Stress Strain Modeling by Transformed Equations of Ultrasonic wave

ARASH ZIAIE
Civil Engineering Department
University of Kerman
22th of Bahman Ave, Kerman University, Kerman
IRAN
ZIAIE111@YAHOO.COM
KAVEH KUMARCI
Sama Organization (Affiliated With Islamic Azad University), Shahr-e-Kord Branch
IRAN
KUMARCI KAVEH@YAHOO.COM
ARASH KYIOUMARSI
Electric Engineering Department
University of Isfahan
Hezar Jerib Bold., Isfahan University, Isfahan
IRAN
KIYOUMARSI_ARASH@YAHOO.COM

Abstract: - The equations of ultrasonic wave propagation in Cartesian coordinates are functions of 27 partial displacement derivatives, which first derived and then transformed into cylindrical coordinates. The new obtained functions are functions of 27 partial displacements of first and second order derivatives in cylindrical coordinates too and they will be linearized using a perturbation method based on the Taylor series expansion. A displacement wave, which propagates in a body, composed of two general part; static displacement part, and also small dynamic displacement part. Happening of the small dynamic displacement of a particle around its static situation, Taylor series expansion can be written around this point. Using this determined static situation and considering only the two first components of Taylor series expansion, the equations of motion will be linearized. Tremendously lengthy algebraic operations involved in the derivation and linearization process, all of the mathematical manipulations are performed using Mathematica.

Key-Words: perturbation- acoustoelasticity- strained cylindrical solids- ultrasonic- wave propagation- Mathematica

1 Introduction

One of the most useful nondestructive methods for stress analyzing, is using Ultrasonic wave propagation in bodies, which is based on difference of wave propagation properties in different stress fields. Using theoretical methods the relation between velocity of wave propagation and strain (stress) value can be obtained.

Considering the importance of this object many scientists are focusing on theoretical and experimental stress analyzing methods with Ultrasonic waves.

Biot[1] in 1940 was the first person who considered the subject scientifically. Hughes and Kelly [2], Henneke[3] and Green[4] are the other pioneers of this method. The most perfect form of equations governing the wave propagation in strained (stressed) bodies that ever obtained is named general motion equation.

The equations of wave propagation in isotropic elastic solids in Cartesian coordinates have been derived by several authors, ether in tensorial notation [5] or matrix notation [4].

The basic concepts and essential expressions used in deriving the equations of wave motion are presented by Murnaghan [6] for the case of finite deformation. In this case the initial and final coordinates of a material point in the undeformed and deformed states respectively cannot be interchanged, as opposed to the case of infinitesimal deformation.

The equations of motion obtained in Lagrangian coordinates are non-linear partial differential equations of several variables.

Obviously the non-linear equations are of little use in applied sciences.

The equations of motion in Cartesian coordinates may be used for the case of one-dimensional solids

(bars), two-dimensional solids (plates) or three-dimensional prismatic solids.
However, for curved objects such as rods and cylindrical shells, the equations may be either derived basically in the cylindrical coordinates or transformed from Cartesian coordinates into cylindrical ones.
A survey of the literature reveals that many researchers have experimented on acoustoelastic effects in objects of cylindrical shapes.
Ultrasonic measurements have been made on a set of steel products including cylindrical forgings and railroad wheels, as done by Deputat and co-workers [10,11] , Murayama et al. [12] and Clark et al. [13] Residual stresses in stainless steel pipe induced by welding processes are determined using longitudinal and Rayleigh waves by Talana et al. [14] .
The acoustoelastic method has been applied to obtain the residual stress distribution in a shrink fit specimen of steel by Kobori and Iwashimizu [15].
In situ measurements of biaxial stresses in pressure tanks and water pipes have been carried out by Fukuoka and Toda [16] using oblique incident shcar waves polarized in the horizontal direction.
Critically refracted longitudinal waves have been used to determine the residual stresses in ductile iron round bars by Srinivasan et al. [17] and also in tempered steel piping samples by Leon-Sala- manca and Nixon [18].
Residual stress evaluation in turbine shafts as well as applied stress evaluation in screws and bolts has been carricd out using guided shear and Lamb waves by Wilband et al. [19].
In recent years more effects and more investigations are arranged in this field, for example stress, strain determination and texture using of ultrasonic velocity measurements is studied by MacDonald [20]. Also Finite Element Modeling of Transient Ultrasonic Waves in Linear Viscoelastic Media is studied by Stucky et al. [21].
Aiello et al., also, have an investigation in finite element analysis of elastic transient ultrasonic wave propagation for NDT applications[22].
Cannas et al. have also a new model of neural NDT by means of Reflected Longitudinal and Torsional Waves Modes in Long and Inaccessible Pipes[23].
Compact ultrasonic beamformer based on Delta-Sigma modulation, is a sample of new studies in ultrasonic field which is done by Lie et al. [24].
In the present paper, an analytical approach to the problem is presented.
The equations of wave motion are first rederived in Cartesian coordinates in a new compact and concise form in order to save space and then transformed into cylindrical coordinates and finally linearized using a simple perturbation method based on the Taylor series expansion.
Because of the tremendously lengthy algebraic operations involed in the derivation and linearization process, and also the huge volume of the mathematical manipulations in the present work, all of the mathematical manipulations are performed using a computer software package which is named as Mathematica [8].

