# Analytical solutions for a nonlinear coupled pendulum

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*Abstract:* - In this paper, the motion of two pendulums coupled by an elastic spring is studied. By extending the linear equivalence method (LEM), the solutions of its simplified set of nonlinear equations are written as a linear superposition of Coulomb vibrations. The inverse scattering transform is applied next to exact set of equations. By using the $\Theta$ - function representation, the motion of pendulum is describable as a linear superposition of cnoidal vibrations and additional terms, which include nonlinear interactions among the vibrations. Comparisons between the LEM and cnoidal solutions and comparisons with the solutions obtained by the fourth-order Runge-Kutta scheme are performed. Finally, an interesting phenomenon is put into evidence with consequences for dynamic of pendulums.

*Key-Words:* cnoidal method, linear equivalence method, cnoidal vibrations, Coulomb vibrations, coupled pendulum.

## 1 Introduction

Since the original paper by Korteweg and deVries, there has remained an open fundamental question [1]: *if the linearized equation can be solved by an ordinary Fourier series as a linear superposition of sine waves, can the equation itself be solved by a generalization of Fourier series which uses the cnoidal wave as the fundamental basis function?*

This paper belongs of a series of papers, which addresses an original, practical and concrete resolution of this old problem. Starting with the above idea, we attach to the coupled pendulum's motion equations two sets of nonlinear differential equations: an exact one and a simplified one. The capability of the linear equivalence method (LEM) [2]-[8] is extended to the analysis of the simplified system of equations. The LEM representation of the solutions is describable as a linear superposition of Coulomb vibrations.

The analysis of these LEM solutions allows us to solve further the exact nonlinear system of equations by using a generalization of Fourier series (the cnoidal method). The cnoidal method uses the cnoidal waves as the fundamental basis function [9]-[11]. The $\Theta$ -function representation of the solutions is derived as a linear superposition of Jacobean elliptic functions (cnoidal vibrations) and additional terms, which include nonlinear interactions among the vibrations. The cnoidal vibrations are much richer than sine vibrations; i. e. the modulus $m$ of the cnoidal vibration $(0 \le m \le 1)$ can be varied to obtain a sine vibration $(m \cong 0)$, Stokes vibration $(m \cong 0.5)$ or soliton vibration $(m \cong 1)$.

In order to clarify the essence of the proposed methods, we note that both methods are applicable to the analysis of complex dynamical systems that have non-simple-harmonic solutions.

It is the case of the nonlinear differential equations having algebraic nonlinearities [12]

$$\dot{z}_n = Az_n + \sum_{i=1}^{N} F_{in}(z_n) \qquad (1)$$

where

$$Az_n = \sum_{p=1}^{N} a_{np} z_p \,, \quad F_{1n}(z) = \sum_{p,q=1}^{N} b_{npq} z_p z_q \,,$$

$$F_{2n}(z) = \sum_{p,q,r=1}^{N} c_{npqr} z_p z_q z_r \,,$$

$$F_{3n}(z) = \sum_{p,q,r,l=1}^{N} d_{npqr} z_p z_q z_r z_l \,,$$

$$F_{4n}(z) = \sum_{p,q,r,l,m=1}^{N} e_{npqrlm} z_p z_q z_r z_l z_m \,,$$

with $n = 1, 2, \ldots.$ The general theory of integrable conservative systems due to Hamilton-Jacobi and the formulation of Morino [13] and Smith and

Morino [14] predict only simple harmonic limit-cycle solutions.

The singular perturbation method known as the *Lie transformation* method introduced by Morino, Mastroddi and. Cutroni [15] can be extended to the analysis of dynamical systems capable of producing not-so-regular vibrations, because not only the zero-divisor terms, but certain small-divisor terms are included into analysis.
So, the Lie transformation method is applicable to the Bolotin systems of equations, but is not adequate for the more complex systems as (1), because of the non-standard type of involved nonlinearities.

There are many other nonlinear differential equations like (1) of physical importance that admit such kind of solutions. We think that the cnoidal method can be successfully applied to a wider class of nonlinear equations [16], [17].

