\sigma_i^2}{2} E_y Y_i(t)^m \\ & + \frac{m(m+1)}{2} E_y \left\{ Y_i(t)^m \sum_{k=1}^{n} \sigma_k^2 Y_k(t)^2 \right\} \\ & - m^2 \sigma_i^2 E_y Y_i(t)^{m+1}. \end{aligned}$$

Integrating both sides of the formula above with respect to t gives

$$\begin{aligned}
&E_y Y_i(t)^m \\
&= \frac{mJ}{n}\int_0^t E_y Y_i(s)^{m-1} ds - mJ\int_0^t E_y Y_i(s)^m ds \\
&\quad + \frac{m(m-1)\sigma_i^2}{2}\int_0^t E_y Y_i(s)^m ds \\
&\quad + \frac{m(m+1)}{2}\int_0^t E_y\left\{Y_i(s)^m \sum_{k=1}^n \sigma_k^2 Y_k(s)^2\right\} ds \\
&\quad - m^2\sigma_i^2\int_0^t E_y Y_i(s)^{m+1} ds + y_i^m .
\end{aligned}$$

We divide the both sides of the formula above by t, and then let t approach to infinity. By using (2), it yields (4) (5) (6) for $m = 1, 2, 3$, respectively.

To show (7), let $f(y_1, y_2, \cdots, y_n) = y_i y_k^2$, where $k \neq i$. We apply Ito formula and (8) to evaluate $f(Y_1(t), Y_2(t), \cdots, Y_n(t)) = Y_i(t)Y_k(t)^2$ first, and then take the expectation on the both sides, finally, differentiate the both sides with respect to t gives

$$\begin{aligned}
\frac{d}{dt}E_y Y_i(t)Y_k(t)^2 &= \frac{J}{n}E_y Y_i(t)^2 + \frac{2J}{n}E_y Y_i(t)Y_k(t) \\
&\quad - \left(3J - \sigma_k^2\right)E_y Y_i(t)Y_k(t)^2 \\
&\quad + 6E_y\left\{Y_i(t)Y_k(t)^2 \sum_{l=1}^n \sigma_l^2 Y_l(t)^2\right\} \\
&\quad - 6\sigma_k^2 E_y Y_i(t)Y_k(t)^3 \\
&\quad - 3\sigma_i^2 E_y Y_i(t)^2 Y_k(t)^2,
\end{aligned}$$

Integrating both sides of the formula above with respect to t, and then dividing the both sides by t, finally, letting t approach to infinity gives

$$\begin{aligned}
\int_{\mathbb{S}} y_i y_k^2 d\pi &= \frac{J}{n\left(3J - \sigma_k^2\right)}\int_{\mathbb{S}} y_i^2 d\pi \\
&\quad + \frac{2J}{n\left(3J - \sigma_k^2\right)}\int_{\mathbb{S}} y_i y_k d\pi \\
&\quad + \frac{6}{3J - \sigma_k^2}\int_{\mathbb{S}} y_i y_k^2\left(\sum_{l=1}^n \sigma_l^2 y_l^2\right) d\pi \\
&\quad - \frac{6\sigma_k^2}{3J - \sigma_k^2}\int_{\mathbb{S}} y_i y_k^3 d\pi \\
&\quad - \frac{3\sigma_i^2}{3J - \sigma_k^2}\int_{\mathbb{S}} y_i^2 y_k^2 d\pi .
\end{aligned}$$

We add up the both sides of the formula above from $k = 1$ to $k = n$ with $k \neq i$ first, and then add $\sigma_i^2 \int_{\mathbb{S}} y_i^3 d\pi$ to the both sides to obtain (7) because

$$f_i = \sum_{k=1, k\neq i}^n \int_{\mathbb{S}} \sigma_k^2 y_i y_k^2 d\pi + \sigma_i^2 \int_{\mathbb{S}} y_i^3 d\pi .$$

This completes the proof.

Lemma 2.1.2

Under the assumption of Lemma 2.1.1, if $0 < \sigma_1 < \sigma_2 < \cdots < \sigma_{n-1} < \sigma_n$ *and* $J > \dfrac{994 n^2 \sigma_n^4}{15 \min\left\{\sigma_j^2 - \sigma_i^2 : 1 \le i < j \le n\right\}}$, *then we have the following evaluations* (i)-(ii) *for each fixed* $i, j,\ 1 \le i < j \le n$;

(i)

$$m_i - m_j = \frac{4J\left(\sigma_j^2 - \sigma_i^2\right)}{n^2\left(2J - \sigma_j^2\right)\left(2J - \sigma_i^2\right)} + \frac{1}{J}\left\{f_i - f_j + o_i - o_j\right\},$$

where

$$o_i = \frac{\sigma_i^2}{2J - \sigma_i^2}\left\{4\sigma_i^2 v_i + \frac{2\sigma_i^2}{n}u_i - 3g_i - \frac{2}{n}f_i\right\}.$$