2 Problem Formulation

The principle of acoustoelastisity is based on the deflection of wave-velocity changes by a change in internal stress field.
When ultrasonic waves traverse a stressed body, the total displacements of any particle consist of a static displacement (due to the stress) and a dynamic one (due to the wave propagation).
In this paper the goal is to derive the equations relating ultrasonic wave velocities to the strains (and hence stresses) objects.
In order to derive and analyze the required relationships for different bodies, the equations of particle motion should be expressed in cylindrical coordinates or another form such as spherical coordinates.
These equations in isotropic elastic solids for limited deflections have been derived by several authors [6], using tensorial notation in a continuum mechanics media [7] and supposing Cartesian coordinates (x_1, x_2, x_3), the motion equation component in 1 direction (u-direction) derived as:

$$
\begin{aligned}
\rho_0 \ddot{u}_1 = (2\mu+\lambda)[& u_{1,11} + u_{2,12} + u_{3,13} + u_{1,1}(3u_{1,11} \\
& + u_{1,22} + u_{1,33} + u_{2,12} + u_{3,13}) + u_{2,1}u_{2,11} + u_{3,1}u_{3,11} \\
& + u_{1,2}(u_{2,22} + 2u_{1,12} + u_{3,23}) \\
& + u_{2,2}(u_{1,11} + u_{1,22} + u_{1,33} + u_{2,12}) + u_{3,2}u_{3,12} \\
& + u_{1,3}(u_{3,33} + 2u_{1,13} + u_{2,23}) + u_{2,3}u_{2,13} \\
& + u_{3,3}(u_{1,11} + u_{1,22} + u_{1,33} + u_{3,13})] \\
+ \mu[& u_{1,22} + u_{1,33} + u_{2,12} - u_{3,13} - u_{1,1}(u_{2,12} + u_{3,13}) \\
& + u_{2,1}(u_{2,22} + u_{2,33} + 2u_{1,12}) \\
& + u_{3,1}(u_{3,22} + u_{3,33} + 2u_{1,13}) + u_{1,2}(u_{2,11} + u_{2,33} - u_{3,23}) \\
& + u_{2,2}(-2u_{1,11} - 2u_{1,33} - u_{2,12}) \\
& + u_{3,2}(2u_{1,23} - u_{3,12}) + u_{3,3}(-2u_{1,11} - 2u_{1,22} - u_{3,13})]
\end{aligned}
$$

$$
\begin{aligned}
&+2(l+2m)[(u_{1,1}+u_{2,2}+u_{3,3})(u_{1,11}+u_{2,12}+u_{3,13})] \\
&+m[u_{1,1}(u_{1,22}+u_{1,33}-3u_{2,12}-3u_{3,13}) \\
&+(u_{1,2}+u_{2,1})(u_{2,11}+u_{2,22}+2u_{1,12}+2u_{3,13}) \\
&+u_{3,1}(u_{3,11}+u_{3,33}+2u_{1,13}+2u_{2,23}) \\
&+u_{2,2}(-4u_{1,11}+u_{1,22}-u_{1,33}-5u_{3,13}-3u_{2,12}) \\
&+u_{3,2}(u_{2,13}+u_{3,12})+u_{1,3}(+u_{3,11}+u_{3,33}+2u_{1,13}+u_{2,33}) \\
&+u_{2,3}(u_{2,13}+u_{3,12}) \\
&+u_{3,3}(-4u_{1,11}+u_{1,33}+3u_{3,13}-5u_{2,12})] \\
&+(n/4)[(u_{1,2}+u_{2,1})(u_{2,33}-u_{3,23})+(u_{1,3}+u_{3,1})(u_{3,22}-u_{2,23}) \\
&+(u_{2,3}+u_{3,2})(2u_{1,23}-u_{2,13}-u_{3,12}) \\
&+2u_{2,2}(u_{3,13}-u_{1,33})+2u_{3,3}(u_{2,12}-u_{1,22})]
\end{aligned}
$$

