Exact Test Critical Values for Correlation Testing with Application

CHING-HUI CHANG [1*], JYH-JIUAN LIN [2] AND NABENDU PAL [3]

[1] Department of Applied Statistics and Information Science
Ming Chuan University
5 De-Ming Road, Gui-Shan District, Taoyuan 333
TAIWAN (ROC)
chchang@mcu.edu.tw

[2] Department of Statistics
Tamkang University
151 Ying-Chuan Road, Tamsui, Taipei County 251
TAIWAN (ROC)
117604@mail.tku.edu.tw

[3] Department of Mathematics
University of Louisiana at Lafayette
Lafayette, Louisiana 70504 - 1010
USA
nxp3695@louisiana.edu

Abstract: Tables of critical values for the exact test method based on the maximum likelihood estimator (MLE) have been obtained to test a hypothesis on the correlation coefficient between the components of a bivariate normal random vector. The exact test is then compared, in terms of size and power, with the other popular methods, namely - the 'z - test', the 'modified z - test' and the 't - test'. While these popular methods have almost identical size and power to the exact test for large samples, their small sample performance is far from satisfactory as evident from our extensive numerical computations. Thus, our tables of critical values are useful when sample size is not large (i.e., ≤ 30). Also, it is demonstrated through a real-life dataset how the tables of critical values can be used for interval estimation.

Key-Words: Asymptotic variance, confidence interval, size, power.

1 Introduction

Let $\underset{\sim}{X} = (X_1, X_2)'$ follows a bivariate normal distribution with $E(X_i) = \mu_i$, $Var(X_i) = \sigma_i^2$, $i = 1, 2$, and $Cov(X_1, X_2) = \sigma_{12}$. The objective of this article is to propose a new test method for testing a hypothesis on the simple correlation coefficient $\rho = \sigma_{12} / (\sigma_1 \sigma_2)$.

Assume that we have *iid* observations $\underset{\sim}{X}_i = (X_{i1}, X_{i2})'$, $i = 1, 2, \cdots, n$, on $\underset{\sim}{X}$. Define

$$\underset{\sim}{\overline{X}}_i = \sum_{j=1}^{n} X_{ij} / n, \quad s_i^2 = \sum_{j=1}^{n} (X_{ij} - \overline{X}_i)^2 / n, \quad i = 1, 2;$$

$$\text{and} \quad s_{12} = \sum_{j=1}^{n} (X_{1j} - \overline{X}_1)(X_{2j} - \overline{X}_2) / n. \qquad (1)$$

The sample correlation between X_1 and X_2, denoted by $r = s_{12} / (s_1 s_2)$, which is also the maximum likelihood estimator (MLE) of ρ, is commonly used to estimate the strength of association between X_1 and X_2.

The probability distribution of r is quite complicated and has several representations. Fisher [3] obtained the *pdf* of r as

$$f(r \mid \rho, n) = \{(1 - r^2)^{(n-4)/2} (1 - \rho^2)^{(n-1)/2}\} \times \sum_{j=0}^{\infty} \{(2\rho r)^j / j!\} \{\Gamma((n + j - 1)/2)\}^2 \times \{\sqrt{\pi}\, \Gamma((n-1)/2) \Gamma((n-2)/2)\}^{-1}. \qquad (2)$$

The above expression is helpful in deriving moments of r in a series form. However, the following representation due to Hotelling [5] is useful for the probability computations.

AMS 2000 Subject Classifications: Primary 62F03, Secondary 62E17.

[*] Corresponding author.

$$\begin{aligned} f(r\mid\rho,n) = {} & \{(1-r^2)^{(n-4)/2}(1-\rho^2)^{(n-1)/2}\}\times \\ & \{\sqrt{2\pi}\,\Gamma(n-0.5)\}^{-1}(n-2)\times \\ & \Gamma(n-1)(1-r\rho)^{-(n-1.5)}\times \\ & {}_2F_1(0.5, 0.5; n-0.5; (1+r\rho)/2), \end{aligned} \qquad (3)$$

where

$$\begin{aligned} {}_2F_1(a,b;c;z) = {} & \sum_{k=0}^{\infty}\{\Gamma(a+k)\Gamma(b+k)\Gamma(c)z^k\}\times \\ & \{\Gamma(a)\Gamma(b)\Gamma(c+k)k!\}^{-1} \end{aligned}$$

is a Gaussian hypergeometric function. The representation in (3) is advantageous because the hypergeometric function converges more rapidly than the series expression in (2). For large *n*, the first term alone (in (3)) is a good approximation.

The *cdf* of r is defined as $F(r^*\mid\rho,n)= P(r\le r^*)$. Then it is easily seen that

$$F(r^*\mid\rho,n)=1-F(-r^*\mid-\rho,n). \qquad (4)$$

David [2] tabulates $F(r^*\mid\rho,n)$ for $\rho=0$ (0.1) 0.90, $n=3$ (1) 25, 50, 100, 200, 400; and $r^*=-1$ (0.05) 1.0. For point estimation of ρ, and for a comparison of various point estimators one can see Pal and Lim [10, 11].

In this article we focus on testing $H_0\colon\rho=\rho_0$ vs. a suitable alternative ($H_A\colon\rho>\rho_0$ or $\rho<\rho_0$ or $\rho\ne\rho_0$). The acceptance region for H_0 can be inverted to obtain an interval estimate for ρ.

Though $\hat\rho_{ML}=r$ can be used as a point estimate of ρ, one encounters some difficulties when it comes to interval estimation of ρ or testing a hypothesis on ρ. Being the MLE, $\hat\rho_{ML}=r$ is consistent, asymptotically normal, and has the asymptotic variance $v(\rho)=\lim_{n\to\infty}\{\sqrt{n}\,Var(r)\}$ $=(1-\rho^2)^2$. Then $\sqrt{n}(r-\rho)/\sqrt{v(\rho)}$ converges weakly to $N(0,1)$ as $n\to\infty$. But when $|\rho|$ is close to 1, the *pdf* of r ((2) or (3)) is heavily skewed, and hence the asymptotic formula is unsatisfactory unless n is very large (see Winterbottom [15]).

Fisher [4] introduced his famous *z*-transformation which is also known as the variance stabilizing transformation. This *z*-transformation suggests that r be transformed to $Z(r)$ given as

$$Z(r)=ln\sqrt{(1+r)/(1-r)}. \qquad (5)$$

It can be shown that $Z(r)$ is approximately normally distributed with mean $Z(\rho)= ln\sqrt{(1+\rho)/(1-\rho)}$, and variance $1/(n-3)$. Therefore, for testing above H_0, the test statistic

$$\Delta_F=\sqrt{(n-3)}\,\{Z(r)-Z(\rho_0)\} \qquad (6)$$

follows approximately $N(0,1)$ distribution under H_0.

The above *z*-transformation works reasonably well for $n>30$. For smaller sample sizes (5) should be used with caution, if at all. An alternative method due to Hotelling [5], which is a modification over the Fisher's *z*-transformation, suggests using

$$Z^*(r)=Z(r)-\{3Z(r)+r\}/(4n). \qquad (7)$$

Define $Z^*(\rho)$ as

$$Z^*(\rho)=Z(\rho)-\{3Z(\rho)+\rho\}/(4n). \qquad (8)$$

Then for $n\ge 10$, $Z^*(r)$ is approximately normally distributed with mean $Z^*(\rho)$ and variance $1/(n-1)$. Thus, Hotelling's modified *z*-test statistic for testing H_0 is

$$\Delta_H=\sqrt{(n-1)}\,\{Z^*(r)-Z^*(\rho_0)\}. \qquad (9)$$

Recall the *pdf* of r in (2) or (3). It can be shown that if $\rho=0$, then $\{\sqrt{(n-2)}\,r/\sqrt{(1-r^2)}\}$ follows *t*-distribution with *df* $(n-2)$ (an exact distribution). Therefore, one can use this *t*-test to test $H_0\colon\rho=0$ against a suitable alternative. This popular *t*-test was extended beyond the non-null case by Kraemer [7]. She showed that

$$\begin{aligned} \Delta_K = {} & \sqrt{(n-2)}\,(r-\rho_*)/\sqrt{(1-r^2)(1-\rho_*^2)} \\ & \sim t_{(n-2)} \text{ (approx.)}, \end{aligned} \qquad (10)$$

where $\rho_*=\rho_*(\rho,n)$ satisfies certain conditions. In particular it is possible to take $\rho_*=\rho$, and the approximation in (10) is quite satisfactory for $n>30$. Kraemer's [7] result was an extension of Samiuddin's [13] work. By inverting the acceptance region of the test based on Δ_K (with $\rho_*=\rho$) one can get the confidence interval for ρ suggested by Jeyaratnam [6].

For testing zero correlation only (i.e., taking $\rho_0=0$), recently Shieh [14] has shown the power calculation and sample size determination using the non-central *t*-distribution (i.e., using distribution of $\sqrt{(n-2)}r/\sqrt{(1-r^2)}$ under H_A). However, the above paper does not deal with comparison of the four tests ($\Delta_F, \Delta_H, \Delta_K$, and the exact test) which is the main focus of our work. Nevertheless, Shieh's paper is a very good exposition since it also covers multiple correlation coefficient in a multivariate normal setup.

While it is expected that the large sample behavior of the three test statistics ($\Delta_F, \Delta_H, \Delta_K$) would be similar, it is not clear about the same for small *n*. Note that all the three tests described above are essentially trying to approximate the distribution of r, through some suitable transformation, so that the cut-off point(s) can be found using the standard statistical tables.

The objectives of this note are the following:

(i) To provide suitable tables of critical values of the exact test based on the distribution of *r*. With the computational resources available today it is now easy to get the critical values. Even though one can use David's [2] tables, as mentioned in Anderson [1] (Section 4.2), it is a bit cumbersome and not very accurate.

(ii) Compare the exact test with those three mentioned earlier in terms of size and power. Our comprehensive numerical computations not only help compare Δ_F, Δ_H and Δ_K, it is also seen that for small n (≤ 30) the exact test is always preferable.

(iii) We will also see how easily the confidence interval, with a fixed confidence level, is obtained from the tables of critical values. This is demonstrated through a real-life dataset.

2 Tables of Critical Values for the Exact Test

Consider testing

H_0: $\rho = \rho_0$ vs.

H_A: (a) $\rho > \rho_0$ or (b) $\rho < \rho_0$ or (c) $\rho \neq \rho_0$. (11)

The exact test based on *r* has the rejection region

$$\Delta_0 = r \begin{cases} > c_1 & \text{if } H_A\text{: (a) } \rho > \rho_0 \\ < c_2 & \text{if } H_A\text{: (b) } \rho < \rho_0 \\ < c_2 \text{ or } > c_1 & \text{if } H_A\text{: (c) } \rho \neq \rho_0, \end{cases} \quad (12)$$

where the critical value(s) c_1 and/or c_2 is/are suitably chosen subject to size restriction. For the time being it is enough to focus on the one-sided alternative $H_A\!: \rho > \rho_0$. Then c_1 is chosen such that

$$\alpha = P(r > c_1 \mid \rho_0, n)$$
$$\text{i.e., } 1-\alpha = F(c_1 \mid \rho_0, n) = \int_{-1}^{c_1} f(r \mid \rho_0, n)\,dr. \quad (13)$$

The above $c_1 = c_1(\alpha, \rho_0, n)$ is tabulated in Tables 1-8 for various values of α (= 0.99, 0.975, 0.95, 0.90, 0.10, 0.05, 0.025, 0.01), ρ_0 (= – 0.90 (0.10) 0.90) and *n* (= 5 (1) 30). The large values of α (= 0.99, 0.975, 0.95, 0.90) have been used so that the same tables can be used for left hand side or two sided alternative. Also, with *n* fixed, the value of c_1 for a ρ_0 that is not listed can be found by interpolation which turns out to be satisfactory.

Remark 2.1. How do we use the Tables 1-8 for the left hand side alternative $H_A\!: \rho < \rho_0$? Note that the cut-off point c_2 in (12) is such that

$$\alpha = P(r < c_2 \mid \rho_0, n) = F(c_2 \mid \rho_0, n)$$
$$= 1 - F(-c_2 \mid -\rho_0, n);$$
$$\text{i.e., } 1-\alpha = F(-c_2 \mid -\rho_0, n).$$

Therefore,

$$c_2 = c_2(\alpha, \rho_0, n) = -c_1(1-\alpha, -\rho_0, n). \quad (14)$$

For a two sided alternative, choose c_1 and c_2 such that the tail probabilities are equal to $(\alpha/2)$.