(ii)

$$\left|f_i - f_j + o_i - o_j\right| < \frac{994\sigma_n^4}{15J}.$$

Proof:

By (4), it is not hard to see

$$m_i - m_j = \frac{4J\left(\sigma_j^2 - \sigma_i^2\right)}{n^2\left(2J - \sigma_j^2\right)\left(2J - \sigma_i^2\right)} + \frac{1}{J}\left\{f_i - f_j + o_i - o_j\right\}.$$

This establishes the first part of this Lemma. Obviously, (7) gives

$$\begin{aligned}
f_i - f_j &= \sigma_i^2 v_i - \sigma_j^2 v_j \\
&\quad + \frac{J\sigma_j^2 u_j}{n(3J - \sigma_j^2)} - \frac{J\sigma_i^2 u_i}{n(3J - \sigma_i^2)} \qquad (9)\\
&\quad + I_1(i) - I_1(j) + I_2(i) - I_2(j) \\
&\quad + I_3(i) - I_3(j) + I_4(i) - I_4(j),
\end{aligned}$$

where

$$I_1(i) = \sum_{k=1,k\neq i}^{n} \frac{2J\sigma_k^2}{n(3J-\sigma_k^2)} \int_{\mathbb{S}} y_i y_k \, d\pi,$$
$$I_2(i) = \sum_{k=1,k\neq i}^{n} \frac{6\sigma_k^2}{3J-\sigma_k^2} \int_{\mathbb{S}} y_i y_k^2 \left(\sum_{l=1}^{n} \sigma_l^2 y_l^2 \right) d\pi,$$
$$I_3(i) = \sum_{k=1,k\neq i}^{n} \frac{6\sigma_k^4}{3J-\sigma_k^2} \int_{\mathbb{S}} y_i y_k^3 \, d\pi,$$
$$I_4(i) = \sum_{k=1,k\neq i}^{n} \frac{3\sigma_i^2\sigma_k^2}{3J-\sigma_k^2} \int_{\mathbb{S}} y_i^2 y_k^2 \, d\pi.$$

In order to evaluate $I_1(i)$ for details, we do a routine computation as above by employing (8) and Ito formula to obtain that for $k \neq i$,

$$\begin{aligned}\frac{d}{dt} E_y Y_i(t) Y_k(t) &= \frac{J}{n}\left\{E_y Y_i(t) + E_y Y_k(t)\right\} \\ &\quad - 2JE_y Y_i(t) Y_k(t) \\ &\quad + 6E_y\left\{Y_i(t)Y_k(t)\sum_{l=1}^{n}\sigma_l^2 Y_l(t)^2\right\} \\ &\quad - 2\sigma_k^2 E_y Y_i(t) Y_k(t)^2 \\ &\quad - 2\sigma_i^2 E_y Y_i(t)^2 Y_k(t),\end{aligned}$$

which implies

$$\begin{aligned}&E_y Y_i(t) Y_k(t) \\ &= \frac{J}{n}\int_0^t \left\{E_y Y_i(s) + E_y Y_k(s)\right\} ds \\ &\quad - 2J\int_0^t E_y Y_i(s) Y_k(s) \, ds \\ &\quad + 6\int_0^t E_y \left\{Y_i(s)Y_k(s)\sum_{l=1}^{n}\sigma_l^2 Y_l(s)^2\right\} ds \\ &\quad - 2\sigma_k^2 \int_0^t E_y Y_i(s) Y_k(s)^2 \, ds \\ &\quad - 2\sigma_i^2 \int_0^t E_y Y_i(s)^2 Y_k(s) \, ds + y_i y_k.\end{aligned}$$

By (2) and (4), we obtain

$$\begin{aligned}&\int_{\mathbb{S}} y_i y_k \, d\pi \\ &= \frac{m_i + m_k}{2n} \\ &\quad + \frac{3}{2J}\int_{\mathbb{S}} y_i y_k \left(\sum_{l=1}^{n}\sigma_l^2 y_l^2\right) d\pi \\ &\quad - \frac{\sigma_i^2}{J}\int_{\mathbb{S}} y_i^2 y_k \, d\pi - \frac{\sigma_k^2}{J}\int_{\mathbb{S}} y_i y_k^2 \, d\pi \\ &= \frac{1}{n^2} + r_{ik},\end{aligned}$$

where

$$\begin{aligned}r_{ik} &= \frac{f_i + f_k - \sigma_i^2 u_i - \sigma_k^2 u_k}{2nJ} \\ &\quad + \frac{3}{2J}\int_{\mathbb{S}} y_i y_k \left(\sum_{l=1}^{n}\sigma_l^2 y_l^2\right) d\pi \\ &\quad - \frac{\sigma_i^2}{J}\int_{\mathbb{S}} y_i^2 y_k \, d\pi \\ &\quad - \frac{\sigma_k^2}{J}\int_{\mathbb{S}} y_i y_k^2 \, d\pi.\end{aligned} \tag{10}$$