Where ρ_0 is density of the body before deflection, (1, 2, 3) are principal direction in Cartesian coordinates system, (,) is the partial derivative and u_i is the component of displacement in i direction.
The equations of motion in the other two directions, 2 and 3 can be obtained by a circular permutation of the subscripts 1, 2 and 3 in equation (1).
There are 27 partial derivatives($U_{1,1}, U_{1,2}, ..., U_{3,33}$) of displacement functions in each of these non-linear partial differential equations.
These equations are achieved in Cartesian coordinates system and they could be derived in other coordinates system.
Here, because of the main purpose of the paper, the relation between Ultrasonic waves velocity and strain (stress) in cylindrical shells, the motion equations convert to cylindrical coordinate system.

3 Problem Solution

3.1 The Principals of Transforming From Cartesian Coordinates System in to Cylindrical

As it mentioned before, the cylindrical coordinates system is proper to derive the properties of cylindrical and cone shells which have curvature in one direction and because the equations of motion are functions of partial first and second order derivatives, we need to change them from Cartesian into cylindrical coordinates system.
By using the relation between independent parameters in cylindrical coordinates (θ, z) and Cartesian coordinates(x, y, z), transforming relations are determined. Because of writing in tensorial form in many equations, the triplet (x_1, x_2, x_3) is used instead of (x, y, z) [7].
The relations of a point coordinate in Cartesian and cylindrical systems may be written as below;

$$
\begin{aligned}
x_1 &= r\cos\theta, \\
x_2 &= r\sin\theta, \\
x_3 &= z
\end{aligned}
$$

And so the variables of cylindrical coordinate system are as follows:

$$
\begin{aligned}
r &= (x_1^2 + x_2^2)^{1/2}, \\
\theta &= \arctan\left(\frac{x_2}{x_1}\right), \\
z &= x_3
\end{aligned}
\qquad (9)
$$

So:

$$
\frac{\partial r}{\partial x_1} = \frac{x_1}{r} = \cos\theta, \quad
\frac{\partial r}{\partial x_2} = \frac{x_2}{r} = \sin\theta, \quad
\frac{\partial r}{\partial x_3} = 0
$$

And also:

$$
\frac{\partial \theta}{\partial x_1} = \frac{-\sin\theta}{r}, \quad
\frac{\partial \theta}{\partial x_2} = \frac{\cos\theta}{r}, \quad
\frac{\partial \theta}{\partial x_3} = 0
$$

$$
\frac{\partial z}{\partial x_1} = 0, \quad \frac{\partial z}{\partial x_2} = 0, \quad \frac{\partial z}{\partial x_3} = 1
$$

So where f is a function of first order partial derivatives in Cartesian coordinate, by using the chain rule of differentiation we can derive the first order partial derivatives in cylindrical coordinate as follow:

$$\frac{\partial f}{\partial x_1}=\frac{\partial f}{\partial r}.\frac{\partial r}{\partial x_1}+\frac{\partial f}{\partial \theta}.\frac{\partial \theta}{\partial x_1}+\frac{\partial f}{\partial z}.\frac{\partial z}{\partial x_1}$$

Substituting the above values in this equation we can write:

$$\frac{\partial f}{\partial x_1}=\cos\theta\frac{\partial f}{\partial r}-\frac{\sin\theta}{r}\frac{\partial f}{\partial \theta} \qquad (2)$$

At the way the principal of other first order partial derivatives may achieved:

$$\frac{\partial f}{\partial x_2}=\sin\theta\frac{\partial f}{\partial r}+\frac{\cos\theta}{r}\frac{\partial f}{\partial \theta} \qquad (3)$$

$$\frac{\partial f}{\partial x_3}=\frac{\partial f}{\partial z} \qquad (4)$$

For deriving the second order partial derivatives principals in cylindrical coordinate system, we suppose that f is a function of second order partial in Cartesian coordinate system, so we have:

$$\frac{\partial^2 f}{\partial x_1^2}=\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_1}\right)$$