3 Size and Power Comparison

In this section we compare the exact test (Δ_0 in (12)) with the other three test methods (based on Δ_F, Δ_H and Δ_K) in terms of size (which is fixed at α for the exact test) and power. For convenience we consider only the case of testing $H_0\!: \rho = \rho_0$ vs. $H_A\!: \rho > \rho_0$ at level α.

(i) The power of the exact test, denoted by β_0, is

$$\beta_0 = P(r > c_1 \mid \rho, n) = \int_{c_1}^{1} f(r \mid \rho, n)\,dr, \quad \rho \geq \rho_0. \quad (15)$$

At $\rho = \rho_0$ we attain the size α of the test.

(ii) The power of the Fisher's *z*-statistic (i.e., base on Δ_F) is given as

$$\beta_F = P(\Delta_F > z_\alpha) = \int_{a}^{1} f(r \mid \rho, n)\,dr, \quad \rho \geq \rho_0, \quad (16)$$

where $a = \{exp(b) - 1\}/\{exp(b) + 1\}$, and $b =$ $2(\,Z(\rho_0) + z_\alpha / \sqrt{n-3}\,)$.

(iii) The power of the modified *z*-test (i.e., Hotelling's Δ_H) is given as

$$\beta_H = P(\Delta_H > z_\alpha) = \int_{c}^{1} f(r \mid \rho, n)\,dr, \quad \rho \geq \rho_0; \quad (17)$$

where c is the solution of $J(r)=0$, $J(r)=ln((1+r)/(1-r))-\{(2r+8nb^*)/(4n-3)\}$, and $b^*=Z^*(\rho_0)+z_\alpha/\sqrt{n-1}$.

(iv) The power of the *t*-test (i.e., Kraemer's Δ_K) is given as

$$\beta_K=P(\Delta_K>t_{(n-2),\,\alpha})=\int_d^1 f(r\,|\,\rho,n)dr,\quad \rho\ge\rho_0; \tag{18}$$

where d is the solution of $h(r)=0$ and $h(r)>0$ for $r>\rho_0$, $h(r)=(u+1-\rho_0^2)r^2-2u\rho_0 r+u\rho_0^2+\rho_0^2-1$, and $u=(n-2)/(t_{(n-2),\,\alpha})^2$.

The Tables 9-13 provide the values of β_0, β_F, β_H and β_K for various values of ρ_0, $\rho\ge\rho_0$ and some selected values of n. These are some of the results of our extensive computations.

Remark 3.1. The following interesting trends have been observed from our computations.

(a) When $\rho_0=0$, Δ_K boils down to an exact test, and that's why we have $\beta_0=\beta_K$. For very small sample sizes $(5\le n\le 10)$, the tests Δ_F and Δ_H are off considerably in attaining the level $\alpha=0.05$. This has been observed for other values of α too (but not reported here).

(b) Fisher's *z*-test (i.e., Δ_F) is found to be conservative (i.e., β_F at ρ_0 is $<\alpha$) in a neighborhood of $\rho_0=0$, whereas Hotelling's modified *z*-test (i.e., Δ_H) is consistently liberal (i.e., β_H at ρ_0 is $>\alpha$) for all ρ_0.

(c) For ρ_0 near 0, the size of the three approximate tests inch toward α as *n* goes up. But as ρ_0 moves away from 0, even moderately large n $(20\le n\le 30)$ may not be very helpful. For example, when $\rho_0=0.60$, $\alpha=0.05$ and $n=15$, size of Δ_F, Δ_H and Δ_K are 0.057, 0.060 and 0.058 respectively.

(d) It has been also observed that for $n>30$, the three approximate tests perform reasonably well (unless $|\rho_0|$ is very close to 1 for which a larger *n* may be deemed necessary).

4 Real life Examples and Interval Estimation

4.1 An Example

The Trial Urban District Assessment (TUDA), a special project under the US National Assessment of Educational Progress (NAEP), began assessing performance in selected large urban districts in 2002. In 2005, eleven urban school districts participated in the TUDA at grades 4 and 8. The Table 14 provides the list of school districts and the district average score in reading and mathematics for the year 2005. (Source: National Center for Education Statistics, http://nces.ed.gov)

Let us denote the true correlation coefficient between average math and average reading score (for a district) by ρ_4 and ρ_8 for the 4th and the 8th grade respectively. The sample correlation coefficients, based on *n* = 11, are found to be

$$\hat\rho_4=r_4=0.9755 \text{ and } \hat\rho_8=r_8=0.9738.$$

If we have to test $H_0\!:\rho=\rho_0$ vs. $H_A\!:\rho>\rho_0$, then at 5% level (using the Table 6 with *n* = 11) the alternative is accepted with ρ_0 as high as 0.90, where ρ could be either ρ_4 or ρ_8. It is interesting to see such a high correlation between math and reading scores. But we have to keep in mind that the data in Table 14 is for districtwise averages, and may not say much about an individual student's performance. It is well known from past studies that girls tend to perform better in reading, whereas boys tend to excel in math. The districtwise average score takes all representative boys and girls into consideration. What the above Table 14 says is that overall performance in math and reading are highly correlated, and a district performing well in one area is highly expected to do so in other area too. For more applications, Lee and Chuang [8] used the correlation coefficients to raise the accuracy of gene clusters in the field of bioinformatics. Mankar and Ghatol [9] also adopted correlation coefficient as an performance parameter in biomedical applications recently. Rashwan, Ismail and Fouad [12] studied the change of the environment by introducing the correlation parameter in the fusion process.

4.2 Interval Estimation

Suppose we want to find a 95% two-sided confidence interval for ρ (where ρ could be either ρ_4 or ρ_8). The acceptance region for testing $H_0\!:\rho=\rho_0$ vs. $H_A\!:\rho\ne\rho_0$ is the interval $(c_2,\ c_1)$, where $(\alpha/2)=0.025=P(r>c_1\,|\,\rho_0,n)$ (see (13)) and $c_2=-c_1(1-(\alpha/2),-\rho_0,n)=-c_1(0.975,-\rho_0,n)$. We now demonstrate inverting the acceptance region to get the desired confidence interval for ρ. Using Table 7 we plot $c_1=c_1(0.025,\rho_0,n=11)$ against ρ_0. On the same coordinates we then plot $c_2=c_2(0.025,\rho_0,n=11)=-c_1(0.975,-\rho_0,n=11)$

Table 1. Cut-off point c_1 for $\alpha = 0.99$ for various n and ρ_0.

n \ ρ_0	-0.90	-0.80	-0.70	-0.60	-0.50	-0.40	-0.30	-0.20	-0.10	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
5	-0.9972	-0.9940	-0.9902	-0.9858	-0.9807	-0.9746	-0.9673	-0.9585	-0.9477	-0.9343	-0.9174	-0.8956	-0.8668	-0.8275	-0.7718	-0.6890	-0.5577	-0.3315	0.1011
6	-0.9945	-0.9883	-0.9811	-0.9729	-0.9633	-0.9522	-0.9391	-0.9236	-0.9049	-0.8822	-0.8542	-0.8191	-0.7741	-0.7152	-0.6357	-0.5245	-0.3621	-0.1119	0.2972
7	-0.9917	-0.9824	-0.9718	-0.9597	-0.9459	-0.9299	-0.9113	-0.8895	-0.8637	-0.8329	-0.7955	-0.7496	-0.6922	-0.6191	-0.5238	-0.3962	-0.2197	0.0337	0.4127
8	-0.9890	-0.9767	-0.9629	-0.9472	-0.9294	-0.9090	-0.8856	-0.8583	-0.8264	-0.7887	-0.7437	-0.6893	-0.6225	-0.5393	-0.4336	-0.2965	-0.1141	0.1357	0.4883
9	-0.9865	-0.9715	-0.9546	-0.9357	-0.9143	-0.8900	-0.8622	-0.8303	-0.7932	-0.7498	-0.6985	-0.6374	-0.5635	-0.4730	-0.3604	-0.2177	-0.0333	0.2107	0.5414
10	-0.9842	-0.9666	-0.9470	-0.9252	-0.9006	-0.8728	-0.8412	-0.8051	-0.7636	-0.7155	-0.6591	-0.5926	-0.5132	-0.4174	-0.3000	-0.1541	0.0305	0.2683	0.5809
11	-0.9820	-0.9621	-0.9401	-0.9155	-0.8880	-0.8572	-0.8223	-0.7826	-0.7373	-0.6851	-0.6245	-0.5537	-0.4700	-0.3701	-0.2493	-0.1016	0.0820	0.3138	0.6114
12	-0.9800	-0.9580	-0.9337	-0.9067	-0.8766	-0.8429	-0.8051	-0.7623	-0.7137	-0.6581	-0.5940	-0.5195	-0.4324	-0.3294	-0.2063	-0.0575	0.1247	0.3508	0.6357
13	-0.9782	-0.9542	-0.9278	-0.8986	-0.8661	-0.8300	-0.7895	-0.7440	-0.6925	-0.6339	-0.5667	-0.4893	-0.3994	-0.2939	-0.1691	-0.0199	0.1605	0.3815	0.6556
14	-0.9765	-0.9507	-0.9223	-0.8911	-0.8565	-0.8181	-0.7753	-0.7273	-0.6732	-0.6120	-0.5423	-0.4624	-0.3702	-0.2628	-0.1368	0.0125	0.1912	0.4074	0.6721
15	-0.9749	-0.9474	-0.9173	-0.8841	-0.8476	-0.8072	-0.7622	-0.7120	-0.6557	-0.5923	-0.5203	-0.4383	-0.3441	-0.2352	-0.1083	0.0408	0.2177	0.4296	0.6861
16	-0.9734	-0.9443	-0.9126	-0.8777	-0.8394	-0.7971	-0.7502	-0.6980	-0.6397	-0.5742	-0.5003	-0.4165	-0.3207	-0.2105	-0.0830	0.0658	0.2409	0.4488	0.6982
17	-0.9720	-0.9414	-0.9082	-0.8717	-0.8318	-0.7877	-0.7391	-0.6851	-0.6250	-0.5577	-0.4821	-0.3966	-0.2995	-0.1883	-0.0604	0.0880	0.2614	0.4657	0.7086
18	-0.9706	-0.9388	-0.9040	-0.8661	-0.8246	-0.7790	-0.7288	-0.6732	-0.6114	-0.5425	-0.4654	-0.3785	-0.2802	-0.1682	-0.0400	0.1079	0.2796	0.4806	0.7178
19	-0.9694	-0.9362	-0.9002	-0.8609	-0.8180	-0.7709	-0.7192	-0.6621	-0.5988	-0.5285	-0.4500	-0.3619	-0.2625	-0.1499	-0.0215	0.1258	0.2960	0.4940	0.7260
20	-0.9682	-0.9339	-0.8965	-0.8560	-0.8117	-0.7633	-0.7102	-0.6517	-0.5871	-0.5155	-0.4358	-0.3466	-0.2463	-0.1332	-0.0047	0.1421	0.3108	0.5059	0.7332
21	-0.9671	-0.9316	-0.8931	-0.8513	-0.8058	-0.7562	-0.7018	-0.6421	-0.5762	-0.5034	-0.4225	-0.3324	-0.2314	-0.1178	0.0108	0.1570	0.3242	0.5167	0.7398
22	-0.9661	-0.9295	-0.8899	-0.8470	-0.8003	-0.7495	-0.6939	-0.6330	-0.5660	-0.4921	-0.4102	-0.3192	-0.2176	-0.1035	0.0250	0.1706	0.3364	0.5265	0.7457
23	-0.9651	-0.9275	-0.8868	-0.8428	-0.7951	-0.7431	-0.6865	-0.6245	-0.5564	-0.4815	-0.3987	-0.3069	-0.2047	-0.0904	0.0381	0.1831	0.3477	0.5355	0.7511
24	-0.9641	-0.9255	-0.8839	-0.8389	-0.7901	-0.7371	-0.6794	-0.6164	-0.5474	-0.4716	-0.3880	-0.2955	-0.1927	-0.0781	0.0502	0.1947	0.3580	0.5437	0.7560
25	-0.9632	-0.9237	-0.8811	-0.8352	-0.7854	-0.7315	-0.6728	-0.6088	-0.5389	-0.4622	-0.3779	-0.2847	-0.1815	-0.0667	0.0615	0.2054	0.3676	0.5513	0.7605
26	-0.9624	-0.9220	-0.8785	-0.8316	-0.7809	-0.7261	-0.6665	-0.6017	-0.5309	-0.4534	-0.3683	-0.2746	-0.1710	-0.0560	0.0721	0.2154	0.3765	0.5583	0.7647
27	-0.9615	-0.9203	-0.8760	-0.8282	-0.7767	-0.7210	-0.6605	-0.5949	-0.5233	-0.4451	-0.3594	-0.2651	-0.1611	-0.0460	0.0820	0.2248	0.3848	0.5649	0.7685
28	-0.9608	-0.9187	-0.8736	-0.8250	-0.7726	-0.7161	-0.6548	-0.5884	-0.5161	-0.4372	-0.3509	-0.2561	-0.1518	-0.0366	0.0912	0.2335	0.3925	0.5709	0.7721
29	-0.9600	-0.9172	-0.8713	-0.8219	-0.7688	-0.7114	-0.6494	-0.5822	-0.5092	-0.4297	-0.3428	-0.2476	-0.1430	-0.0277	0.0999	0.2417	0.3997	0.5766	0.7754
30	-0.9593	-0.9157	-0.8690	-0.8189	-0.7651	-0.7070	-0.6443	-0.5764	-0.5027	-0.4226	-0.3352	-0.2396	-0.1347	-0.0193	0.1082	0.2494	0.4065	0.5819	0.7785