## 2 Formulation of the problem

Fig. 1 shows a coupled pendulum consisted from two straight rods $O_1Q_1$, $O_2Q_2$ of masses $M_1$, $M_2$, lengths $O_1Q_1 = O_2Q_2 = a$, and mass centres $C_1, C_2$ with $O_1C_1 = l_1$, $O_2C_2 = l_2$ and $O_1O_2 = l$. The rods are linked together by an elastic spring $Q_1Q_2$, $Q_1 \in O_1C_1$, $Q_2 \in O_2C_2$ characterised by an elastic constant $k$. The elastic force in the spring is given by $k\,|O_1O_2 - Q_1Q_2|$.

The kinetic energy $T$ of the system

$$T = \frac{1}{2}(I_1\dot{\theta}_1^2 + I_2\dot{\theta}_2^2) \ , \qquad (2)$$

where $\theta_1$ and $\theta_2$ are the displacement angles in rapport to the verticals, the dot means differentiation with respect to time, $I_1$ is the mass moment of inertia of $O_1O_2$ with respect to $C_1$ and $I_2$ is the mass moment of inertia of $O_3O_4$ with respect to $C_2$.

The elastic potential

$$U = g(M_1 l_1 \cos\theta_1 + M_2 l_2 \cos\theta_2) - \frac{k}{2}(O_1O_2 - Q_1Q_2)^2,$$

with

$$Q_1Q_2^2 = [O_1O_2 + a(\sin\theta_2 - \sin\theta_1)]^2 + a^2(\cos\theta_2 - \cos\theta_1)^2 =$$

$$O_1O_2^2 + 2aO_1O_2(\sin\theta_2 - \sin\theta_1) + 2a^2[1 - \cos(\theta_2 - \theta_1)].$$

From Lagrange equations we derive the motion equations of the pendulum

$$I_1\ddot{\theta}_1 + M_1 g l_1 \sin\theta_1 + \frac{k}{2}\frac{\partial}{\partial\theta_1}(O_1O_2 - Q_1Q_2)^2 = 0 \ ,$$

$$I_2\ddot{\theta}_2 + M_2 g l_2 \sin\theta_2 + \frac{k}{2}\frac{\partial}{\partial\theta_2}(O_1O_2 - Q_1Q_2)^2 = 0 \ , \quad (3)$$

with $g$ the gravitational acceleration. Equations (3) are coupled and nonlinear. By substituting (2) into (3) we have

$$I_1\ddot{\theta}_1 + M_1 g l_1 \sin\theta_1 - kH[-al\cos\theta_1 - a^2\sin(\theta_2 - \theta_1)] = 0,$$

$$I_2\ddot{\theta}_2 + M_2 g l_2 \sin\theta_2 - kH[al\cos\theta_2 + a^2\sin(\theta_2 - \theta_1)] = 0, \quad (4)$$

where

$$\mathrm{H}(\theta_1, \theta_2) = \frac{l - \Psi(\theta_1, \theta_2)}{\Psi(\theta_1, \theta_2)}, \quad \Psi(\theta_1, \theta_2) = Q_1Q_2 = \sqrt{A} \ ,$$

$$A = l^2 + 2al(\sin\theta_2 - \sin\theta_1) + 2a^2(1 - \cos(\theta_2 - \theta_1))$$

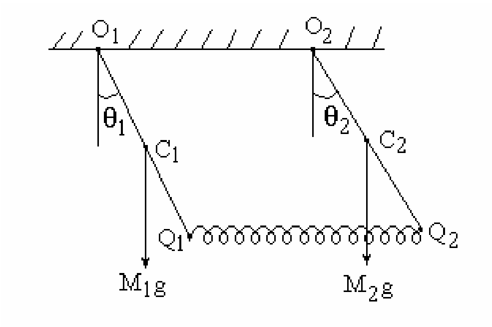


Fig. 1. The coupled pendulum.