Now using equation (3), $\frac{\partial^2 f}{\partial x_1^2}$ in cylindrical coordinate system may be written as follows:

$$\frac{\partial^2 f}{\partial x_1^2}=\frac{\partial}{\partial x_1}\left(\cos\theta.\frac{\partial f}{\partial r}-\frac{\sin\theta}{r}.\frac{\partial f}{\partial \theta}\right)$$

$$=\cos\theta\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial r}\right)-\frac{\sin\theta}{r}.\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial \theta}\right)$$

$$=\cos\theta\left(\frac{\partial^2 f}{\partial r^2}.\frac{\partial r}{\partial x_1}+\frac{\partial^2 f}{\partial r\partial\theta}.\frac{\partial \theta}{\partial x_1}+\frac{\partial^2 f}{\partial r\partial z}.\frac{\partial z}{\partial x_1}\right)$$

$$-\frac{\sin\theta}{r}\left(\frac{\partial^2 f}{\partial r\partial\theta}.\frac{\partial r}{\partial x_1}+\frac{\partial^2 f}{\partial \theta^2}.\frac{\partial \theta}{\partial x_1}+\frac{\partial^2 f}{\partial \theta\partial z}.\frac{\partial z}{\partial x_1}\right)$$

Substituting the value of $\frac{\partial r}{\partial x_1}$, $\frac{\partial \theta}{\partial x_1}$ and $\frac{\partial z}{\partial x_1}$ in above relation and summarizing the equation we have:

$$\frac{\partial^2 f}{\partial x_1^2}=\cos^2\theta\frac{\partial^2 f}{\partial r^2}+\frac{\sin 2\theta}{r^2}.\frac{\partial f}{\partial \theta}+\frac{\sin^2\theta}{r}.\frac{\partial f}{\partial r}$$

$$-\frac{\sin 2\theta}{r}.\frac{\partial^2 f}{\partial r\partial\theta}+\frac{\sin^2\theta}{r^2}.\frac{\partial f}{\partial \theta^2}$$

And for the other instance $\frac{\partial^2 f}{\partial x_1\partial x_2}$ in cylindrical coordinate system will be:

$$\frac{\partial^2 f}{\partial x_1\partial x_2}=$$

$$\left(\begin{array}{l}\sin\theta\left(\frac{\partial^2 f}{\partial r^2}.\frac{\partial r}{\partial x_1}+\frac{\partial^2 f}{\partial r\partial\theta}.\frac{\partial \theta}{\partial x_1}+\frac{\partial^2 f}{\partial r\partial z}.\frac{\partial z}{\partial x_1}\right)\\ +\frac{\cos\theta}{r}\left(\frac{\partial^2 f}{\partial r\partial\theta}.\frac{\partial r}{\partial x_1}+.\frac{\partial^2 f}{\partial \theta^2}.\frac{\partial \theta}{\partial x_1}+\frac{\partial^2 f}{\partial \theta\partial z}.\frac{\partial z}{\partial x_1}\right)\end{array}\right)$$

Substituting the value of $\frac{\partial r}{\partial x_1}, \frac{\partial \theta}{\partial x_1}, \frac{\partial z}{\partial x_1}$, and summarizing the equation we have:

$$\frac{\partial^2 f}{\partial x_1\partial x_2}=\frac{1}{2}\sin 2\theta\frac{\partial^2 f}{\partial r^2}-\frac{\cos 2\theta}{r^2}\frac{\partial f}{\partial \theta}-\frac{\sin 2\theta}{2r}\frac{\partial f}{\partial r}$$

$$+\frac{\cos 2\theta}{r}\frac{\partial^2 f}{\partial r\partial\theta}-\frac{\sin 2\theta}{2r^2}\frac{\partial^2 f}{\partial \theta^2}$$

And at the same way all of the 27 partial derivatives may be obtained.

By substituting these partial derivatives in Cartesian motion equation, the cylindrical form of motion equation will obtained.

And also motion equation in other direction of cylindrical coordinates ($\rho_0\ddot{u}_\theta, \rho_0\ddot{u}_z$), will be derived. Each of these motion equation components in cylindrical coordinate system are contain of 27 partial derivatives of first and second order.

Due to the tremendously lengthy form of the resulting equations, however, they cannot be presented here in their original form, because of the limited space. Nevertheless, it is possible to change their form to make them shorter.

To accomplish this, the following change of notation is adopted:

$$(r,\theta,z) \rightarrow (1,2,3)$$

And by using tensorial notation in a continuum mechanics media [7] we will have:

$$u_r = u_{1,} ,$$

$$u_\theta = u_2 ,$$

$$\frac{\partial u_r}{\partial \theta} = u_{1,2} ,$$

$$\frac{\delta^2 u_z}{\delta r \partial \theta} = u_{3.12}$$

3.2 Linearization Process in Cylindrical Coordinate System

Motion equations in cylindrical coordinate system are equations with nonlinear partial derivatives which for extension and solving some special method should be used.