Table 2. Cut-off point c_1 for $\alpha = 0.975$ for various n and ρ_0 .

n \ ρ_0	-0.90	-0.80	-0.70	-0.60	-0.50	-0.40	-0.30	-0.20	-0.10	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
5	-0.9947	-0.9885	-0.9814	-0.9732	-0.9635	-0.9521	-0.9386	-0.9223	-0.9026	-0.8783	-0.8482	-0.8099	-0.7606	-0.6955	-0.6072	-0.4839	-0.3056	-0.0384	0.3733
6	-0.9909	-0.9806	-0.9688	-0.9553	-0.9398	-0.9218	-0.9008	-0.8760	-0.8466	-0.8114	-0.7687	-0.7162	-0.6508	-0.5680	-0.4612	-0.3205	-0.1312	0.1291	0.4927
7	-0.9873	-0.9731	-0.9571	-0.9389	-0.9182	-0.8944	-0.8671	-0.8353	-0.7982	-0.7545	-0.7025	-0.6399	-0.5639	-0.4703	-0.3534	-0.2053	-0.0147	0.2343	0.5636
8	-0.9841	-0.9664	-0.9466	-0.9242	-0.8990	-0.8704	-0.8378	-0.8004	-0.7571	-0.7067	-0.6477	-0.5779	-0.4946	-0.3941	-0.2716	-0.1204	0.0683	0.3067	0.6108
9	-0.9812	-0.9604	-0.9372	-0.9112	-0.8822	-0.8494	-0.8124	-0.7702	-0.7219	-0.6664	-0.6020	-0.5268	-0.4383	-0.3332	-0.2074	-0.0552	0.1306	0.3598	0.6446
10	-0.9786	-0.9549	-0.9288	-0.8997	-0.8673	-0.8310	-0.7901	-0.7440	-0.6917	-0.6319	-0.5633	-0.4839	-0.3916	-0.2835	-0.1556	-0.0034	0.1793	0.4005	0.6700
11	-0.9762	-0.9501	-0.9212	-0.8894	-0.8540	-0.8146	-0.7705	-0.7211	-0.6653	-0.6021	-0.5300	-0.4475	-0.3523	-0.2419	-0.1129	0.0388	0.2185	0.4328	0.6900
12	-0.9741	-0.9456	-0.9144	-0.8801	-0.8421	-0.8000	-0.7531	-0.7008	-0.6421	-0.5760	-0.5011	-0.4160	-0.3186	-0.2066	-0.0770	0.0739	0.2507	0.4591	0.7061
13	-0.9721	-0.9416	-0.9083	-0.8716	-0.8313	-0.7868	-0.7375	-0.6827	-0.6215	-0.5529	-0.4758	-0.3885	-0.2894	-0.1762	-0.0462	0.1038	0.2779	0.4811	0.7194
14	-0.9703	-0.9379	-0.9026	-0.8640	-0.8216	-0.7749	-0.7235	-0.6664	-0.6031	-0.5324	-0.4533	-0.3643	-0.2637	-0.1497	-0.0196	0.1294	0.3010	0.4997	0.7306
15	-0.9686	-0.9345	-0.8974	-0.8570	-0.8127	-0.7641	-0.7107	-0.6517	-0.5865	-0.5140	-0.4331	-0.3427	-0.2410	-0.1263	0.0037	0.1518	0.3211	0.5157	0.7401
16	-0.9671	-0.9314	-0.8927	-0.8505	-0.8045	-0.7542	-0.6990	-0.6384	-0.5714	-0.4973	-0.4150	-0.3233	-0.2207	-0.1055	0.0244	0.1715	0.3387	0.5296	0.7484
17	-0.9656	-0.9285	-0.8882	-0.8445	-0.7970	-0.7451	-0.6883	-0.6261	-0.5577	-0.4821	-0.3986	-0.3058	-0.2025	-0.0869	0.0428	0.1889	0.3542	0.5418	0.7556
18	-0.9643	-0.9258	-0.8841	-0.8390	-0.7900	-0.7367	-0.6785	-0.6148	-0.5450	-0.4683	-0.3836	-0.2899	-0.1859	-0.0701	0.0594	0.2046	0.3681	0.5527	0.7620
19	-0.9631	-0.9233	-0.8803	-0.8339	-0.7835	-0.7289	-0.6694	-0.6044	-0.5334	-0.4555	-0.3699	-0.2754	-0.1708	-0.0548	0.0744	0.2187	0.3805	0.5624	0.7677
20	-0.9619	-0.9209	-0.8767	-0.8290	-0.7775	-0.7216	-0.6609	-0.5948	-0.5227	-0.4438	-0.3572	-0.2620	-0.1570	-0.0409	0.0880	0.2315	0.3917	0.5712	0.7728
21	-0.9608	-0.9187	-0.8733	-0.8245	-0.7718	-0.7148	-0.6530	-0.5858	-0.5127	-0.4329	-0.3455	-0.2497	-0.1443	-0.0281	0.1005	0.2432	0.4020	0.5792	0.7775
22	-0.9597	-0.9166	-0.8702	-0.8203	-0.7665	-0.7084	-0.6456	-0.5774	-0.5034	-0.4227	-0.3347	-0.2383	-0.1326	-0.0163	0.1120	0.2539	0.4113	0.5864	0.7817
23	-0.9587	-0.9146	-0.8672	-0.8163	-0.7615	-0.7024	-0.6386	-0.5696	-0.4947	-0.4132	-0.3246	-0.2277	-0.1217	-0.0054	0.1226	0.2638	0.4199	0.5930	0.7855
24	-0.9578	-0.9127	-0.8644	-0.8125	-0.7568	-0.6968	-0.6321	-0.5622	-0.4865	-0.4044	-0.3151	-0.2178	-0.1116	0.0048	0.1324	0.2729	0.4278	0.5991	0.7890
25	-0.9569	-0.9109	-0.8617	-0.8089	-0.7523	-0.6915	-0.6259	-0.5553	-0.4788	-0.3961	-0.3063	-0.2086	-0.1021	0.0142	0.1416	0.2814	0.4352	0.6048	0.7923
26	-0.9561	-0.9092	-0.8591	-0.8056	-0.7481	-0.6864	-0.6201	-0.5487	-0.4716	-0.3882	-0.2979	-0.1999	-0.0932	0.0230	0.1501	0.2893	0.4420	0.6100	0.7953
27	-0.9553	-0.9076	-0.8567	-0.8023	-0.7441	-0.6817	-0.6146	-0.5425	-0.4648	-0.3809	-0.2901	-0.1917	-0.0849	0.0313	0.1581	0.2966	0.4483	0.6149	0.7981
28	-0.9545	-0.9061	-0.8544	-0.7993	-0.7403	-0.6771	-0.6094	-0.5366	-0.4583	-0.3739	-0.2827	-0.1840	-0.0771	0.0391	0.1656	0.3035	0.4543	0.6194	0.8007
29	-0.9538	-0.9046	-0.8522	-0.7964	-0.7367	-0.6728	-0.6044	-0.5311	-0.4522	-0.3673	-0.2757	-0.1768	-0.0697	0.0464	0.1726	0.3100	0.4598	0.6237	0.8031
30	-0.9531	-0.9032	-0.8501	-0.7936	-0.7332	-0.6687	-0.5997	-0.5258	-0.4464	-0.3610	-0.2691	-0.1699	-0.0627	0.0533	0.1792	0.3161	0.4651	0.6276	0.8054

Table 3. Cut-off point c_1 for $\alpha = 0.95$ for various n and ρ_0 .