Therefore,

$$\begin{aligned}I_1(i) - I_1(j) &= \frac{2J\sigma_j^2}{n^3(3J-\sigma_j^2)} \\ &\quad - \frac{2J\sigma_i^2}{n^3(3J-\sigma_i^2)} \\ &\quad + s_{ij},\end{aligned} \tag{11}$$

where

$$s_{ij} = \sum_{k=1,k\neq i}^{n} \frac{2J\sigma_k^2}{n(3J-\sigma_k^2)} r_{ik} - \sum_{k=1,k\neq j}^{n} \frac{2J\sigma_k^2}{n(3J-\sigma_k^2)} r_{jk}.$$

It is clear that

$$\left|r_{jk}\right| \leq \left(\frac{4}{2nJ} + \frac{7}{2J}\right)\sigma_n^2 \leq \frac{9\sigma_n^2}{2J},$$

due to

$$\left|f_i\right| \leq \sigma_n^2, \ \left|\sigma_i^2\right| \leq \sigma_n^2, \ \left|u_i\right| \leq 1, \ n \geq 2,$$
$$\left|\int_{\mathbb{S}} y_i y_k \left(\sum_{l=1}^{n}\sigma_l^2 y_l^2\right) d\pi\right| \leq \sigma_n^2,$$
$$\left|\frac{\sigma_i^2}{J}\int_{\mathbb{S}} y_i^2 y_k \, d\pi\right| \leq \frac{\sigma_n^2}{J},$$
$$\left|\frac{\sigma_k^2}{J}\int_{\mathbb{S}} y_i y_k^2 \, d\pi\right| \leq \frac{\sigma_n^2}{J}.$$

Since

$$\begin{aligned}J &> \frac{994n^2\sigma_n^4}{15\min\left\{\sigma_j^2 - \sigma_i^2 : 1 \leq i < j \leq n\right\}} \\ &> \frac{994n^2\sigma_n^4}{15\sigma_n^2} > 2\sigma_n^2,\end{aligned}$$

we have

$$\frac{1}{J-\sigma_n^2} < \frac{2}{J}, \tag{12}$$

$$\frac{1}{2J-\sigma_n^2} < \frac{2}{3J}, \tag{13}$$

$$\frac{1}{3J-\sigma_n^2} < \frac{2}{5J}. \tag{14}$$

We always keep (12) (13) (14) and $n \geq 2$ in mind to establish the inequalities (15)-(26) below.

On the other hand, we use $\left|r_{jk}\right| \leq \frac{9\sigma_n^2}{2J}$ to evaluate $\left|s_{ij}\right|$ as follows.

$$\begin{aligned} \left|s_{ij}\right| &\leq \left|\sum_{k=1,k\neq i}^{n} \frac{2J\sigma_k^2}{n(3J-\sigma_k^2)} r_{ik}\right| \\ &\quad + \left|\sum_{k=1,k\neq j}^{n} \frac{2J\sigma_k^2}{n(3J-\sigma_k^2)} r_{jk}\right| \\ &\leq \frac{9\sigma_n^2}{2J}\left|\sum_{k=1,k\neq i}^{n} \frac{2J\sigma_k^2}{n(3J-\sigma_k^2)}\right| \\ &\quad + \frac{9\sigma_n^2}{2J}\left|\sum_{k=1,k\neq j}^{n} \frac{2J\sigma_k^2}{n(3J-\sigma_k^2)}\right| \\ &\leq \left(\frac{9\sigma_n^2}{J}\right)\left(\frac{2J\sigma_n^2}{3J-\sigma_n^2}\right) \\ &\leq \frac{36\sigma_n^4}{5J}. \end{aligned} \tag{15}$$

By

$$\left|\frac{6\sigma_k^2}{3J-\sigma_k^2}\right| \leq \frac{6\sigma_n^2}{3J-\sigma_n^2} \leq \frac{12\sigma_n^2}{5J},$$

$$\left|\sum_{k=1,k\neq i}^{n} \int_{\mathbb{S}} y_i y_k^2 \left(\sum_{l=1}^{n} \sigma_l^2 y_l^2\right) d\pi\right| \leq \sigma_n^2,$$

$$\left|\frac{6\sigma_k^4}{3J-\sigma_k^2}\right| \leq \frac{6\sigma_n^4}{3J-\sigma_n^2} \leq \frac{12\sigma_n^4}{5J},$$

$$\left|\frac{3\sigma_i^2\sigma_k^2}{3J-\sigma_k^2}\right| \leq \frac{3\sigma_n^4}{3J-\sigma_n^2} \leq \frac{6\sigma_n^4}{5J},$$

$$\left|\sum_{k=1,k\neq i}^{n} \int_{\mathbb{S}} y_i y_k^3 d\pi\right| \leq 1,$$

$$\left|\sum_{k=1,k\neq i}^{n} \int_{\mathbb{S}} y_i^2 y_k^2 d\pi\right| \leq 1,$$