To accomplish this, cylindrical forms of motion equations are changed to linear form using Taylor series expansion around the displaced static situation. The total displacement at any moment at any point in the elastic medium is composed of two parts: the static part caused by the applied stress and the dynamic part due to the propagating stress wave.

So, Assuming a low amplitude plane wave propagating in an initially isotropic elastic medium, the displacement components in cylindrical coordinates can written as:

$$u_n = \eta_n a_n + m_n A \exp\left[i(\omega t - k_j x_j)\right], n = 1,2,3 \quad (5)$$

Where u_n are the total displacement components (u_1 , u_2 , u_3) along the initial coordinates a_n, η_n are the principal strains(η_1 , η_2 , η_3), m_n are the direction cosines of the polarization vector(m_1 , m_2 , m_3), A is the amplitude of the wave, $i = \sqrt{-1},$ ω is the circular frequency of the wave, t is the time, k_j are components of the wave vector (k_1 , k_2 , k_3) and x_j are the current coordinates (x_1 , x_2 , x_3), related to the initial coordinates a_j ; (a_1 , a_2 , a_3) and at this equation the summing convention refer to the dummy index j only.

Short form of equation (5) is as:

$$u_n = U_n^s + u_n^d$$

Where $U_n^s = \eta_n . a_n$ is static displacement and $u_n^d = m_n A \exp[i(\omega t - k_j x_j)]$ is the small dynamic part of displacement.

The first term in equation (5) denotes the large static displacement u_n^s of an initially isotropic medium subjected to a triaxial strain (stress) field.

The second term represents the very small dynamic deformation u_n^d due to the elastic plane wave propagating through the then anisotropic medium, because the isotropy in wave propagation is already removed by the applied strains (stresses).

The first sentence of equation (5) is supposed from initial coordinates without any deformation (Lagrangian coordinates), but the second term is not so and for transferring of this part to the initial coordinates we have:

$$x_j = a_j + u_j$$

In this equation the components of displacement vector u_j related to the initial coordinates a_j as bellow:

$$u_j = \eta_j . a_j , \qquad j = 1,2,3$$

(Summing convention dose not refer to j),

So:

$$x_j = (1 + \eta_j) a_j$$

(Summing convention dose not refer to j) (6)

The components of the wave vector (k_j) may also be written in terms of the wave number (k) and the direction cosines of the wave normal (l_j) as:

$$k_j = \frac{2\pi}{\lambda} l_j = k_j l_j \qquad j = 1,2,3 \qquad (7)$$

Where λ is the wavelength.

Substituting from equations (6) and (7) into equation (5), the three components of the total displacement (u_1 , u_2 , u_3) in cylindrical coordinates related to strain components (η_1 , η_2 , η_3), the direction cosines of the wave normal (l_1, l_2, l_3) and the direction cosines of the polarization vector(m_1 , m_2 , m_3), that all referred to the initial cylindrical coordinates, as:

$$u_1 = \eta_1 a_1 + m_1 e^{i\omega t}.A\exp\left\{-ik\begin{bmatrix}(1+\eta_1)a_1 l_1 \\ +(1+\eta_2)a_2 l_2 + (1+\eta_3)a_3 l_3\end{bmatrix}\right\}$$

$$= \eta_1 a_1 + m_1 e^{i\omega t}.F$$

$$u_2 = \eta_2 a_2 + m_2 e^{i\omega t}.F \;,$$

$$u_3 = \eta_3 a_3 + m_3 e^{i\omega t}.F \quad (8)$$

Where:

$$F = A\exp\{-ik[(1+\eta_1)a_1 l_1 + (1+\eta_2)a_2 l_2 + (1+\eta_3)a_3 l_3]\}$$

Equation (8) then represents the superposition of a very small dynamic displacement upon a large static deformation.
Equation (8) may, therefore, be written in short as:

$$u_1 = U_1^S + u_1^d \qquad u_1^d << U_1^s$$

$$u_2 = U_2^S + u_2^d \qquad u_2^d << U_2^s$$

(9-a)

$$u_3 = U_3^S + u_3^d \qquad u_3^d << U_3^s$$

With the above considerations, it is now possible to proceed with the linearization process.
Referring bake to the first component of the equation of motion, it can be written in the following general form in terms of the 27 variables $u_{i,j}$ and $u_{i,jk}$ as:

$$\rho_0 \ddot{u}_1 = f(u_{1,1}, u_{1,2}, u_{1,3}, u_{2,1}, u_{2,2}, ..., u_{3,3}, u_{1,11}, u_{1,12}, u_{1,13}, u_{1,22}, ..., u_{3,32}, u_{3,33}) \quad (9)$$