n \ ρ_0	-0.90	-0.80	-0.70	-0.60	-0.50	-0.40	-0.30	-0.20	-0.10	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
5	-0.9912	-0.9811	-0.9694	-0.9559	-0.9401	-0.9217	-0.8999	-0.8740	-0.8429	-0.8054	-0.7594	-0.7026	-0.6314	-0.5410	-0.4244	-0.2721	-0.0707	0.1971	0.5505
6	-0.9864	-0.9710	-0.9536	-0.9337	-0.9110	-0.8848	-0.8545	-0.8193	-0.7780	-0.7293	-0.6714	-0.6019	-0.5179	-0.4153	-0.2889	-0.1318	0.0649	0.3118	0.6206
7	-0.9822	-0.9623	-0.9399	-0.9148	-0.8863	-0.8540	-0.8171	-0.7748	-0.7260	-0.6694	-0.6035	-0.5260	-0.4345	-0.3256	-0.1951	-0.0379	0.1526	0.3838	0.6639
8	-0.9785	-0.9547	-0.9282	-0.8986	-0.8654	-0.8281	-0.7860	-0.7382	-0.6838	-0.6215	-0.5499	-0.4670	-0.3708	-0.2583	-0.1262	0.0297	0.2144	0.4335	0.6935
9	-0.9753	-0.9482	-0.9181	-0.8847	-0.8476	-0.8062	-0.7598	-0.7077	-0.6488	-0.5822	-0.5064	-0.4198	-0.3204	-0.2058	-0.0731	0.0810	0.2606	0.4703	0.7151
10	-0.9725	-0.9424	-0.9092	-0.8727	-0.8322	-0.7874	-0.7375	-0.6818	-0.6194	-0.5494	-0.4703	-0.3809	-0.2793	-0.1634	-0.0308	0.1214	0.2967	0.4987	0.7318
11	-0.9700	-0.9373	-0.9014	-0.8620	-0.8187	-0.7710	-0.7181	-0.6595	-0.5942	-0.5214	-0.4399	-0.3483	-0.2451	-0.1284	0.0038	0.1543	0.3258	0.5215	0.7450
12	-0.9678	-0.9327	-0.8945	-0.8526	-0.8068	-0.7565	-0.7011	-0.6400	-0.5723	-0.4973	-0.4137	-0.3205	-0.2161	-0.0990	0.0329	0.1816	0.3498	0.5402	0.7559
13	-0.9658	-0.9286	-0.8882	-0.8442	-0.7962	-0.7437	-0.6861	-0.6228	-0.5531	-0.4762	-0.3909	-0.2964	-0.1911	-0.0737	0.0576	0.2048	0.3700	0.5558	0.7649
14	-0.9639	-0.9249	-0.8826	-0.8366	-0.7867	-0.7322	-0.6727	-0.6075	-0.5361	-0.4575	-0.3709	-0.2752	-0.1693	-0.0517	0.0790	0.2247	0.3874	0.5692	0.7726
15	-0.9622	-0.9215	-0.8774	-0.8297	-0.7780	-0.7218	-0.6606	-0.5938	-0.5208	-0.4409	-0.3531	-0.2565	-0.1501	-0.0325	0.0977	0.2421	0.4024	0.5807	0.7792
16	-0.9607	-0.9184	-0.8727	-0.8235	-0.7701	-0.7124	-0.6496	-0.5814	-0.5071	-0.4259	-0.3371	-0.2398	-0.1329	-0.0153	0.1143	0.2574	0.4156	0.5908	0.7849
17	-0.9593	-0.9155	-0.8684	-0.8177	-0.7629	-0.7037	-0.6396	-0.5701	-0.4946	-0.4124	-0.3227	-0.2248	-0.1176	0.0000	0.1290	0.2710	0.4273	0.5998	0.7900
18	-0.9579	-0.9128	-0.8644	-0.8124	-0.7563	-0.6958	-0.6305	-0.5598	-0.4832	-0.4000	-0.3096	-0.2111	-0.1037	0.0138	0.1423	0.2832	0.4378	0.6077	0.7945
19	-0.9567	-0.9104	-0.8607	-0.8074	-0.7501	-0.6885	-0.6220	-0.5503	-0.4727	-0.3887	-0.2976	-0.1987	-0.0910	0.0263	0.1543	0.2942	0.4472	0.6148	0.7986
20	-0.9556	-0.9081	-0.8573	-0.8028	-0.7444	-0.6817	-0.6142	-0.5415	-0.4631	-0.3783	-0.2867	-0.1873	-0.0795	0.0377	0.1652	0.3042	0.4558	0.6213	0.8022
21	-0.9545	-0.9059	-0.8540	-0.7985	-0.7391	-0.6754	-0.6069	-0.5334	-0.4541	-0.3687	-0.2765	-0.1768	-0.0689	0.0482	0.1752	0.3133	0.4635	0.6272	0.8055
22	-0.9535	-0.9039	-0.8510	-0.7945	-0.7341	-0.6695	-0.6002	-0.5258	-0.4459	-0.3598	-0.2671	-0.1671	-0.0591	0.0578	0.1844	0.3217	0.4707	0.6326	0.8086
23	-0.9525	-0.9020	-0.8482	-0.7908	-0.7295	-0.6640	-0.5938	-0.5187	-0.4381	-0.3515	-0.2584	-0.1581	-0.0500	0.0667	0.1929	0.3294	0.4772	0.6375	0.8113
24	-0.9516	-0.9002	-0.8455	-0.7872	-0.7251	-0.6588	-0.5879	-0.5121	-0.4309	-0.3438	-0.2503	-0.1497	-0.0416	0.0750	0.2007	0.3365	0.4833	0.6420	0.8139
25	-0.9508	-0.8985	-0.8430	-0.7839	-0.7210	-0.6539	-0.5823	-0.5059	-0.4241	-0.3365	-0.2427	-0.1419	-0.0337	0.0827	0.2081	0.3432	0.4889	0.6462	0.8162
26	-0.9500	-0.8969	-0.8406	-0.7807	-0.7170	-0.6493	-0.5771	-0.5000	-0.4177	-0.3297	-0.2355	-0.1346	-0.0263	0.0899	0.2149	0.3493	0.4941	0.6502	0.8184
27	-0.9492	-0.8954	-0.8383	-0.7777	-0.7134	-0.6449	-0.5721	-0.4945	-0.4117	-0.3233	-0.2288	-0.1277	-0.0194	0.0967	0.2213	0.3551	0.4990	0.6538	0.8205
28	-0.9485	-0.8939	-0.8361	-0.7749	-0.7098	-0.6408	-0.5674	-0.4892	-0.4060	-0.3172	-0.2225	-0.1212	-0.0129	0.1030	0.2273	0.3605	0.5036	0.6572	0.8224
29	-0.9478	-0.8925	-0.8341	-0.7721	-0.7065	-0.6369	-0.5629	-0.4843	-0.4006	-0.3115	-0.2165	-0.1151	-0.0068	0.1090	0.2329	0.3656	0.5078	0.6604	0.8242
30	-0.9471	-0.8912	-0.8321	-0.7696	-0.7033	-0.6331	-0.5587	-0.4796	-0.3955	-0.3061	-0.2108	-0.1093	-0.0010	0.1146	0.2382	0.3704	0.5119	0.6634	0.8258

Table 4. Cut-off point c_1 for $\alpha = 0.9$ for various n and ρ_0 .

n \ ρ_0	-0.90	-0.80	-0.70	-0.60	-0.50	-0.40	-0.30	-0.20	-0.10	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
5	-0.9850	-0.9678	-0.9482	-0.9255	-0.8994	-0.8691	-0.8339	-0.7927	-0.7443	-0.6870	-0.6191	-0.5381	-0.4411	-0.3245	-0.1842	-0.0156	0.1857	0.4231	0.6965
6	-0.9791	-0.9555	-0.9290	-0.8991	-0.8652	-0.8267	-0.7827	-0.7325	-0.6748	-0.6084	-0.5317	-0.4428	-0.3395	-0.2194	-0.0796	0.0828	0.2706	0.4861	0.7299
7	-0.9742	-0.9456	-0.9137	-0.8781	-0.8383	-0.7936	-0.7435	-0.6870	-0.6231	-0.5509	-0.4688	-0.3755	-0.2692	-0.1481	-0.0100	0.1473	0.3257	0.5269	0.7519
8	-0.9702	-0.9373	-0.9010	-0.8609	-0.8165	-0.7672	-0.7124	-0.6513	-0.5831	-0.5067	-0.4212	-0.3251	-0.2173	-0.0960	0.0404	0.1934	0.3648	0.5559	0.7676
9	-0.9667	-0.9304	-0.8905	-0.8467	-0.7986	-0.7456	-0.6871	-0.6224	-0.5509	-0.4716	-0.3836	-0.2857	-0.1769	-0.0559	0.0788	0.2284	0.3944	0.5778	0.7796
10	-0.9638	-0.9244	-0.8815	-0.8346	-0.7834	-0.7274	-0.6659	-0.5985	-0.5244	-0.4428	-0.3529	-0.2539	-0.1445	-0.0239	0.1093	0.2561	0.4176	0.5950	0.7890
11	-0.9612	-0.9192	-0.8737	-0.8242	-0.7704	-0.7118	-0.6479	-0.5782	-0.5020	-0.4187	-0.3274	-0.2274	-0.1178	0.0024	0.1342	0.2786	0.4365	0.6090	0.7966
12	-0.9590	-0.9147	-0.8669	-0.8151	-0.7591	-0.6983	-0.6324	-0.5608	-0.4829	-0.3981	-0.3057	-0.2050	-0.0952	0.0245	0.1551	0.2974	0.4523	0.6206	0.8030
13	-0.9570	-0.9107	-0.8608	-0.8071	-0.7491	-0.6865	-0.6188	-0.5455	-0.4662	-0.3802	-0.2870	-0.1857	-0.0759	0.0434	0.1729	0.3134	0.4656	0.6304	0.8084
14	-0.9552	-0.9070	-0.8554	-0.7999	-0.7402	-0.6760	-0.6068	-0.5321	-0.4516	-0.3646	-0.2706	-0.1690	-0.0591	0.0598	0.1883	0.3271	0.4771	0.6389	0.8130
15	-0.9535	-0.9038	-0.8505	-0.7935	-0.7323	-0.6666	-0.5960	-0.5201	-0.4385	-0.3507	-0.2561	-0.1541	-0.0443	0.0741	0.2017	0.3392	0.4872	0.6463	0.8170
16	-0.9520	-0.9008	-0.8461	-0.7876	-0.7251	-0.6581	-0.5863	-0.5094	-0.4269	-0.3383	-0.2432	-0.1410	-0.0311	0.0869	0.2136	0.3498	0.4960	0.6527	0.8206
17	-0.9507	-0.8981	-0.8421	-0.7823	-0.7185	-0.6504	-0.5776	-0.4997	-0.4163	-0.3271	-0.2315	-0.1291	-0.0194	0.0982	0.2243	0.3593	0.5039	0.6585	0.8238
18	-0.9494	-0.8956	-0.8384	-0.7774	-0.7125	-0.6433	-0.5696	-0.4908	-0.4068	-0.3170	-0.2210	-0.1184	-0.0088	0.1085	0.2339	0.3678	0.5109	0.6637	0.8266
19	-0.9482	-0.8933	-0.8349	-0.7729	-0.7070	-0.6369	-0.5622	-0.4827	-0.3980	-0.3077	-0.2114	-0.1087	0.0009	0.1178	0.2425	0.3755	0.5173	0.6684	0.8291
20	-0.9471	-0.8911	-0.8318	-0.7688	-0.7019	-0.6309	-0.5555	-0.4753	-0.3900	-0.2992	-0.2026	-0.0998	0.0097	0.1263	0.2504	0.3826	0.5231	0.6726	0.8315
21	-0.9461	-0.8892	-0.8288	-0.7649	-0.6972	-0.6254	-0.5492	-0.4684	-0.3825	-0.2914	-0.1945	-0.0916	0.0178	0.1341	0.2577	0.3890	0.5285	0.6765	0.8336
22	-0.9452	-0.8873	-0.8261	-0.7613	-0.6928	-0.6203	-0.5434	-0.4620	-0.3757	-0.2841	-0.1870	-0.0840	0.0253	0.1412	0.2643	0.3949	0.5333	0.6801	0.8355
23	-0.9443	-0.8855	-0.8235	-0.7580	-0.6887	-0.6155	-0.5380	-0.4560	-0.3693	-0.2774	-0.1801	-0.0770	0.0322	0.1479	0.2705	0.4003	0.5378	0.6834	0.8373
24	-0.9435	-0.8839	-0.8211	-0.7548	-0.6848	-0.6110	-0.5329	-0.4505	-0.3633	-0.2711	-0.1736	-0.0705	0.0386	0.1540	0.2762	0.4054	0.5420	0.6864	0.8390
25	-0.9427	-0.8824	-0.8188	-0.7518	-0.6812	-0.6068	-0.5282	-0.4453	-0.3577	-0.2653	-0.1676	-0.0644	0.0446	0.1598	0.2815	0.4101	0.5459	0.6892	0.8405
26	-0.9420	-0.8809	-0.8167	-0.7490	-0.6778	-0.6028	-0.5237	-0.4404	-0.3525	-0.2598	-0.1620	-0.0587	0.0502	0.1651	0.2865	0.4145	0.5495	0.6919	0.8419
27	-0.9412	-0.8795	-0.8146	-0.7464	-0.6746	-0.5990	-0.5195	-0.4358	-0.3475	-0.2546	-0.1566	-0.0534	0.0554	0.1702	0.2911	0.4186	0.5529	0.6943	0.8433
28	-0.9406	-0.8782	-0.8127	-0.7439	-0.6716	-0.5955	-0.5155	-0.4314	-0.3429	-0.2497	-0.1516	-0.0484	0.0604	0.1749	0.2955	0.4224	0.5560	0.6966	0.8445
29	-0.9399	-0.8770	-0.8109	-0.7415	-0.6687	-0.5922	-0.5118	-0.4273	-0.3385	-0.2451	-0.1469	-0.0436	0.0650	0.1793	0.2996	0.4260	0.5590	0.6988	0.8457
30	-0.9393	-0.8758	-0.8091	-0.7392	-0.6659	-0.5890	-0.5082	-0.4234	-0.3343	-0.2407	-0.1424	-0.0391	0.0694	0.1835	0.3035	0.4295	0.5618	0.7009	0.8468

Table 5. Cut-off point c_1 for $\alpha = 0.1$ for various n and ρ_0.