we have

$$\left|I_2(i)-I_2(j)\right| \leq \frac{24\sigma_n^4}{5J}, \tag{16}$$

$$\left|I_3(i)-I_3(j)\right| \leq \frac{24\sigma_n^4}{5J}, \tag{17}$$

$$\left|I_4(i)-I_4(j)\right| \leq \frac{12\sigma_n^4}{5J}. \tag{18}$$

Moreover, from (4) (5), we get

$$u_i = \frac{2J}{n^2\left(2J-\sigma_i^2\right)} + t_i, \tag{19}$$

where

$$\begin{aligned} t_i &= \frac{J}{2J-\sigma_i^2}\left\{\frac{3}{J} g_i - \frac{4}{J}\sigma_i^2 v_i\right\} \\ &\quad + \frac{2J}{n\left(2J-\sigma_i^2\right)}\left\{\frac{1}{J} f_i - \frac{1}{J}\sigma_i^2 u_i\right\}. \end{aligned}$$

Note that

$$\begin{aligned} \left|t_i\right| &\leq \frac{J}{2J-\sigma_i^2}\left|\frac{3}{J} g_i - \frac{4}{J}\sigma_i^2 v_i\right| \\ &\quad + \frac{2J}{n\left(2J-\sigma_i^2\right)}\left|\frac{1}{J} f_i - \frac{1}{J}\sigma_i^2 u_i\right| \\ &\leq \frac{7\sigma_n^2}{2J-\sigma_n^2} + \frac{4\sigma_n^2}{n\left(2J-\sigma_n^2\right)} \\ &\leq \frac{7\sigma_n^2}{2J-\sigma_n^2} + \frac{4\sigma_n^2}{2\left(2J-\sigma_n^2\right)} \\ &= \frac{9\sigma_n^2}{2J-\sigma_n^2} \leq \frac{6\sigma_n^2}{J}. \end{aligned} \tag{20}$$

Analogously, from (4) (5) (6), we have

$$v_i = \frac{2J^2}{n^3\left(2J-\sigma_i^2\right)\left(J-\sigma_i^2\right)} + k_i, \tag{21}$$

where

$$\begin{aligned} k_i &= \frac{J}{J-\sigma_i^2}\left\{\frac{2}{J} h_i - \frac{3}{J}\sigma_i^2 w_i\right\} \\ &\quad + \frac{J^2}{n\left(J-\sigma_i^2\right)\left(2J-\sigma_i^2\right)}\left\{\frac{3}{J} g_i - \frac{4}{J}\sigma_i^2 v_i\right\} \\ &\quad + \frac{2J^2}{n^2\left(J-\sigma_i^2\right)\left(2J-\sigma_i^2\right)}\left\{\frac{1}{J} f_i - \frac{1}{J}\sigma_i^2 u_i\right\}. \end{aligned}$$

Also notice that

$$
\begin{aligned}
&|k_i| \\
&\le \frac{J}{J-\sigma_n^2}\left|\frac{2}{J}h_i-\frac{3}{J}\sigma_i^2 w_i\right| \\
&\quad+\frac{J^2}{n\left(J-\sigma_n^2\right)\left(2J-\sigma_n^2\right)}\left|\frac{3}{J}g_i-\frac{4}{J}\sigma_i^2 v_i\right| \\
&\quad+\frac{2J^2}{n^2\left(J-\sigma_n^2\right)\left(2J-\sigma_n^2\right)}\left|\frac{1}{J}f_i-\frac{1}{J}\sigma_i^2 u_i\right| \qquad (22)\\
&\le \frac{5\sigma_n^2}{J-\sigma_n^2} \\
&\quad+\frac{J\sigma_n^2}{n\left(J-\sigma_n^2\right)}\left\{\frac{7}{2J-\sigma_n^2}+\frac{4}{n\left(2J-\sigma_n^2\right)}\right\} \\
&\le \frac{10\sigma_n^2}{J}+\frac{6\sigma_n^2}{J}=\frac{16\sigma_n^2}{J}.
\end{aligned}
$$