Or, in short form, as:

$$\rho_0 \ddot{u}_1 = f(u_{i,j}, u_{i,jk})$$

Noting that $u_{i,jk} = u_{i,kj}$ similarly, for the two remaining components of the equation of motion,

$$\rho_0 \ddot{u}_2 = g(u_{i.j}, u_{i,jk}) \quad (10)$$

$$\rho_0 \ddot{u}_3 = h(u_{i,j}, u_{i,jk}) + \left[(\xi\partial/\partial u_{1,1} + \zeta\partial/\partial u_{1,2} + \gamma\partial/\partial u_{1,3} + ... + \varepsilon\partial/\partial u_{1,11} + ... + \rho\partial/\partial u_{3,33})\; f\right]\Big|_{stat}.$$

(11)

The three functions f, g and h in equations (9), (10) and (11) are analytic functions and may be expanded Taylor series around bellow static deformation values:

$$(U_{1,1}^S, U_{1,2}^S, U_{1,3}^S, U_{2,1}^S, ..., U_{1,11}^S, U_{1,12}^S, ..., U_{3,33}^S)$$

So Taylor series will be written as:

$$\rho_0 \ddot{u}_1 = f(u_{1,1}, u_{1,2}, u_{1,3}, u_{2,1}, u_{2,2}, ..., u_{1,11}, u_{1,12}, ..., u_{3,33})$$
$$= f(u_{1,1}^s, u_{1,2}^s, u_{1,3}^s, u_{2,1}^s, u_{2,2}^2, ..., u_{1,11}^s, u_{1,12}^s, ..., u_{3,33}^s)$$
$$+ \left[(\xi\partial/\partial u_{1,1} + \zeta\partial/\partial u_{1,2} + \gamma\partial/\partial u_{1,3} + ... + \varepsilon\partial/\partial u_{1,11} + ... + \rho\partial/\partial u_{3,33})\; f\right]\Big|_{stat}.$$

(12)

$$+ \frac{1}{2!}\left[(\xi\partial/\partial u_{1,1} + \zeta\partial/\partial u_{1,2} + \gamma\partial/\partial u_{1,3} + ... + \varepsilon\partial/\partial u_{1,11} + ... + \rho\partial/\partial u_{3,33})^2 f\right]\Big|_{stat}.$$

+Negligible HOT

Where we use equation (9), to expanded Taylor series as bellow:

$$\xi = u_{1.1} - U_{1.1}^s = u_{1,1}^d$$

$$\zeta = u_{1,2} - U_{1,2}^s = u_{1,2}^d$$

$$\gamma = u_{1,3} - U_{1,3}^s = u_{1,3}^d$$

.
.
.

$$\varepsilon = u_{1,11} - U_{1,11}^s = u_{1,11}^d$$

.
.
.

$$\rho = u_{3,33} - U^s_{3,33} = u^d_{3,33}$$

The symbol $\big|_{stat}$ implies evaluation of the partial derivatives for the static values u^s_n and f, g functions in equation (10), (11) may be extend at the same way.

Saving the two first sentences of right part of equation (12) we can substitute related derivatives and using equation (8) the static displacements and their derivatives are obtained as follow:

$$U^s_1 = \eta_1\alpha_1,$$
$$U^s_2 = \eta_2\alpha_2,$$
$$U^s_3 = \eta_3\alpha_3$$
$$U^s_{1,1} = \eta_1,$$
$$U^s_{2,2} = \eta_2,$$
$$U^s_{3,3} = \eta_3,$$
$$U^s_{1,2} = U^s_{1,3} = 0,$$
$$U^s_{2,1} = U^s_{2,3} = 0,$$
$$U^s_{3,1} = U^s_{3,2} = 0$$

Thus the values of all second order derivatives in static situation are equal to zero.

Now considering the mentioned points and equation (12) the equation of motion will be achieved.

First term of this equation is related to motion equation in static deformation state and is equal to zero.