n \ ρ_0	-0.90	-0.80	-0.70	-0.60	-0.50	-0.40	-0.30	-0.20	-0.10	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
5	-0.6965	-0.4231	-0.1857	0.0156	0.1842	0.3245	0.4411	0.5381	0.6191	0.6870	0.7443	0.7927	0.8339	0.8691	0.8994	0.9255	0.9482	0.9678	0.9850
6	-0.7299	-0.4861	-0.2706	-0.0828	0.0796	0.2194	0.3395	0.4428	0.5317	0.6084	0.6748	0.7325	0.7827	0.8267	0.8652	0.8991	0.9290	0.9555	0.9791
7	-0.7519	-0.5269	-0.3257	-0.1473	0.0100	0.1481	0.2692	0.3755	0.4688	0.5509	0.6231	0.6870	0.7435	0.7936	0.8383	0.8781	0.9137	0.9456	0.9742
8	-0.7676	-0.5559	-0.3648	-0.1934	-0.0404	0.0960	0.2173	0.3251	0.4212	0.5067	0.5831	0.6513	0.7124	0.7672	0.8165	0.8609	0.9010	0.9373	0.9702
9	-0.7796	-0.5778	-0.3944	-0.2284	-0.0788	0.0559	0.1769	0.2857	0.3836	0.4716	0.5509	0.6224	0.6871	0.7456	0.7986	0.8467	0.8905	0.9304	0.9667
10	-0.7890	-0.5950	-0.4176	-0.2561	-0.1093	0.0239	0.1445	0.2539	0.3529	0.4428	0.5244	0.5985	0.6659	0.7274	0.7834	0.8346	0.8815	0.9244	0.9638
11	-0.7966	-0.6090	-0.4365	-0.2786	-0.1342	-0.0024	0.1178	0.2274	0.3274	0.4187	0.5020	0.5782	0.6479	0.7118	0.7704	0.8242	0.8737	0.9192	0.9612
12	-0.8030	-0.6206	-0.4523	-0.2974	-0.1551	-0.0245	0.0952	0.2050	0.3057	0.3981	0.4829	0.5608	0.6324	0.6983	0.7591	0.8151	0.8669	0.9147	0.9590
13	-0.8084	-0.6304	-0.4656	-0.3134	-0.1729	-0.0434	0.0759	0.1857	0.2870	0.3802	0.4662	0.5455	0.6188	0.6865	0.7491	0.8071	0.8608	0.9107	0.9570
14	-0.8130	-0.6389	-0.4771	-0.3271	-0.1883	-0.0598	0.0591	0.1690	0.2706	0.3646	0.4516	0.5321	0.6068	0.6760	0.7402	0.7999	0.8554	0.9070	0.9552
15	-0.8170	-0.6463	-0.4872	-0.3392	-0.2017	-0.0741	0.0443	0.1541	0.2561	0.3507	0.4385	0.5201	0.5960	0.6666	0.7323	0.7935	0.8505	0.9038	0.9535
16	-0.8206	-0.6527	-0.4960	-0.3498	-0.2136	-0.0869	0.0311	0.1410	0.2432	0.3383	0.4269	0.5094	0.5863	0.6581	0.7251	0.7876	0.8461	0.9008	0.9520
17	-0.8238	-0.6585	-0.5039	-0.3593	-0.2243	-0.0982	0.0194	0.1291	0.2315	0.3271	0.4163	0.4997	0.5776	0.6504	0.7185	0.7823	0.8421	0.8981	0.9507
18	-0.8266	-0.6637	-0.5109	-0.3678	-0.2339	-0.1085	0.0088	0.1184	0.2210	0.3170	0.4068	0.4908	0.5696	0.6433	0.7125	0.7774	0.8384	0.8956	0.9494
19	-0.8291	-0.6684	-0.5173	-0.3755	-0.2425	-0.1178	-0.0009	0.1087	0.2114	0.3077	0.3980	0.4827	0.5622	0.6369	0.7070	0.7729	0.8349	0.8933	0.9482
20	-0.8315	-0.6726	-0.5231	-0.3826	-0.2504	-0.1263	-0.0097	0.0998	0.2026	0.2992	0.3900	0.4753	0.5555	0.6309	0.7019	0.7688	0.8318	0.8911	0.9471
21	-0.8336	-0.6765	-0.5285	-0.3890	-0.2577	-0.1341	-0.0178	0.0916	0.1945	0.2914	0.3825	0.4684	0.5492	0.6254	0.6972	0.7649	0.8288	0.8892	0.9461
22	-0.8355	-0.6801	-0.5333	-0.3949	-0.2643	-0.1412	-0.0253	0.0840	0.1870	0.2841	0.3757	0.4620	0.5434	0.6203	0.6928	0.7613	0.8261	0.8873	0.9452
23	-0.8373	-0.6834	-0.5378	-0.4003	-0.2705	-0.1479	-0.0322	0.0770	0.1801	0.2774	0.3693	0.4560	0.5380	0.6155	0.6887	0.7580	0.8235	0.8855	0.9443
24	-0.8390	-0.6864	-0.5420	-0.4054	-0.2762	-0.1540	-0.0386	0.0705	0.1736	0.2711	0.3633	0.4505	0.5329	0.6110	0.6848	0.7548	0.8211	0.8839	0.9435
25	-0.8405	-0.6892	-0.5459	-0.4101	-0.2815	-0.1598	-0.0446	0.0644	0.1676	0.2653	0.3577	0.4453	0.5282	0.6068	0.6812	0.7518	0.8188	0.8824	0.9427
26	-0.8419	-0.6919	-0.5495	-0.4145	-0.2865	-0.1651	-0.0502	0.0587	0.1620	0.2598	0.3525	0.4404	0.5237	0.6028	0.6778	0.7490	0.8167	0.8809	0.9420
27	-0.8433	-0.6943	-0.5529	-0.4186	-0.2911	-0.1702	-0.0554	0.0534	0.1566	0.2546	0.3475	0.4358	0.5195	0.5990	0.6746	0.7464	0.8146	0.8795	0.9412
28	-0.8445	-0.6966	-0.5560	-0.4224	-0.2955	-0.1749	-0.0604	0.0484	0.1516	0.2497	0.3429	0.4314	0.5155	0.5955	0.6716	0.7439	0.8127	0.8782	0.9406
29	-0.8457	-0.6988	-0.5590	-0.4260	-0.2996	-0.1793	-0.0650	0.0436	0.1469	0.2451	0.3385	0.4273	0.5118	0.5922	0.6687	0.7415	0.8109	0.8770	0.9399
30	-0.8468	-0.7009	-0.5618	-0.4295	-0.3035	-0.1835	-0.0694	0.0391	0.1424	0.2407	0.3343	0.4234	0.5082	0.5890	0.6659	0.7392	0.8091	0.8758	0.9393

Table 6. Cut-off point c_1 for $\alpha = 0.05$ for various n and ρ_0 .

n \ ρ_0	-0.90	-0.80	-0.70	-0.60	-0.50	-0.40	-0.30	-0.20	-0.10	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
5	-0.5505	-0.1971	0.0707	0.2721	0.4244	0.5410	0.6314	0.7026	0.7594	0.8054	0.8429	0.8740	0.8999	0.9217	0.9401	0.9559	0.9694	0.9811	0.9912
6	-0.6206	-0.3118	-0.0649	0.1318	0.2889	0.4153	0.5179	0.6019	0.6714	0.7293	0.7780	0.8193	0.8545	0.8848	0.9110	0.9337	0.9536	0.9710	0.9864
7	-0.6639	-0.3838	-0.1526	0.0379	0.1951	0.3256	0.4345	0.5260	0.6035	0.6694	0.7260	0.7748	0.8171	0.8540	0.8863	0.9148	0.9399	0.9623	0.9822
8	-0.6935	-0.4335	-0.2144	-0.0297	0.1262	0.2583	0.3708	0.4670	0.5499	0.6215	0.6838	0.7382	0.7860	0.8281	0.8654	0.8986	0.9282	0.9547	0.9785
9	-0.7152	-0.4703	-0.2606	-0.0810	0.0731	0.2058	0.3204	0.4198	0.5064	0.5822	0.6488	0.7077	0.7598	0.8062	0.8476	0.8847	0.9181	0.9482	0.9753
10	-0.7318	-0.4987	-0.2967	-0.1214	0.0308	0.1634	0.2793	0.3809	0.4703	0.5494	0.6194	0.6818	0.7375	0.7874	0.8322	0.8727	0.9092	0.9424	0.9725
11	-0.7450	-0.5215	-0.3258	-0.1543	-0.0038	0.1284	0.2451	0.3483	0.4399	0.5214	0.5942	0.6595	0.7181	0.7710	0.8187	0.8620	0.9014	0.9373	0.9700
12	-0.7559	-0.5402	-0.3498	-0.1816	-0.0329	0.0990	0.2161	0.3205	0.4137	0.4973	0.5723	0.6400	0.7011	0.7565	0.8068	0.8526	0.8945	0.9327	0.9678
13	-0.7649	-0.5558	-0.3700	-0.2048	-0.0576	0.0737	0.1911	0.2964	0.3909	0.4762	0.5531	0.6228	0.6861	0.7437	0.7962	0.8442	0.8882	0.9286	0.9658
14	-0.7726	-0.5692	-0.3874	-0.2247	-0.0790	0.0517	0.1693	0.2752	0.3709	0.4575	0.5361	0.6075	0.6727	0.7322	0.7867	0.8366	0.8826	0.9249	0.9639
15	-0.7792	-0.5807	-0.4024	-0.2421	-0.0977	0.0325	0.1501	0.2565	0.3531	0.4409	0.5208	0.5938	0.6606	0.7218	0.7780	0.8297	0.8774	0.9215	0.9622
16	-0.7849	-0.5908	-0.4156	-0.2574	-0.1143	0.0153	0.1329	0.2398	0.3371	0.4259	0.5071	0.5814	0.6496	0.7124	0.7701	0.8235	0.8727	0.9184	0.9607
17	-0.7900	-0.5998	-0.4273	-0.2710	-0.1290	0.0000	0.1176	0.2248	0.3227	0.4124	0.4946	0.5701	0.6396	0.7037	0.7629	0.8177	0.8684	0.9155	0.9593
18	-0.7945	-0.6077	-0.4378	-0.2832	-0.1423	-0.0138	0.1037	0.2111	0.3096	0.4000	0.4832	0.5598	0.6305	0.6958	0.7563	0.8124	0.8644	0.9128	0.9579
19	-0.7986	-0.6148	-0.4472	-0.2942	-0.1543	-0.0263	0.0910	0.1987	0.2976	0.3887	0.4727	0.5503	0.6220	0.6885	0.7501	0.8074	0.8607	0.9104	0.9567
20	-0.8022	-0.6213	-0.4558	-0.3042	-0.1652	-0.0377	0.0795	0.1873	0.2867	0.3783	0.4631	0.5415	0.6142	0.6817	0.7444	0.8028	0.8573	0.9081	0.9556
21	-0.8055	-0.6272	-0.4635	-0.3133	-0.1752	-0.0482	0.0689	0.1768	0.2765	0.3687	0.4541	0.5334	0.6069	0.6754	0.7391	0.7985	0.8540	0.9059	0.9545
22	-0.8086	-0.6326	-0.4707	-0.3217	-0.1844	-0.0578	0.0591	0.1671	0.2671	0.3598	0.4459	0.5258	0.6002	0.6695	0.7341	0.7945	0.8510	0.9039	0.9535
23	-0.8113	-0.6375	-0.4772	-0.3294	-0.1929	-0.0667	0.0500	0.1581	0.2584	0.3515	0.4381	0.5187	0.5938	0.6640	0.7295	0.7908	0.8482	0.9020	0.9525
24	-0.8139	-0.6420	-0.4833	-0.3365	-0.2007	-0.0750	0.0416	0.1497	0.2503	0.3438	0.4309	0.5121	0.5879	0.6588	0.7251	0.7872	0.8455	0.9002	0.9516
25	-0.8162	-0.6462	-0.4889	-0.3432	-0.2081	-0.0827	0.0337	0.1419	0.2427	0.3365	0.4241	0.5059	0.5823	0.6539	0.7210	0.7839	0.8430	0.8985	0.9508
26	-0.8184	-0.6502	-0.4941	-0.3493	-0.2149	-0.0899	0.0263	0.1346	0.2355	0.3297	0.4177	0.5000	0.5771	0.6493	0.7170	0.7807	0.8406	0.8969	0.9500
27	-0.8205	-0.6538	-0.4990	-0.3551	-0.2213	-0.0967	0.0194	0.1277	0.2288	0.3233	0.4117	0.4945	0.5721	0.6449	0.7134	0.7777	0.8383	0.8954	0.9492
28	-0.8224	-0.6572	-0.5036	-0.3605	-0.2273	-0.1030	0.0129	0.1212	0.2225	0.3172	0.4060	0.4892	0.5674	0.6408	0.7098	0.7749	0.8361	0.8939	0.9485
29	-0.8242	-0.6604	-0.5078	-0.3656	-0.2329	-0.1090	0.0068	0.1151	0.2165	0.3115	0.4006	0.4843	0.5629	0.6369	0.7065	0.7721	0.8341	0.8925	0.9478
30	-0.8258	-0.6634	-0.5119	-0.3704	-0.2382	-0.1146	0.0010	0.1093	0.2108	0.3061	0.3955	0.4796	0.5587	0.6331	0.7033	0.7696	0.8321	0.8912	0.9471