Defining the dimensionless variable $\tau = t\sqrt{\frac{k}{M_1}}$ and introducing the notations

$$\frac{M_1^2 g l_1}{I_1 k} = w, \quad \frac{M_1 M_2 g l_2}{I_2 k} = \beta w, \quad \Phi = \frac{\Psi}{l}$$

$$\frac{a}{l} = \xi, \quad \frac{AM_1}{kI_1} = \delta, \quad \frac{alM_1}{I_1} = \alpha, \quad \frac{alM_1}{I_2} = \tilde{\alpha},$$

the equations (4) are reduced to the dimensionless equations

$$\ddot{\theta}_1 + w\sin\theta_1 + \gamma\alpha[\cos\theta_1 + \xi\sin(\theta_2 - \theta_1)] = 0 \ ,$$

$$\ddot{\theta}_2 + \beta w\sin\theta_2 - \gamma\tilde{\alpha}[\cos\theta_2 + \xi\sin(\theta_2 - \theta_1)] = 0 \ , \quad (5)$$

where the dot means the differentiation with respect to $\tau$ and

$$\gamma(\theta_1, \theta_2) = \Phi^{-1/2} - 1 \ , \qquad (6)$$

$$\Phi(\theta_1,\theta_2) = 1 + 2\xi(\sin\theta_2 - \sin\theta_1) + 2\xi^2(1-\cos(\theta_2-\theta_1)).$$

The associated initial conditions are

$$\theta_1(0) = \theta_1^0,\ \ \theta_2(0) = \theta_2^0,\qquad \dot\theta_1(0) = \theta_{p1}^0,\ \ \dot\theta_2(0) = \theta_{p2}^0. \tag{7}$$

We begin with constructing a simplified form of (6) by taking

$$\gamma(\theta_1,\theta_2) = -\xi(\sin\theta_2 - \sin\theta_1) - \xi^2(1-\cos(\theta_2-\theta_1)), \tag{8}$$

with

$$|\,2\xi(\sin\theta_2 - \sin\theta_1) + 2\xi^2(1-\cos(\theta_2-\theta_1))\,| < 1,$$

verified for $\xi \le 0.3$. By using (8) the substantial simplifications are made that restricts the accuracy of the approach. But the simplified set of equations solved by the LEM method will help us to understand and to solve the exact set of equations by using the cnoidal method. By noting

$$\theta_1 = z_1,\quad \theta_2 = z_2,\ \dot\theta_1 = z_3,\quad \dot\theta_2 = z_4,$$

the equations (9) become

$$\dot z_1 = z_3,\ \dot z_2 = z_4,$$

$$\dot z_3 = -w\sin z_1 - \gamma\alpha[\cos z_1 + \xi\sin(z_2 - z_1)],$$

$$\dot z_4 = -\beta w\sin z_2 + \gamma\tilde\alpha[\cos z_2 + \xi\sin(z_2 - z_1)], \tag{9}$$

with the initial conditions (7)

$$z_1(0) = z_1^0,\ z_2(0) = z_2^0,\ z_3(0) = z_3^0,\ z_4(0) = z_4^0, \tag{10}$$

We refer to (5) and (7) (or (9) and (10)) as the exact motion equations for the pendulum. Substituting for $\gamma(z_1,z_2)$ given by (8) in (9) we can obtain a simplified form of the pendulum motion equations. In addition, for $|z_p| \le \frac{\pi}{2}$ and approximating the trigonometric functions by polynomials of five-order

$$\sin z = z + \tilde a z^3 + \tilde b z^5 + \varepsilon(z),\quad |\varepsilon(z)| \le 2\times 10^{-4},$$
$$\tilde a = -0.16605,\quad \tilde b = 0.00761$$
$$\cos z = 1 + \tilde c z^2 + \tilde d z^4 + \varepsilon(z),\quad |\varepsilon(z)| \le 9\times 10^{-4},$$
$$\tilde c = -0.49670,\ \tilde d = 0.03705,$$

the system of equations (9) can be written as

$$\dot z_1 = z_3,\ \dot z_2 = z_4,$$

$$\dot z_3 = -wP(z_1) - \alpha Q_1(z_1,z_2)\gamma(z_1,z_2),$$

$$\dot z_4 = -\beta wP(z_2) + \tilde\alpha Q_2(z_1,z_2)\gamma(z_1,z_2), \tag{11}$$