Thus:

$$f(U^s_{1,1}, U^s_{1,2}, U^s_{1,3}, U^s_{2,1}, ..., U^s_{1,11}, U^s_{1,12}, ..., U^s_{3,33}) = 0$$

This point could be deduced from the physics of the problem, so equation (12) wall be summarized as:

$$\rho_0 \ddot{u}_1 = \left[\begin{array}{l} (u^d{}_{1,1}\frac{\partial}{\partial u_{1,1}} + u^d{}_{1,2}\frac{\partial}{\partial u_{1,2}} + u^d{}_{1,3}\frac{\partial}{\partial u_{1,3}} + \\ ... + u^d{}_{1,11}\frac{\partial}{\partial u_{1,11}} + ... + u^d{}_{3,33}\frac{\partial}{\partial u_{3,33}}) f \end{array} \right]_{stat}$$

$$+\text{Negligible HOT} \tag{13}$$

At the same way the motion equations in other two directions (2, 3) will be summarized and the results are similar to equation (13).

The 27 partial derivatives required in the second part of equation (13) are all determined and substitute in the equation at static position to linearizing the equation. Ensuring the correctness, the software package Mathematica is employed for determination of these derivatives.

For example,

$$\frac{\partial f}{\partial u_{1,1}}\Big|_{stat} = \eta_3\lambda\cos^2\theta + \eta_3 m\cos^2\theta + \mu\cos^2\theta - \frac{\eta_3 n\cos^2\theta}{2}$$
$$+2\eta_1\lambda\cos^3\theta + 2\eta_1 m\cos^3\theta + 4\eta_1\mu\cos^3\theta + 2\eta_3 l\sin^2\theta$$
$$+\lambda\sin^2\theta + \eta_3\lambda\sin^2\theta + 2\mu\sin^2\theta + 4\eta_1 l\cos\theta\sin^2\theta$$
$$+2\eta_1\lambda\cos\theta\sin^2\theta + 4\eta_1 m\cos\theta\sin^2\theta + 4\eta_1\mu\cos\theta\sin^2\theta$$
$$+3\eta_1 m\cos\theta\sin^2\theta + \eta_2\mu\cos\theta\sin^2\theta$$

The other derivates in this direction and the other directions are also determined at the same way. For substituting $u^d_{1,1}$ to $u^d_{3,33}$ in equations (13) we use equation (8) as follow:

$$u^d_{1,1} = m_1[-ik(1+\eta_1)l_1]e^{i\omega t}.F$$
$$u^d_{1,2} = m_1[-ik(1+\eta_2)l_2]e^{i\omega t}.F \tag{14}$$

.
.
.

$$u^d_{1,12} = m_1[-k^2(1+\eta_1)(1+\eta_2)l_1 l_2]e^{i\omega t}.F$$

.
.
.

$$u^d_{3,33} = m_3[-k^2(1+\eta_3)^2 l_3^2]e^{i\omega t}.F$$

At the other hand time-rate of change of deformation is achieved from equation (8) as:

$$\ddot{u}_1 = -m_1\omega^2 e^{i\omega t}.F \tag{15}$$

Substituting from equations (14) to (15) into equation (13), canceling the common multiplier $(-Fe^{i\omega t})$ from both sides, dividing the resulting equation by k^2 and taking $\omega^2/k^2 = V^2$, the end result will be as:

$$\rho_0 m_1 V^2 = F(\eta_1, \eta_2, \eta_3, l, m, n, \lambda, \mu, l_1, l_2, l_3, m_1, m_2, m_3, \theta)$$

Because of being extremely lengthy, one more summarizes process is applied on the equation before of presentation.

Considering a perfect symmetric shell, stress, strain and the related equations are independent from the value of θ, so we can suppose $\theta = 0$ for more simplification. And finally the linearized equation of motion in 1 direction, after the simplification and neglecting of the higher order strain statements, will be as bellow:

$$\begin{aligned} m_1 \rho_0 V^2 &= m_1 \left[(\lambda + ml_2^2 + ml_3^2 + 2ll_1^2)\gamma \right] \\ &+ 4ml_1^2\eta_1 + \lambda l_1^2 (l + 4\eta_1) + 2\mu l_1^2 (1 + 5\eta_1) \\ &+ \mu l_2^2 (l + 2\eta_1 + 4\eta_2) + \mu l_3^2 (l + 2\eta_1 + 4\eta_3) \\ &- \frac{n}{2} (l_2^2 \eta_3 + l_3^2 \eta_2)] \\ &+ m_2 \left[l_1 l_2 (2\mu + 2\lambda + 4l\gamma + 2m\gamma - 4m\eta_3 + 4\lambda\eta_1 \right. \\ &+ 4\lambda\eta_2 + 4\lambda\eta_1 + 4\mu\eta_2 + n\eta_3)] \\ &+ m_3 [l_1 l_3 (2\mu + 2\lambda + 4l\gamma + 2m\gamma - 4m\eta_2 + 4\lambda\eta_1 \\ &+ 4\lambda\eta_3 + 4\mu\eta_1 + 4\mu\eta_3 + n\eta_3)] \qquad (16) \end{aligned}$$