Table 7. Cut-off point c_1 for $\alpha = 0.025$ for various n and ρ_0 .

n \ ρ_0	-0.90	-0.80	-0.70	-0.60	-0.50	-0.40	-0.30	-0.20	-0.10	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
5	-0.3733	0.0384	0.3056	0.4839	0.6072	0.6955	0.7606	0.8099	0.8482	0.8783	0.9026	0.9223	0.9386	0.9521	0.9635	0.9732	0.9814	0.9885	0.9947
6	-0.4927	-0.1291	0.1312	0.3205	0.4612	1.5604	0.6508	0.7162	0.7687	0.8114	0.8466	0.8760	0.9008	0.9218	0.9398	0.9553	0.9688	0.9806	0.9909
7	-0.5636	-0.2343	0.0147	0.2053	0.3534	0.4703	0.5639	0.6399	0.7025	0.7545	0.7982	0.8353	0.8671	0.8944	0.9182	0.9389	0.9571	0.9731	0.9873
8	-0.6108	-0.3067	-0.0683	0.1204	0.2716	0.3941	0.4946	0.5779	0.6477	0.7067	0.7571	0.8004	0.8378	0.8704	0.8990	0.9242	0.9466	0.9664	0.9841
9	-0.6446	-0.3598	-0.1306	0.0552	0.2074	0.3332	0.4383	0.5268	0.6020	0.6664	0.7219	0.7702	0.8124	0.8494	0.8822	0.9112	0.9372	0.9604	0.9812
10	-0.6700	-0.4005	-0.1793	0.0034	0.1556	0.2834	0.3916	0.4839	0.5633	0.6319	0.6917	0.7440	0.7901	0.8310	0.8673	0.8997	0.9288	0.9549	0.9786
11	-0.6900	-0.4328	-0.2185	-0.0388	0.1129	0.2419	0.3523	0.4475	0.5300	0.6021	0.6653	0.7211	0.7705	0.8146	0.8540	0.8894	0.9212	0.9501	0.9762
12	-0.7061	-0.4591	-0.2507	-0.0739	0.0770	0.2066	0.3186	0.4160	0.5011	0.5760	0.6421	0.7008	0.7531	0.8000	0.8421	0.8801	0.9144	0.9456	0.9741
13	-0.7194	-0.4811	-0.2779	-0.1038	0.0462	0.1762	0.2894	0.3885	0.4758	0.5529	0.6215	0.6827	0.7375	0.7868	0.8313	0.8716	0.9083	0.9416	0.9721
14	-0.7306	-0.4997	-0.3010	-0.1294	0.0196	0.1497	0.2637	0.3643	0.4533	0.5324	0.6031	0.6664	0.7235	0.7749	0.8216	0.8640	0.9026	0.9379	0.9703
15	-0.7401	-0.5157	-0.3211	-0.1518	-0.0037	0.1263	0.2410	0.3427	0.4331	0.5140	0.5865	0.6517	0.7107	0.7641	0.8127	0.8570	0.8974	0.9345	0.9686
16	-0.7484	-0.5296	-0.3387	-0.1715	-0.0244	0.1055	0.2207	0.3233	0.4150	0.4973	0.5714	0.6384	0.6990	0.7542	0.8045	0.8505	0.8927	0.9314	0.9671
17	-0.7556	-0.5418	-0.3542	-0.1889	-0.0428	0.0869	0.2025	0.3058	0.3986	0.4821	0.5577	0.6261	0.6883	0.7451	0.7970	0.8445	0.8882	0.9285	0.9656
18	-0.7620	-0.5527	-0.3681	-0.2046	-0.0594	0.0701	0.1859	0.2899	0.3836	0.4683	0.5450	0.6148	0.6785	0.7367	0.7900	0.8390	0.8841	0.9258	0.9643
19	-0.7677	-0.5624	-0.3805	-0.2187	-0.0744	0.0548	0.1708	0.2754	0.3699	0.4555	0.5334	0.6044	0.6694	0.7289	0.7835	0.8339	0.8803	0.9233	0.9631
20	-0.7728	-0.5712	-0.3917	-0.2315	-0.0880	0.0409	0.1570	0.2620	0.3572	0.4438	0.5227	0.5948	0.6609	0.7216	0.7775	0.8290	0.8767	0.9209	0.9619
21	-0.7775	-0.5792	-0.4020	-0.2432	-0.1005	0.0281	0.1443	0.2497	0.3455	0.4329	0.5127	0.5858	0.6530	0.7148	0.7718	0.8245	0.8733	0.9187	0.9608
22	-0.7817	-0.5864	-0.4113	-0.2539	-0.1120	0.0163	0.1326	0.2383	0.3347	0.4227	0.5034	0.5774	0.6456	0.7084	0.7665	0.8203	0.8702	0.9166	0.9597
23	-0.7855	-0.5930	-0.4199	-0.2638	-0.1226	0.0054	0.1217	0.2277	0.3246	0.4132	0.4947	0.5696	0.6386	0.7024	0.7615	0.8163	0.8672	0.9146	0.9587
24	-0.7890	-0.5991	-0.4278	-0.2729	-0.1324	-0.0048	0.1116	0.2178	0.3151	0.4044	0.4865	0.5622	0.6321	0.6968	0.7568	0.8125	0.8644	0.9127	0.9578
25	-0.7923	-0.6048	-0.4352	-0.2814	-0.1416	-0.0142	0.1021	0.2086	0.3063	0.3961	0.4788	0.5553	0.6259	0.6915	0.7523	0.8089	0.8617	0.9109	0.9569
26	-0.7953	-0.6100	-0.4420	-0.2893	-0.1501	-0.0230	0.0932	0.1999	0.2979	0.3882	0.4716	0.5487	0.6201	0.6864	0.7481	0.8056	0.8591	0.9092	0.9561
27	-0.7981	-0.6149	-0.4483	-0.2966	-0.1581	-0.0313	0.0849	0.1917	0.2901	0.3809	0.4648	0.5425	0.6146	0.6817	0.7441	0.8023	0.8567	0.9076	0.9553
28	-0.8007	-0.6194	-0.4543	-0.3035	-0.1656	-0.0391	0.0771	0.1840	0.2827	0.3739	0.4583	0.5366	0.6094	0.6771	0.7403	0.7993	0.8544	0.9061	0.9545
29	-0.8031	-0.6237	-0.4598	-0.3100	-0.1726	-0.0464	0.0697	0.1768	0.2757	0.3673	0.4522	0.5311	0.6044	0.6728	0.7367	0.7964	0.8522	0.9046	0.9538
30	-0.8054	-0.6276	-0.4651	-0.3161	-0.1792	-0.0533	0.0627	0.1699	0.2691	0.3610	0.4464	0.5258	0.5997	0.6687	0.7332	0.7936	0.8501	0.9032	0.9531

Table 8. Cut-off point c_1 for $\alpha = 0.01$ for various n and ρ_0.

n \ ρ_0	-0.90	-0.80	-0.70	-0.60	-0.50	-0.40	-0.30	-0.20	-0.10	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
5	-0.1011	0.3315	0.5577	0.6890	0.7718	0.8275	0.8668	0.8956	0.9174	0.9343	0.9477	0.9585	0.9673	0.9746	0.9807	0.9858	0.9902	0.9940	0.9972
6	-0.2972	0.1119	0.3621	0.5245	0.6357	0.7152	0.7741	0.8191	0.8542	0.8822	0.9049	0.9236	0.9391	0.9522	0.9633	0.9729	0.9811	0.9883	0.9945
7	-0.4127	-0.0337	0.2197	0.3962	0.5238	0.6191	0.6922	0.7496	0.7955	0.8329	0.8637	0.8895	0.9113	0.9299	0.9459	0.9597	0.9718	0.9824	0.9917
8	-0.4883	-0.1357	0.1141	0.2965	0.4336	0.5393	0.6225	0.6893	0.7437	0.7887	0.8264	0.8583	0.8856	0.9090	0.9294	0.9472	0.9629	0.9767	0.9890
9	-0.5414	-0.2107	0.0333	0.2177	0.3604	0.4730	0.5635	0.6374	0.6985	0.7498	0.7932	0.8303	0.8622	0.8900	0.9143	0.9357	0.9546	0.9715	0.9865
10	-0.5809	-0.2683	-0.0305	0.1541	0.3000	0.4174	0.5132	0.5926	0.6591	0.7155	0.7636	0.8051	0.8412	0.8728	0.9006	0.9252	0.9470	0.9666	0.9842
11	-0.6114	-0.3138	-0.0820	0.1016	0.2493	0.3701	0.4700	0.5537	0.6245	0.6851	0.7373	0.7826	0.8223	0.8572	0.8880	0.9155	0.9401	0.9621	0.9820
12	-0.6357	-0.3508	-0.1247	0.0575	0.2063	0.3294	0.4324	0.5195	0.5940	0.6581	0.7137	0.7623	0.8051	0.8429	0.8766	0.9067	0.9337	0.9580	0.9800
13	-0.6556	-0.3815	-0.1605	0.0199	0.1691	0.2939	0.3994	0.4893	0.5667	0.6339	0.6925	0.7440	0.7895	0.8300	0.8661	0.8986	0.9278	0.9542	0.9782
14	-0.6721	-0.4074	-0.1912	-0.0125	0.1368	0.2628	0.3702	0.4624	0.5423	0.6120	0.6732	0.7273	0.7753	0.8181	0.8565	0.8911	0.9223	0.9507	0.9765
15	-0.6861	-0.4296	-0.2177	-0.0408	0.1083	0.2352	0.3441	0.4383	0.5203	0.5923	0.6557	0.7120	0.7622	0.8072	0.8476	0.8841	0.9173	0.9474	0.9749
16	-0.6982	-0.4488	-0.2409	-0.0658	0.0830	0.2105	0.3207	0.4165	0.5003	0.5742	0.6397	0.6980	0.7502	0.7971	0.8394	0.8777	0.9126	0.9443	0.9734
17	-0.7086	-0.4657	-0.2614	-0.0880	0.0604	0.1883	0.2995	0.3966	0.4821	0.5577	0.6250	0.6851	0.7391	0.7877	0.8318	0.8717	0.9082	0.9414	0.9720
18	-0.7178	-0.4806	-0.2796	-0.1079	0.0400	0.1682	0.2802	0.3785	0.4654	0.5425	0.6114	0.6732	0.7288	0.7790	0.8246	0.8661	0.9040	0.9388	0.9706
19	-0.7260	-0.4940	-0.2960	-0.1258	0.0215	0.1499	0.2625	0.3619	0.4500	0.5285	0.5988	0.6621	0.7192	0.7709	0.8180	0.8609	0.9002	0.9362	0.9694
20	-0.7332	-0.5059	-0.3108	-0.1421	0.0047	0.1332	0.2463	0.3466	0.4358	0.5155	0.5871	0.6517	0.7102	0.7633	0.8117	0.8560	0.8965	0.9339	0.9682
21	-0.7398	-0.5167	-0.3242	-0.1570	-0.0108	0.1178	0.2314	0.3324	0.4225	0.5034	0.5762	0.6421	0.7018	0.7562	0.8058	0.8513	0.8931	0.9316	0.9671
22	-0.7457	-0.5265	-0.3364	-0.1706	-0.0250	0.1035	0.2176	0.3192	0.4102	0.4921	0.5660	0.6330	0.6939	0.7495	0.8003	0.8470	0.8899	0.9295	0.9661
23	-0.7511	-0.5355	-0.3477	-0.1831	-0.0381	0.0904	0.2047	0.3069	0.3987	0.4815	0.5564	0.6245	0.6865	0.7431	0.7951	0.8428	0.8868	0.9275	0.9651
24	-0.7560	-0.5437	-0.3580	-0.1947	-0.0502	0.0781	0.1927	0.2955	0.3880	0.4716	0.5474	0.6164	0.6794	0.7371	0.7901	0.8389	0.8839	0.9255	0.9641
25	-0.7605	-0.5513	-0.3676	-0.2054	-0.0615	0.0667	0.1815	0.2847	0.3779	0.4622	0.5389	0.6088	0.6728	0.7315	0.7854	0.8352	0.8811	0.9237	0.9632
26	-0.7647	-0.5583	-0.3765	-0.2154	-0.0721	0.0560	0.1710	0.2746	0.3683	0.4534	0.5309	0.6017	0.6665	0.7261	0.7809	0.8316	0.8785	0.9220	0.9624
27	-0.7685	-0.5649	-0.3848	-0.2248	-0.0820	0.0460	0.1611	0.2651	0.3594	0.4451	0.5233	0.5949	0.6605	0.7210	0.7767	0.8282	0.8760	0.9203	0.9615
28	-0.7721	-0.5709	-0.3925	-0.2335	-0.0912	0.0366	0.1518	0.2561	0.3509	0.4372	0.5161	0.5884	0.6548	0.7161	0.7726	0.8250	0.8736	0.9187	0.9608
29	-0.7754	-0.5766	-0.3997	-0.2417	-0.0999	0.0277	0.1430	0.2476	0.3428	0.4297	0.5092	0.5822	0.6494	0.7114	0.7688	0.8219	0.8713	0.9172	0.9600
30	-0.7785	-0.5819	-0.4065	-0.2494	-0.1082	0.0193	0.1347	0.2396	0.3352	0.4226	0.5027	0.5764	0.6443	0.7070	0.7651	0.8189	0.8690	0.9157	0.9593

Table 9. Powers of the four tests for $n = 5$ and $\alpha = 0.05$.