By (19) (20) (21) (22) (23), we have

$$
\begin{aligned}
&\frac{J\sigma_j^2 u_j}{n\left(3J-\sigma_j^2\right)}-\frac{J\sigma_i^2 u_i}{n\left(3J-\sigma_i^2\right)}+\sigma_i^2 v_i-\sigma_j^2 v_j \\
&=\frac{2J^2\sigma_j^2}{n^3\left(2J-\sigma_j^2\right)\left(3J-\sigma_j^2\right)}+\frac{J\sigma_j^2 t_j}{n\left(3J-\sigma_j^2\right)} \\
&\quad-\frac{2J^2\sigma_i^2}{n^3\left(2J-\sigma_i^2\right)\left(3J-\sigma_i^2\right)}-\frac{J\sigma_i^2 t_i}{n\left(3J-\sigma_i^2\right)} \\
&\quad+\frac{2J^2\sigma_i^2}{n^3\left(2J-\sigma_i^2\right)\left(J-\sigma_i^2\right)}+\sigma_i^2 k_i \qquad (23)\\
&\quad-\frac{2J^2\sigma_j^2}{n^3\left(2J-\sigma_j^2\right)\left(J-\sigma_j^2\right)}-\sigma_j^2 k_j \\
&=\frac{-4J^3\sigma_j^2}{n^3\left(J-\sigma_j^2\right)\left(2J-\sigma_j^2\right)\left(3J-\sigma_j^2\right)} \\
&\quad+\frac{4J^3\sigma_i^2}{n^3\left(J-\sigma_i^2\right)\left(2J-\sigma_i^2\right)\left(3J-\sigma_i^2\right)} \\
&\quad+w,
\end{aligned}
$$

where

$$w=\frac{J\sigma_j^2 t_j}{n\left(3J-\sigma_j^2\right)}-\frac{J\sigma_i^2 t_i}{n\left(3J-\sigma_i^2\right)}+\sigma_i^2 k_i-\sigma_j^2 k_j.$$

Plainly, by (14) (20) (23), we have

$$|w|\le\frac{12\sigma_n^4}{5J}+\frac{32\sigma_n^4}{J}=\frac{172\sigma_n^4}{5J}.$$

Therefore, (11) (24) gives

$$
\begin{aligned}
&\frac{J\sigma_j^2 u_j}{n\left(3J-\sigma_j^2\right)}-\frac{J\sigma_i^2 u_i}{n\left(3J-\sigma_i^2\right)}+\sigma_i^2 v_i-\sigma_j^2 v_j \\
&\quad+I_1(i)-I_1(j) \\
&=\frac{-4J^3\sigma_j^2}{n^3\left(J-\sigma_j^2\right)\left(2J-\sigma_j^2\right)\left(3J-\sigma_j^2\right)} \\
&\quad+\frac{4J^3\sigma_i^2}{n^3\left(J-\sigma_i^2\right)\left(2J-\sigma_i^2\right)\left(3J-\sigma_i^2\right)} \\
&\quad+\frac{2J\sigma_j^2}{n^3\left(3J-\sigma_j^2\right)}-\frac{2J\sigma_i^2}{n^3\left(3J-\sigma_i^2\right)} \\
&\quad+s_{ij}+w \\
&=\frac{-2J\sigma_j^4}{n^3\left(J-\sigma_j^2\right)\left(2J-\sigma_j^2\right)} \\
&\quad+\frac{2J\sigma_i^4}{n^3\left(J-\sigma_i^2\right)\left(2J-\sigma_i^2\right)} \\
&\quad+s_{ij}+w,
\end{aligned}
$$

which implies

$$
\begin{aligned}
|F|&\le\frac{4J\sigma_n^4}{n^3\left(J-\sigma_n^2\right)\left(2J-\sigma_n^2\right)}+|s_{ij}|+|w| \\
&\le\frac{4J\sigma_n^4}{n^3\left(J-\sigma_n^2\right)\left(2J-\sigma_n^2\right)}+\frac{208\sigma_n^4}{5J} \qquad (24)\\
&\le\frac{2\sigma_n^4}{3J}+\frac{208\sigma_n^4}{5J}=\frac{634\sigma_n^4}{15J},
\end{aligned}
$$

where

$$
\begin{aligned}
F&=\frac{J\sigma_j^2 u_j}{n\left(3J-\sigma_j^2\right)}-\frac{J\sigma_i^2 u_i}{n\left(3J-\sigma_i^2\right)}+\sigma_i^2 v_i-\sigma_j^2 v_j \\
&\quad+I_1(i)-I_1(j).
\end{aligned}
$$

By (16) (17) (18) (24), we get

$$
\begin{aligned}
\left|f_i-f_j\right|&\le|F|+|I_2(i)-I_2(j)|+|I_3(i)-I_3(j)| \\
&\quad+|I_4(i)-I_4(j)| \qquad (25)\\
&=\frac{634\sigma_n^4}{15J}+\frac{60\sigma_n^4}{5J}=\frac{814\sigma_n^4}{15J}.
\end{aligned}
$$

On the other hand, by the definition of o_i in the first part of this Lemma , we have

$$\begin{aligned}\left|o_i - o_j\right| &\le \left|o_i\right| + \left|o_j\right| \\ &\le 2\left(\frac{\sigma_n^2}{2J-\sigma_n^2}\right)\left(9\sigma_n^2\right) \\ &\le \frac{12\sigma_n^4}{J},\end{aligned} \tag{26}$$

due to

$$\left|\frac{\sigma_i^2}{2J-\sigma_i^2}\right| \le \frac{\sigma_n^2}{2J-\sigma_n^2},$$
$$\left|4\sigma_i^2 v_i\right| \le 4\sigma_n^2,$$
$$\left|\frac{2\sigma_i^2}{n}u_i\right| \le \sigma_n^2,$$
$$\left|-3g_i\right| \le 3\sigma_n^2,$$
$$\left|\frac{2}{n}f_i\right| \le \sigma_n^2.$$

Combine (25) (26), we obtain

$$\left|f_i - f_j + o_i - o_j\right| \le \frac{814\sigma_n^4}{15J} + \frac{12\sigma_n^4}{J} = \frac{994\sigma_n^4}{15J}.$$

This completes the proof.