Where:

$$\gamma = \eta_1 + \eta_2 + \eta_3$$

The new quantity V is the velocity of wave propagation in the strained (stressed) solid and the last equation is linearized in 1 direction.

Performing the same lengthy process as above, the other two linearized components of the equation of motion in (2 , 3) direction are obtained:

$$\begin{aligned} m_2 \gamma_0 V_2^2 &= m_1 [l_1 l_2 (2\mu + 2\lambda + 4l\gamma + 2m\gamma - 4m\eta_3 \\ &+ 4\lambda\eta_2 + 4\mu\eta_1 + 4\mu\eta_2 + n\eta_3)] \qquad (17) \\ &+ m_2 [(\lambda + ml_1^2 + ml_3^2 + 2ll_2^2)\gamma \\ &+ 4ml_2^2\eta_2 + \lambda l_2^2 (l + 4\eta_2) + 2\mu l_2^2 (1 + 5\eta_2) \\ &- n / 2 (l_1^2 \eta_3 + l_3^2 \eta_1)] \end{aligned}$$

4 Conclusion

As it mentioned before, When ultrasonic waves traverse a stressed body, the total displacements of any particle consist of a static displacement (due to the stress) and a dynamic one (due to the wave propagation).

In other words because of the vibration of particles is happened due to wave propagation around the static position of it, the Taylor series expansion and the partial nonlinear derivatives may be written around this point.

Considering the two first terms of Taylor series in last equations, the equations are changed to linear form as they expressed here.

This results in an eigenvalue problem which is solved for the three eigenvalues to yield one quasi-longitudinal and two quasi-shear wave velocities.

The next exact numerical calculations imply that linearizing error of these equations and neglecting of the next terms of Taylor series is so tiny.

For example the value of this error for a steel cylinder is less than 1×10^{-5} and for an aluminum cylinder is less than 1.5×10^{-5}.

The obtained linearized equation of this paper can be used widely in physics, mathematics and engineering. For example the equations (16) to (18) are applied to stress analyzing by ultrasonic waves transmitting as the following form:

$$\begin{cases} (\lambda_{11} - \rho_0 V^2) m_1 + \lambda_{12} m_2 + \lambda_{13} m_3 = 0 \\ \lambda_{21} m_1 + (\lambda_{22} - \rho_0 V^2) m_2 + \lambda_{23} m_3 = 0 \\ \lambda_{31} m_1 + \lambda_{32} m_2 + (\lambda_{33} - \rho_0 V^2) m_3 = 0 \end{cases}$$

So the solution of motion equations for measuring of the wave velocities from elasticity constants and strains (stresses) affects will tend to determination of eigen values of the following matrix:

$$\begin{bmatrix} \lambda_{11} - \rho_0 V^2 & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} - \rho_0 V^2 & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} - \rho_0 V^2 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = 0$$

Thus for any $\begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}$ vector the determinant of the matrix should be equal to zero:

$$\begin{vmatrix} \lambda_{11}-\rho_0 V^2 & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22}-\rho_0 V^2 & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33}-\rho_0 V^2 \end{vmatrix} = 0$$

At the general form there are three roots of ($\rho_0 V^2$) for this equation, so $\rho_0 V^2$ and the resulting value of propagated wave's velocity will achieved.

One of the three mentioned values is related to quasi-longitudinal and the other two roots are belonged to quasi-transverse wave.

So obtaining of this determinant the relation between wave velocity and the related strain (stress) will achieved and the goal of this investigation, which is to derive the equations relating ultrasonic wave velocities to the strains (and hence stresses) objects, will be reached.

It should be mentioned that as we said before, because of the tremendously lengthy algebraic operations involed in the derivation and linearization process, and also the huge volume of the mathematical manipulations in the present work, all of the mathematical manipulations are performed using a software package Mathematica.

These equations, (the obtained equations at the final stages of this study), are not restricted as to the angle of incidence of the wave, so that by varying the incident angle, numerous information regarding the mechanical properties of the body is obtainable easily.

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