with

$$P(z) = z + \tilde a z^3 + \tilde b z^5,$$
$$Q_1(z_1,z_2) = R_1(z_1) + R_2(z_1,z_2),$$
$$Q_2(z_1,z_2) = R_1(z_2) + R_2(z_1,z_2),$$
$$R_1(z) = 1 + \tilde c z^2 + \tilde d z^4,$$
$$\gamma(z_1,z_2) = -1 + f^{-1/2}(z_1,z_2),$$

$$R_2(z_1,z_2) = -\xi z_1 + \xi z_2 + \tilde a\xi z_2^3 - 3\tilde a\xi z_2^2 z_1 + 3\tilde a\xi z_2 z_1^2 - \tilde a\xi z_1^3 + \tilde b\xi z_2^5 - 5\tilde b\xi z_2^4 z_1 + 10\tilde b\xi z_2^3 z_1^2 - 10\tilde b\xi z_2^2 z_1^3 + 5\tilde b\xi z_2 z_1^4 - \tilde b\xi z_1^5,$$

$$f(z_1,z_2) = 1 - 2\xi z_1 + 2\xi z_2 + 4\tilde c\xi^2 z_1 z_2 - 2\tilde c\xi^2 z_1^2 - 2\tilde c\xi^2 z_2^2 - 2\tilde a\xi z_1^3 + 2\tilde a\xi z_2^3 - 2\tilde d\xi^2 z_2^4 - 2\tilde d\xi^2 z_1^4 + 8\tilde d\xi^2 z_2^3 z_1 - 12\tilde d\xi^2 z_2^2 z_1^2 + 8\tilde d\xi^2 z_2 z_1^3 + 2\tilde b\xi z_2^5 - 2\tilde b\xi z_1^5.$$

On inserting (8) in (11), a system (1) is obtained for $n = 1,2,3,4$ and $N = 4$. We refer to (1) and (10) as the simplified motion equations for the pendulum. The matrix $A = a_{np}$ is

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\alpha\xi - w & \alpha\xi & 0 & 0 \\ \tilde\alpha\xi & -\tilde\alpha\xi - \beta w & 0 & 0 \end{bmatrix}, \tag{12}$$

where

$$\tilde a = -0.16605,\quad \tilde b = 0.00761,$$
$$\tilde c = -0.49670,\quad \tilde d = 0.03705,$$

$$a_{13} = 1,\quad a_{24} = 1,$$

$$a_{31} = -\alpha\xi - w,\quad a_{32} = \alpha\xi,$$
$$a_{41} = \tilde\alpha\xi,\quad a_{42} = -\tilde\alpha\xi - \beta w,$$

$$a_{35} = \delta,\ a_{56} = 1,$$
$$b_{312} = -2\alpha\xi^2(1-\tilde c),\ b_{322} = -\alpha\xi^2(\tilde c - 1),$$
$$b_{412} = -2\tilde\alpha\xi^2(\tilde c - 1),\quad b_{422} = -\tilde\alpha\xi^2(1-\tilde c),$$
$$c_{3111} = -\alpha\xi\tilde a - \alpha\xi\tilde c + \alpha\xi^3\tilde c - w\tilde a,$$
$$c_{3112} = -\alpha\xi\tilde c(3\xi^2 - 1),\ c_{3122} = 3\alpha\xi^3\tilde c,$$
$$c_{4122} = -\tilde\alpha\xi\tilde c(3\xi^2 - 1),\ c_{4112} = 3\tilde\alpha\xi^3\tilde c,$$
$$c_{3222} = -\alpha\xi(\xi^2\tilde c - \tilde a),\ c_{4111} = -\tilde\alpha\xi(\xi^2\tilde c - \tilde a),$$
$$c_{4222} = -\tilde\alpha\xi\tilde a - \tilde\alpha\xi\tilde c + \tilde\alpha\tilde c\xi^3 - \beta w\tilde a,$$
$$d_{31111} = -\alpha\xi^2(\tilde d + \tilde c^2 - 2\tilde a),$$
$$d_{31112} = -\alpha\xi^2(\tilde a - 4\tilde d - 2\tilde c^2),$$
$$d_{31122} = -\alpha\xi^2(6\tilde d + \tilde c^2 - 6\tilde a),\ d_{31222} = -\alpha\xi^2(5\tilde a - 4\tilde d),$$