2.2 Showing main result

In order to present our main result in the theoretical framework, we conclude it as the following Theorem 2.2.1.

Theorem 2.2.1

Under the assumption of Lemma 2.1.1, if $0 < \sigma_1 < \sigma_2 < \cdots < \sigma_{n-1} < \sigma_n$ *and*

$$J > \frac{994n^2\sigma_n^4}{15\min\left\{\sigma_j^2 - \sigma_i^2 : 1 \le i < j \le n\right\}},$$ *then*

$$m_1 > m_2 > \cdots > m_{n-1} > m_n.$$

Proof:

In light of Lemma 2.1.1 and Lemma 2.1.2, for any fixed $i, j,\ 1 \le i < j \le n$, we obtain

$$\begin{aligned}J\left(m_i - m_j\right) &= \frac{4J^2\left(\sigma_j^2 - \sigma_i^2\right)}{n^2\left(2J-\sigma_j^2\right)\left(2J-\sigma_i^2\right)} \\ &\quad + \left\{f_i - f_j + o_i - o_j\right\} \\ &\ge \frac{4J^2\left(\sigma_j^2 - \sigma_i^2\right)}{n^2\left(2J-\sigma_j^2\right)\left(2J-\sigma_i^2\right)} \\ &\quad - \frac{994\sigma_n^4}{15J} \\ &\ge \frac{1}{n^2}\min\left\{\sigma_j^2 - \sigma_i^2 : 1 \le i < j \le n\right\} \\ &\quad - \frac{994\sigma_n^4}{15J}.\end{aligned} \tag{27}$$

On the other hand, since J satisfies the condition (3) and by (27), we get

$$J\left(m_i - m_j\right) > 0,$$

which establishes $m_i > m_j$ for any fixed i, j, $1 \le i < j \le n$. In consequence, we have

$$m_1 > m_2 > \cdots > m_{n-1} > m_n.$$

This completes the proof.

3 Numerical study

For the reason that the visualization is becoming increasingly popular as a manner of enhancing one's understanding of a stochastic differential equation, we perform our main result for $n = 2$ by numerical analysis in this section.

Now, assume $\sigma_1 = 0.2$, $\sigma_2 = 0.6$, $J = 110$, $X_1(0) = 40000$, $X_2(0) = 60000$. It is trivial that $J = 110$ satisfies the condition (3). Now, the mean-field Bouchaud-Mezard model becomes

$$dX_1(t) = 0.2X_1(t)dB_1(t) + 55\left\{X_2(t) - X_1(t)\right\}dt,$$
$$dX_2(t) = 0.6X_2(t)dB_2(t) + 55\left\{X_1(t) - X_2(t)\right\}dt.$$

Refer to [2][6][7], we take a sample size $N = 7000$ and a time mesh 10^{-6} to run the simulation. We use the method of the Euler approximation to generate an approximate sample path of $Y_1(t)$ as depicted in Fig. 1, where $Y_2(t)$ is obtained by the value $1 - Y_1(t)$.

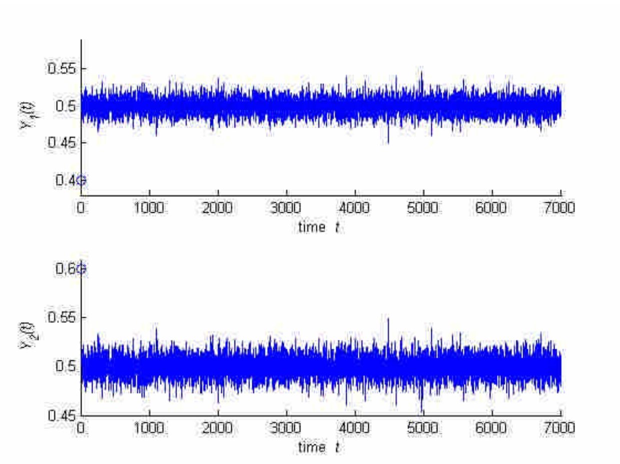

Fig. 1 The sample paths of $Y_1(t)$ and $Y_2(t)$ plotted versus t

The average $\frac{1}{n}\sum_{k=1}^{n} Y_1(k)$ is computed and compared with $\frac{1}{n}\sum_{k=1}^{n} Y_2(k)$ recursively until $n = N$. The result is plotted in Fig. 2.