ρ_0	power	ρ																			
		0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
0.00	β_0	0.05000	0.05837	0.06798	0.07901	0.09172	0.10637	0.12332	0.14298	0.16586	0.19261	0.22399	0.26099	0.30482	0.35699	0.41934	0.49402	0.58330	0.68879	0.80886	0.93038
	β_F	0.04384	0.05127	0.05982	0.06968	0.08108	0.09427	0.10961	0.12748	0.14840	0.17301	0.20210	0.23667	0.27801	0.32777	0.38802	0.46134	0.55069	0.65883	0.78576	0.91951
	β_H	0.06515	0.07574	0.08780	0.10156	0.11726	0.13520	0.15574	0.17930	0.20638	0.23758	0.27361	0.31533	0.36374	0.41998	0.48534	0.56107	0.64805	0.74590	0.85065	0.94872
	β_K	0.05000	0.05837	0.06797	0.07901	0.09172	0.10637	0.12332	0.14298	0.16586	0.19260	0.22399	0.26099	0.30482	0.35699	0.41933	0.49401	0.58330	0.68879	0.80886	0.93038
0.20	β_0					0.05000	0.05862	0.06878	0.08081	0.09516	0.11238	0.13322	0.15867	0.19010	0.22937	0.27912	0.34311	0.42668	0.53724	0.68336	0.86484
	β_F					0.04790	0.05620	0.06598	0.07758	0.09143	0.10808	0.12827	0.15298	0.18356	0.22189	0.27061	0.33355	0.41623	0.52641	0.67354	0.85901
	β_H					0.07240	0.08437	0.09832	0.11466	0.13388	0.15661	0.18364	0.21599	0.25500	0.30240	0.36046	0.43210	0.52092	0.63076	0.76337	0.90848
	β_K					0.05459	0.06392	0.07489	0.08785	0.10325	0.12168	0.14391	0.17093	0.20412	0.24534	0.29717	0.36321	0.44846	0.55950	0.70318	0.87628
0.40	β_0									0.05000	0.05981	0.07197	0.08723	0.10669	0.13198	0.16561	0.21159	0.27661	0.37243	0.52015	0.75273
	β_F									0.05178	0.06191	0.07444	0.09015	0.11017	0.13613	0.17058	0.21754	0.28370	0.38069	0.52908	0.75977
	β_H									0.07981	0.09464	0.11272	0.13500	0.16278	0.19794	0.24315	0.30240	0.38167	0.48996	0.63958	0.83800
	β_K									0.05897	0.07035	0.08437	0.10187	0.12404	0.15259	0.19016	0.24081	0.31115	0.41222	0.56238	0.78503
0.60	β_0													0.05000	0.06314	0.08132	0.10750	0.14721	0.21189	0.32838	0.56942
	β_F													0.05553	0.06996	0.08985	0.11832	0.16117	0.23016	0.35214	0.59594
	β_H													0.08743	0.10886	0.13772	0.17781	0.23576	0.32378	0.46586	0.70734
	β_K													0.06321	0.07940	0.10158	0.13309	0.18001	0.25444	0.38289	0.62848
0.80	β_0																	0.05000	0.07675	0.13292	0.29225
	β_F																	0.05919	0.09020	0.15419	0.32882
	β_H																	0.09533	0.14162	0.23150	0.44698
	β_K																	0.06735	0.10201	0.17250	0.35877

Table 10. Powers of the four tests for $n = 10$ and $\alpha = 0.05$.

ρ_0	power	ρ																			
		0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
0.00	β_0	0.05000	0.06554	0.08493	0.10885	0.13805	0.17329	0.21530	0.26472	0.32203	0.38734	0.46031	0.53989	0.62409	0.70980	0.79270	0.86739	0.92815	0.97031	0.99263	0.99949
	β_F	0.04891	0.06417	0.08324	0.10680	0.13560	0.17041	0.21197	0.26097	0.31788	0.38289	0.45567	0.53524	0.61964	0.70579	0.78937	0.86492	0.92660	0.96958	0.99242	0.99947
	β_H	0.05267	0.06889	0.08905	0.11385	0.14400	0.18026	0.22331	0.27375	0.33196	0.39798	0.47136	0.55093	0.63460	0.71922	0.80047	0.87312	0.93171	0.97200	0.99311	0.99952
	β_K	0.05000	0.06554	0.08493	0.10885	0.13805	0.17329	0.21530	0.26472	0.32203	0.38734	0.46031	0.53989	0.62409	0.70980	0.79270	0.86739	0.92815	0.97031	0.99263	0.99949
0.20	β_0					0.05000	0.06633	0.08738	0.11433	0.14857	0.19165	0.24525	0.31098	0.39008	0.48285	0.58777	0.70025	0.81130	0.90704	0.97160	0.99747
	β_F					0.05211	0.06900	0.09072	0.11844	0.15354	0.19756	0.25213	0.31876	0.39858	0.49170	0.59640	0.70790	0.81716	0.91055	0.97290	0.99761
	β_H					0.05666	0.07473	0.09784	0.12716	0.16406	0.20999	0.26649	0.33490	0.41606	0.50975	0.61383	0.72320	0.82874	0.91739	0.97539	0.99788
	β_K					0.05327	0.07045	0.09253	0.12066	0.15623	0.20075	0.25582	0.32292	0.40311	0.49639	0.60095	0.71192	0.82022	0.91237	0.97357	0.99768
0.40	β_0									0.05000	0.06909	0.09536	0.13140	0.18063	0.24732	0.33628	0.45175	0.59430	0.75432	0.90205	0.98764
	β_F									0.05519	0.07586	0.10409	0.14250	0.19447	0.26408	0.35573	0.47287	0.61484	0.77069	0.91065	0.98908
	β_H									0.06076	0.08308	0.11334	0.15415	0.20883	0.28126	0.37539	0.49386	0.63485	0.78623	0.91858	0.99036
	β_K									0.05640	0.07744	0.10611	0.14506	0.19763	0.26788	0.36011	0.47757	0.61936	0.77423	0.91248	0.98938
0.60	β_0													0.05000	0.07662	0.11885	0.18642	0.29443	0.46247	0.69896	0.93608
	β_F													0.05815	0.08816	0.13499	0.20843	0.32287	0.49509	0.72683	0.94527
	β_H													0.06497	0.09769	0.14810	0.22597	0.34497	0.51961	0.74682	0.95142
	β_K													0.05942	0.08994	0.13745	0.21175	0.32709	0.49983	0.73076	0.94651
0.80	β_0																	0.05000	0.10932	0.26133	0.64666
	β_F																	0.06103	0.12975	0.29717	0.68619
	β_H																	0.06930	0.14459	0.32191	0.71099
	β_K																	0.06235	0.13215	0.30125	0.69041

Table 11. Powers of the four tests for $n = 15$ and $\alpha = 0.05$.

ρ_0	power	ρ																			
		0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
0.00	β_0	0.05000	0.07086	0.09828	0.13346	0.17754	0.23139	0.29542	0.36936	0.45200	0.54101	0.63285	0.72289	0.80587	0.87664	0.93125	0.96811	0.98873	0.99743	0.99974	1.00000
	β_F	0.04947	0.07016	0.09737	0.13233	0.17616	0.22976	0.29356	0.36732	0.44986	0.53886	0.63081	0.72108	0.80439	0.87555	0.93055	0.96774	0.98858	0.99739	0.99973	1.00000
	β_H	0.05100	0.07219	0.09999	0.13560	0.18014	0.23446	0.29891	0.37319	0.45603	0.54504	0.63667	0.72627	0.80862	0.87866	0.93254	0.96879	0.98900	0.99750	0.99975	1.00000
	β_K	0.05000	0.07086	0.09828	0.13346	0.17754	0.23139	0.29542	0.36936	0.45200	0.54101	0.63285	0.72289	0.80587	0.87664	0.93125	0.96811	0.98873	0.99743	0.99974	1.00000
0.20	β_0					0.05000	0.07203	0.10205	0.14210	0.19431	0.26055	0.34199	0.43830	0.54685	0.66180	0.77383	0.87111	0.94254	0.98296	0.99768	0.99996
	β_F					0.05211	0.07487	0.10576	0.14682	0.20012	0.26744	0.34980	0.44670	0.55528	0.66955	0.78014	0.87546	0.94486	0.98378	0.99781	0.99996
	β_H					0.05404	0.07745	0.10913	0.15109	0.20536	0.27363	0.35679	0.45417	0.56274	0.67635	0.78565	0.87922	0.94686	0.98448	0.99792	0.99996
	β_K					0.05266	0.07561	0.10673	0.14804	0.20162	0.26922	0.35181	0.44885	0.55743	0.67152	0.78174	0.87655	0.94544	0.98399	0.99784	0.99996
0.40	β_0									0.05000	0.07601	0.11408	0.16866	0.24473	0.34674	0.47615	0.62729	0.78231	0.91039	0.98178	0.99943
	β_F									0.05465	0.08251	0.12293	0.18026	0.25926	0.36383	0.49453	0.64460	0.79555	0.91744	0.98362	0.99950
	β_H									0.05716	0.08601	0.12764	0.18639	0.26687	0.37268	0.50394	0.65335	0.80212	0.92088	0.98450	0.99954
	β_K									0.05522	0.08331	0.12401	0.18167	0.26101	0.36587	0.49671	0.64663	0.79708	0.91825	0.98383	0.99951
0.60	β_0													0.05000	0.08690	0.15004	0.25490	0.41850	0.64268	0.87346	0.99139
	β_F													0.05711	0.09790	0.16635	0.27738	0.44570	0.66819	0.88705	0.99277
	β_H													0.06037	0.10288	0.17362	0.28722	0.45733	0.67878	0.89248	0.99329
	β_K													0.05771	0.09882	0.16769	0.27920	0.44787	0.67017	0.88808	0.99286
0.80	β_0																	0.05000	0.13600	0.37142	0.83358
	β_F																	0.05951	0.15627	0.40621	0.85507
	β_H																	0.06366	0.16486	0.42024	0.86303
	β_K																	0.06012	0.15756	0.40834	0.85630

Table 12. Powers of the four tests for $n = 20$ and $\alpha = 0.05$.