$$d_{32222} = -\alpha\xi^2(\tilde{d} - 2\tilde{a}),\ d_{41111} = -\alpha\xi^2(2\tilde{a} - \tilde{d}),$$

$$d_{41112} = -\tilde{\alpha}\xi^2(4\tilde{d} - 5\tilde{a}),\ d_{41122} = -\tilde{\alpha}\xi^2(6\tilde{a} - 6\tilde{d} - \tilde{c}^2),$$

$$d_{41222} = -\tilde{\alpha}\xi^2(-5\tilde{a} + 4\tilde{d} + 2\tilde{c}^2),\ ,d_{42222} = \tilde{\alpha}\xi^2(\tilde{d} + \tilde{c}^2).$$

## 3 The LEM method

We begin with a brief explain of LEM in the spirit of [2], [3]. Consider a nonlinear Cauchy problem

$$\dot{z} = F(z),\quad z(t_0) = z_0,\quad t_0 \in I \subset \mathrm{R}\ , \qquad (13)$$

where $F$ is a vector of $n$ analytic functions $F_j(z) = \sum_{|v|=1}^{\infty} a_{jv} z^v$ with the coefficients $a_{jv} \in I \subset R$. Here $\nu = (\nu_1, \nu_2, ... \nu_n)$ are multi-indexes of length $n$, and $z^v = \prod_{j=1}^{n} z_j^{\nu_j}$.The LEM method consists in introduction of a new variable $v(t,\sigma)$ defined as an exponential transformation of real parameters $\sigma$

$$v(t,\sigma) = e^{<\sigma,z>},\ <\sigma,z> = \sum_{j=1}^{n} \sigma_j z_j\ , \qquad (14)$$

where $\sigma = [\sigma_1, \sigma_2, ... \sigma_n] \in R^n$. Using (14), the system (13) is transformed into a partial linear differential equation

$$\frac{\partial v}{\partial t} - \sum_{j=1}^{N_s} \sigma_j F_j(D) v = 0\,, \qquad (15)$$

where $F_j(t,D)$ is the formal differential operator associated to $F_j(\sigma)$

$$F_j(D) = \sum_{|\nu|=1}^{\infty} a_{j\nu} \frac{\partial^{|\nu|}}{\partial\sigma^{\nu}},\ \frac{\partial^{|\nu|}}{\partial\sigma^{\nu}} = \frac{\partial^{\nu_1+\nu_2...+\nu_n}}{\partial\sigma_1^{\nu_1}\partial\sigma_2^{\nu_2}...\partial\sigma_n^{\nu_n}}\,. \qquad (16)$$

The linear equivalence transformation (14), where $z_n$, $n = 1,2,...,6$, is a solution of (13) and satisfies the partial linear differential equation (15) and initial conditions

$$v(t_0) = \exp(\sigma_1 z_1^0 + \sigma_2 z_2^0 + ... + \sigma_n z_n^0)\,. \qquad (17)$$

Because $z(t)$ satisfies the initial problem (13), the initial solution $z(t)$ is easily obtained from $v(t,\sigma)$. This method is applied now for a simplest problem.Let consider the problem [1]

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^2,\quad y(0) = y_0\,, \qquad (18)$$

By applying the transformation $v = e^{\sigma y}$ to (18) we obtain a linear equivalent equation

$$\frac{\partial v}{\partial t} = \sigma \frac{\partial^2 v}{\partial\sigma^2}\,, \qquad (19)$$

and the condition

$$v(0,\sigma) = e^{\sigma y_0}\,. \qquad (20)$$

For

$$v(t,\sigma) = \phi(t)\psi(\sigma)\,, \qquad (21)$$

we have from (19)