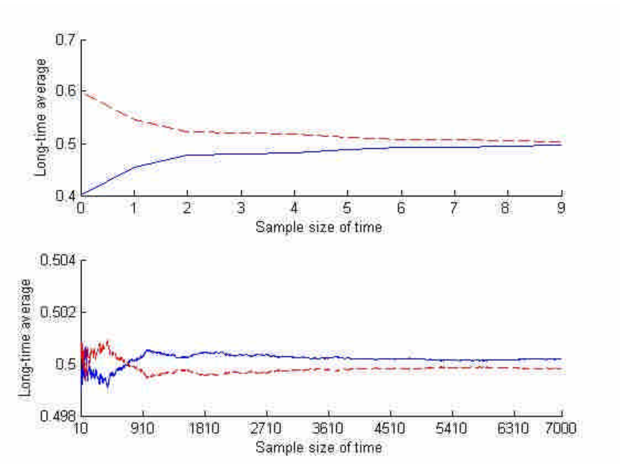

Fig 2. The comparison of the long-time averages based on the data in Fig. 1. The blue solid line represents the long-time average of $\{Y_1(t)\}_{t\ge 0}$, the red dash line represents the long-time average of $\{Y_2(t)\}_{t\ge 0}$

This simulation demonstrates conclusively that

$$\frac{1}{N}\sum_{k=1}^{N} Y_1(k) > \frac{1}{N}\sum_{k=1}^{N} Y_2(k),$$

which coincides with our main result.

4 Discussion

In this section, we address a conjecture that the condition (3) on J may be unnecessary. Indeed, for $n = 2$, it is valid that the stochastic differential equation of $Y_1(t)$ satisfies the following;

$$\begin{aligned} dY_1(t) &= \sqrt{\sigma_1^2 + \sigma_2^2}\, Y_1(t)\left(1 - Y_1(t)\right) dW(t) \\ &\quad + \frac{J}{2}dt - JY_1(t)dt \\ &\quad + Y_1(t)\left(1 - Y_1(t)\right)\left\{\sigma_2^2\left(1 - Y_1(t)\right) - \sigma_1^2 Y_1(t)\right\} dt, \end{aligned}$$

where $W(t) = \frac{\sigma_1 B_1(t) - \sigma_2 B_2(t)}{\sqrt{\sigma_1^2 + \sigma_2^2}}$ is a standard Brownian motion. It is well-known that $Y_1(t)$ is ergodic and the invariant probability measure $\pi(\cdot)$ has the following probability density function;

$$\pi(y) = \frac{Cp(y)}{\sigma_1^2 + \sigma_2^2}\, y^{\frac{2\sigma_2^2}{\sigma_1^2+\sigma_2^2} - 2} (1 - y)^{\frac{2\sigma_1^2}{\sigma_1^2+\sigma_2^2} - 2}, \tag{28}$$

where C is the normalization constant and

$$p(y) = \exp\left\{ -\frac{J}{\left(\sigma_1^2 + \sigma_2^2\right) y} - \frac{J}{\left(\sigma_1^2 + \sigma_2^2\right)(1 - y)} \right\}.$$

By (28), we have

$$\begin{aligned} m_1 &= \int_{\mathbb{S}} y_1 d\pi \\ &= \int_0^1 y\pi(y)dy \\ &= \frac{C\alpha_1}{\sigma_1^2 + \sigma_2^2}, \end{aligned} \tag{29}$$

where

$$\alpha_1 = \int_0^1 y^{\frac{2\sigma_2^2}{\sigma_1^2+\sigma_2^2} - 1} (1 - y)^{\frac{2\sigma_1^2}{\sigma_1^2+\sigma_2^2} - 2} p(y)dy.$$

On the other hand, we have

$$\begin{aligned} m_2 &= \int_{\mathbb{S}} y_2 d\pi \\ &= \int_0^1 (1 - y)\pi(y)dy \\ &= \frac{C\alpha_2}{\sigma_1^2 + \sigma_2^2}, \end{aligned} \tag{30}$$

where

$$\alpha_2 = \int_0^1 y^{\frac{2\sigma_2^2}{\sigma_1^2+\sigma_2^2} - 2} (1 - y)^{\frac{2\sigma_1^2}{\sigma_1^2+\sigma_2^2} - 1} p(y)dy.$$

Assume $\sigma_1 < \sigma_2$. Now, by (29) (30), to show $m_1 > m_2$, it suffices to claim $\alpha_1 > \alpha_2$.