ρ_0	power	ρ																			
		0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
0.00	β_0	0.05000	0.07541	0.11015	0.15596	0.21416	0.28526	0.36857	0.46189	0.56128	0.66132	0.75565	0.83799	0.90351	0.94999	0.97846	0.99282	0.99836	0.99979	0.99999	1.00000
	β_F	0.04966	0.07494	0.10953	0.15516	0.21318	0.28411	0.36729	0.46053	0.55994	0.66008	0.75458	0.83716	0.90294	0.94964	0.97829	0.99276	0.99834	0.99979	0.99999	1.00000
	β_H	0.05049	0.07609	0.11105	0.15712	0.21557	0.28691	0.37041	0.46382	0.56320	0.66310	0.75717	0.83917	0.90433	0.95047	0.97869	0.99291	0.99838	0.99980	0.99999	1.00000
	β_K	0.05000	0.07541	0.11015	0.15596	0.21416	0.28526	0.36857	0.46189	0.56128	0.66132	0.75565	0.83799	0.90351	0.94999	0.97846	0.99282	0.99836	0.99979	0.99999	1.00000
0.20	β_0					0.05000	0.07691	0.11514	0.16756	0.23673	0.32400	0.42851	0.54609	0.66853	0.78414	0.88014	0.94709	0.98350	0.99709	0.99982	1.00000
	β_F					0.05195	0.07966	0.11886	0.17240	0.24270	0.33097	0.43613	0.55377	0.67553	0.78974	0.88392	0.94909	0.98424	0.99724	0.99984	1.00000
	β_H					0.05304	0.08119	0.12092	0.17506	0.24598	0.33478	0.44028	0.55793	0.67931	0.79275	0.88593	0.95015	0.98463	0.99732	0.99984	1.00000
	β_K					0.05230	0.08015	0.11953	0.17326	0.24376	0.33220	0.43748	0.55512	0.67676	0.79072	0.88457	0.94944	0.98437	0.99727	0.99984	1.00000
0.40	β_0									0.05000	0.08195	0.13094	0.20306	0.30387	0.43532	0.59111	0.75206	0.88731	0.96899	0.99684	0.99998
	β_F									0.05416	0.08810	0.13962	0.21463	0.31819	0.45140	0.60687	0.76472	0.89480	0.97163	0.99718	0.99998
	β_H									0.05565	0.09029	0.14268	0.21868	0.32316	0.45692	0.61222	0.76897	0.89727	0.97249	0.99729	0.99998
	β_K									0.05453	0.08863	0.14036	0.21562	0.31941	0.45276	0.60818	0.76577	0.89541	0.97184	0.99720	0.99998
0.60	β_0													0.05000	0.09587	0.17866	0.31789	0.52452	0.76794	0.94928	0.99892
	β_F													0.05631	0.10634	0.19476	0.33979	0.54870	0.78601	0.95514	0.99910
	β_H													0.05833	0.10964	0.19977	0.34649	0.55593	0.79127	0.95679	0.99915
	β_K													0.05669	0.10695	0.19569	0.34105	0.55006	0.78700	0.95546	0.99911
0.80	β_0																	0.05000	0.16032	0.46805	0.92491
	β_F																	0.05841	0.18020	0.49986	0.93535
	β_H																	0.06108	0.18632	0.50920	0.93819
	β_K																	0.05880	0.18109	0.50124	0.93578

Table 13. Powers of the four tests for $n = 25$ and $\alpha = 0.05$.

ρ_0	power	ρ																			
		0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
0.00	β_0	0.05000	0.07951	0.12119	0.17726	0.24897	0.33598	0.43577	0.54343	0.65200	0.75348	0.84038	0.90756	0.95348	0.98045	0.99353	0.99846	0.99977	0.99998	1.00000	1.00000
	β_F	0.04975	0.07916	0.12071	0.17663	0.24820	0.33508	0.43480	0.54246	0.65110	0.75271	0.83979	0.90716	0.95324	0.98034	0.99349	0.99845	0.99977	0.99998	1.00000	1.00000
	β_H	0.05027	0.07990	0.12173	0.17795	0.24982	0.33695	0.43682	0.54449	0.65299	0.75432	0.84103	0.90799	0.95373	0.98057	0.99358	0.99847	0.99978	0.99998	1.00000	1.00000
	β_K	0.05000	0.07951	0.12119	0.17726	0.24897	0.33598	0.43577	0.54343	0.65200	0.75348	0.84038	0.90756	0.95348	0.98045	0.99353	0.99846	0.99977	0.99998	1.00000	1.00000
0.20	β_0					0.05000	0.08131	0.12732	0.19169	0.27696	0.38308	0.50594	0.63636	0.76091	0.86501	0.93823	0.97905	0.99547	0.99953	0.99999	1.00000
	β_F					0.05180	0.08396	0.13101	0.19653	0.28291	0.38988	0.51303	0.64301	0.76636	0.86879	0.94032	0.97990	0.99568	0.99955	0.99999	1.00000
	β_H					0.05251	0.08499	0.13244	0.19841	0.28521	0.39249	0.51574	0.64554	0.76844	0.87022	0.94111	0.98021	0.99576	0.99956	0.99999	1.00000
	β_K					0.05206	0.08433	0.13152	0.19721	0.28374	0.39082	0.51401	0.64392	0.76711	0.86931	0.94061	0.98001	0.99571	0.99956	0.99999	1.00000
0.40	β_0									0.05000	0.08734	0.14674	0.23573	0.35922	0.51425	0.68419	0.83810	0.94331	0.98969	0.99948	1.00000
	β_F									0.05379	0.09320	0.15525	0.24714	0.37305	0.52892	0.69715	0.84690	0.94730	0.99061	0.99954	1.00000
	β_H									0.05480	0.09475	0.15749	0.25012	0.37663	0.53269	0.70044	0.84910	0.94829	0.99084	0.99955	1.00000
	β_K									0.05405	0.09360	0.15583	0.24791	0.37398	0.52990	0.69801	0.84747	0.94756	0.99067	0.99954	1.00000
0.60	β_0													0.05000	0.10410	0.20579	0.37658	0.61429	0.85223	0.98040	0.99987
	β_F													0.05572	0.11415	0.22162	0.39755	0.63510	0.86443	0.98278	0.99989
	β_H													0.05715	0.11662	0.22545	0.40254	0.63995	0.86720	0.98330	0.99990
	β_K													0.05599	0.11462	0.22235	0.39850	0.63603	0.86496	0.98288	0.99990
0.80	β_0																	0.05000	0.18329	0.55253	0.96723
	β_F																	0.05761	0.20281	0.58085	0.97202
	β_H																	0.05955	0.20761	0.58751	0.97306
	β_K																	0.05789	0.20350	0.58182	0.97217

Table 14. 2005 TUDA average score in Mathematics and Reading.

District	4th Grade Score		8th Grade Score	
	Mathematics	Reading	Mathematics	Reading
Charlotte	244	221	281	259
Austin	242	217	281	257
Houston	233	211	267	248
San Diego	232	208	270	253
New York City	231	213	267	251
Boston	229	207	270	253
Atlanta	221	201	245	240
Los Angeles	220	196	250	239
Cleveland	220	197	249	240
Chicago	216	198	258	249
Washington DC	211	191	245	238

(using Table 2) against ρ_0. The resultant diagram is given in Fig. 1.

Then, on the vertical axis place $\hat{\rho} = r$ and draw a horizontal line. The horizontal line passing through $\hat{\rho} = r$ cuts the c_1 and c_2 curves at two points. The corresponding horizontal axis values indicate the 95% confidence interval for ρ.

Using the above approach, we obtain the exact 95% confidence interval for ρ_4 and ρ_8 as (0.89715, 0.99275) and (0.89037, 0.99224) respectively (by using $\hat{\rho}_4 = r_4 = 0.9755$ and $\hat{\rho}_8 = r_8 = 0.9738$ on the vertical axis). Our exact confidence intervals are now compared with the ones obtained from the three approximate methods as shown in the Table 15.

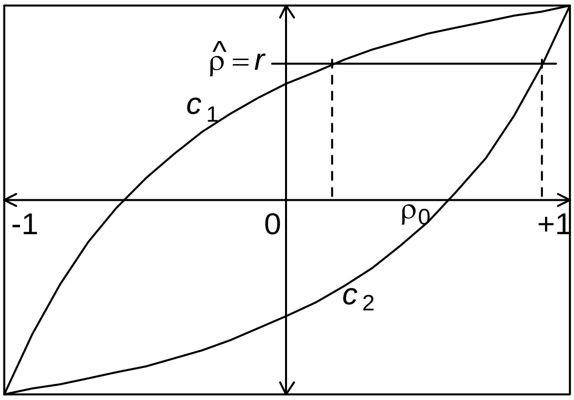

Fig. 1. Plots of c_1 and c_2 against ρ_0 ($n = 11$, $\alpha = 0.05$, $H_A : \rho \neq \rho_0$).

Table 15. 95% confidence intervals for ρ_4 and ρ_8.

Method	ρ_4	ρ_8
Invert Δ_0	(0.89715, 0.99275)	(0.89037, 0.99224)
Invert Δ_F	(0.90551, 0.99382)	(0.89920, 0.99338)
Invert Δ_H	(0.91012, 0.99346)	(0.90409, 0.99301)
Invert Δ_K	(0.90489, 0.99385)	(0.89855, 0.99342)

Concluding Remark. (i) Quite often we use one of the three popular approximate tests while testing on a bivariate normal correlation coefficient. This study shows that while the approximate tests may perform well for large sample sizes, the same is not true for small sample sizes. For this reason we have provided tables of critical values for the exact test for small sample sizes. These tables will be helpful in real life problems where one doesn't have (or can't afford) a large sample size. (ii) The method of exact test for a bivariate normal correlation coefficient discussed above can be extended for partial correlation coefficients and multiple correlation coefficient which are under consideration now, and will be reported in a forthcoming paper.

References:

[1] T. W. Anderson, *An Introduction to Multivariate Statistical Analysis*, 2nd Edition, John Wiley & Sons, New York, 1984.

[2] F. N. David, *Tables of the Ordinates and Probability Integral of the Distribution of the Correlation Coefficient in Small Samples*, Cambridge University, Cambridge, UK, 1938.

[3] R. A. Fisher, Frequency Distribution of the Values of the Correlation Coefficient in Samples from an Indefinitely Large Population, *Biometrika*, Vol. 10, No. 4, 1915, pp. 507 - 521.

[4] R. A. Fisher, On the Probable Error of a Coefficient of Correlation Deduced from a Small Sample, *Metron*, Vol. 1, No. 4, 1921, pp. 3 - 32.

[5] H. Hotelling, New Light on the Correlation Coefficient and its Transforms, *Journal of the Royal Statistical Society, Series B*, Vol. 15, No. 2, 1953, pp. 193 - 232.

[6] S. Jeyaratnam, Confidence Intervals for the

Correlation Coefficient, *Statistics & Probability Letters*, Vol. 15, 1992, pp. 389 - 393.

[7] H. C. Kraemer, Improved Approximation to the Non-Null Distribution of the Correlation Coefficient, *Journal of the American Statistical Association*, Vol. 68, 1973, pp. 1004 - 1008.

[8] C. I. Lee, H. M. Chuang, An Effective Soft Clustering Approach to Mining Gene Expressions from Multi-Source Databases, *Proceedings of the 6th WSEAS Int. Conf. on Artificial Intelligence, Knowledge Engineering and Data Bases*, Corfu Island, Greece, February 16-19, 2007, pp. 165 - 169.

[9] V. R. Mankar, A. A. Ghatol, Use of RBF Neural Network in EMG Signal Noise Removal, *WSEAS Transactions on Circuits and Systems*, Issue 4, Volume 7, April 2008, pp. 259 - 265.

[10] N. Pal, W. K. Lim, Shrinkage Estimation of a Correlation Coefficient and two Examples with Real Life Data-Sets, *Journal of Statistical Computation and Simulation*, Vol. 62, 1999, pp. 357 - 373.

[11] N. Pal, W. K. Lim, Estimation of a Correlation Coefficient: Some Second Order Decision-Theoretic Results, *Statistics & Decisions*, Vol. 18, 2000, pp. 185 - 203.

[12] S. Rashwan, M. A. Ismail, S. Fouad, Detection of Buried Landmines using the Possibilistic Correlation-Dependent Fusion Methods, *Proceedings of the 6th WSEAS International Conference on Signal Processing, Robotics and Automation*, Corfu Island, Greece, February 16-19, 2007, pp. 42 - 47.

[13] M. Samiuddin, On a Test for an Assigned Value of Correlation in a Bivariate Normal Distribution, *Biometrika*, Vol. 57, No. 2, 1970, pp. 461 - 464.

[14] G. Shieh, Exact Interval Estimation, Power Calculation, and Sample Size Determination in Normal Correlation Analysis, *Psychometrika*, Vol. 71, No. 3, 2006, pp. 529-540.

[15] A. Winterbottom, A Note on the Derivation of Fisher's Transformation of the Correlation Coefficient, *The American Statistician*, Vol. 33, 1979, pp. 142 - 143.