Let $t = \frac{1}{2} - y$, then the integration by change of variables gives

$$\alpha_1 = \int_0^1 y^{\frac{2\sigma_2^2}{\sigma_1^2+\sigma_2^2}-1}(1-y)^{\frac{2\sigma_1^2}{\sigma_1^2+\sigma_2^2}-2}p(y)dy \\ = \int_{-0.5}^{0.5}(0.5-t)a(t)p(0.5-t)dt, \tag{31}$$

where

$$a(t) = (0.5-t)^{\frac{2\sigma_2^2}{\sigma_1^2+\sigma_2^2}-2}(0.5+t)^{\frac{2\sigma_1^2}{\sigma_1^2+\sigma_2^2}-2}.$$

Similarly,

$$\alpha_2 = \int_0^1 y^{\frac{2\sigma_2^2}{\left(\sigma_1^2+\sigma_2^2\right)}-2}(1-y)^{\frac{2\sigma_1^2}{\left(\sigma_1^2+\sigma_2^2\right)}-1}p(y)dy \\ = \int_{-0.5}^{0.5}(0.5+t)a(t)p(0.5-t)dt. \tag{32}$$

According to (31) (32), we obtain

$$\begin{aligned}\alpha_1 - \alpha_2 &= -2\int_{-0.5}^{0.5} ta(t)p(0.5-t)dt \\ &= -2\int_0^{0.5} ta(t)p(0.5-t)dt \\ &\quad + 2\int_0^{0.5} ta(-t)p(0.5+t)dt \\ &= 2\int_0^{0.5} t\left\{a(-t)-a(t)\right\}p(0.5+t)dt.\end{aligned}$$

Notice that we use $p(0.5+t) = p(0.5-t)$ for $0 \le t \le 0.5$ to establish the last equality of the formula above.

It remains to claim

$$\int_0^{0.5} t\left\{a(-t)-a(t)\right\}p(0.5+t)dt > 0.$$

To show this claim, it suffices to prove

$$a(-t) - a(t) > 0 \text{ for } 0 < t < 0.5.$$

Since

$$\begin{aligned}a(-t)-a(t) &= (0.5+t)^{\frac{2\sigma_2^2}{\sigma_1^2+\sigma_2^2}-2}(0.5-t)^{\frac{2\sigma_1^2}{\sigma_1^2+\sigma_2^2}-2} \\ &\quad -(0.5-t)^{\frac{2\sigma_2^2}{\sigma_1^2+\sigma_2^2}-2}(0.5+t)^{\frac{2\sigma_1^2}{\sigma_1^2+\sigma_2^2}-2} \\ &= \frac{(0.5+t)^{\frac{2\left(\sigma_2^2-\sigma_1^2\right)}{\sigma_1^2+\sigma_2^2}} - (0.5-t)^{\frac{2\left(\sigma_2^2-\sigma_1^2\right)}{\sigma_1^2+\sigma_2^2}}}{(0.5+t)^{\frac{2\sigma_2^2}{\sigma_1^2+\sigma_2^2}}(0.5-t)^{\frac{2\sigma_2^2}{\sigma_1^2+\sigma_2^2}}},\end{aligned}$$

and

$$0.5+t > 0.5-t, \quad \frac{2\left(\sigma_2^2-\sigma_1^2\right)}{\sigma_1^2+\sigma_2^2} > 0,$$

we obtain $a(-t)-a(t) > 0$ for $0 < t < 0.5$. This turns out that if $\sigma_1 < \sigma_2$, then $m_1 > m_2$. However, in this case, we do not need the condition (3) on J. This arises the conjecture that the condition (3) may be unnecessary for $n \ge 3$. But showing the correctness of this conjecture appears to be not so easy. It is left as an open problem for the further study.

5 Conclusion

This study shows that a strictly increasing sequence $\{\sigma_k\}_{k=1}^n$ of investment volatilities implies a strictly decreasing sequence $\{m_k\}_{k=1}^n$, given appropriate J, based on both theoretical and numerical analyses. The simulation yields conclusively the same result as obtained by theoretical analysis. It reveals a negative correlation between the investment volatilities and the ratio processes. As an empirical application, this negative correlation can be employed to characterize the mean-field Bouchaud-Mezard model.

On the other hand, our main result has the following interpretation in economics. Since the long-time average $\frac{1}{t}\int_0^t Y_i(s)ds$ must converge to m_i as t tends to infinity, for any given positive integer N and $\sigma_i < \sigma_j$, we have

$$\lim_{t\to\infty}\frac{1}{t}\int_N^t \left(Y_i(s) - Y_j(s)\right)ds = m_i - m_j > 0.$$

Hence, for t large enough, it gives

$$\int_N^t \left(Y_i(s) - Y_j(s)\right)ds > 0,$$

which together with the mean-value theorem for integrals prove that $Y_i(\xi) > Y_j(\xi)$ for some $\xi \in (N,t)$. Thus, we immediately obtain $X_i(\xi) > X_j(\xi)$ for some $\xi \in (N,t)$. This indicates that the wealth of an agent who prefers a high volatility becomes smaller than that of an agent who prefers a low volatility for a long time. This interesting result gives a sufficient condition for becoming a rich agent in mean-field Bouchaud-Mezard model. An agent whose spontaneous growth or decrease in wealth due to investment in stock markets is always small will eventually become rich in the mean-field Bouchaud-Mezard model.

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