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \lambda\phi\ ,\quad \sigma\frac{\mathrm{d}^2\psi}{\mathrm{d}\sigma^2} = \lambda\psi(\sigma)\ , \qquad (22)$$

with $\lambda$ an arbitrary constant. For $\psi$ we take the form

$$\psi(\sigma) = \sum_{j=0}^{\infty} \varphi_j \frac{\sigma^j}{j!}\ , \qquad (23)$$

that yields to a recurrent linear algebraic system

$$j\psi_{j+1} = \lambda\psi_j\,,\quad j \in N^*\,. \qquad (24)$$

We see that $\psi_0 = 0$. The other coefficients are determined from

$$\psi_j = \frac{\lambda^{j-1}}{(j-1)!}\psi_1\,,\quad j \in N\,. \qquad (25)$$

For $v(t,\sigma)$ we have the general representation

$$v(t,\sigma) = 1 + \int_{-\infty}^{0} e^{\lambda t} \sum_{j=1}^{\infty} \frac{\lambda^{j-1}}{(j-1)!}\frac{\sigma^j}{j!}\psi_1 d\lambda\,, \qquad (26)$$

for $t \ge 0$. From the initial condition (18) it results

$$\int_{-\infty}^{0} \frac{\lambda^{j-1}}{(j-1)!}\frac{\sigma^j}{j!}\psi_1 \mathrm{d}\lambda = y_0^j\ ,\quad j \in N\,. \qquad (27)$$

So,

$$\psi_1 = e^{-\frac{\lambda}{y_0}}\ , \qquad (28)$$

and the solution.(26) becomes

$$v(t,\sigma) = 1 + \int_{-\infty}^{0} e^{\lambda(t-\frac{1}{y_0})} \sum_{j=1}^{\infty} \frac{\lambda^{j-1}}{(j-1)!}\frac{\sigma^j}{j!}\mathrm{d}\lambda = \\ 1 + \sum_{j=1}^{\infty} -\left(\frac{1}{t-\frac{1}{y_0}}\right)^j \frac{\sigma^j}{j!} = e^{-\frac{y_0}{ty_0-1}\sigma}\,. \qquad (29)$$

that yields to the solution of the problem (18) which is $y(t) = \frac{y_0}{1-ty_0}$.

Now we return to the nonlinear system of equations (1) and apply LEM, by introducing the linear equivalence transformation (LEM) that depends on four parameters $\sigma_i \in R$, $i = 1,2,3,4$.

$$v(t,\sigma_1,\sigma_2,\sigma_3,\sigma_4) = \exp(\sigma_1 z_1 + \sigma_2 z_2 + \sigma_3 z_3 + \sigma_4 z_4), \tag{30}$$

By inserting (30) into (1) we obtain a linear partial differential equation

$$\frac{\partial v}{\partial t} = \sum_{n=1}^{4}(\sum_{p=1}^{4}\sigma_n a_{np}\frac{\partial v}{\partial \sigma_p} + \sum_{p,q=1}^{4}\sigma_n b_{npq}\frac{\partial^2 v}{\partial\sigma_p\partial\sigma_q} + \sum_{p,q,r=1}^{4}\sigma_n c_{npqr}\frac{\partial^3 v}{\partial\sigma_p\partial\sigma_q\partial\sigma_r} + \sum_{p,q,r,l=1}^{4}\sigma_n d_{npqrl}\frac{\partial^4 v}{\partial\sigma_p\partial\sigma_q\partial\sigma_r\partial\sigma_l} + \sum_{p,q,r,l,m=1}^{4}\sigma_n e_{npqrlm}\frac{\partial^5 v}{\partial\sigma_p\partial\sigma_q\partial\sigma_r\partial\sigma_l\partial\sigma_m}), \tag{31}$$

with the initial conditions

$$v(0,\sigma_1,\sigma_2,\sigma_3,\sigma_4) = \exp(\sigma_1 z_1^0 + \sigma_2 z_2^0 + \sigma_3 z_3^0 + \sigma_4 z_4^0), \tag{32}$$

and the boundless condition at $t \to \infty$

$$|v(t,\sigma_1,\sigma_2,\sigma_3,\sigma_4)| \le v_0 \text{ at } t \to \infty . \tag{33}$$

The last condition (33) eliminates the unbounded solutions at $t \to \infty$ of (31). We investigate numerically the existence of bounded solutions for this equation. Taking the solution $v(t,\sigma)$ of the form

$$v(t,\sigma_1,\sigma_2,\sigma_3,\sigma_4) = 1 + \sum_{k=1}^{4}\sum_{i=1} A_k^i \frac{\sigma_k^i}{i!} + \sum_{\substack{k,l=1\\k\ne l}}^{4}\sum_{i,j=1} A_k^i A_l^j \frac{\sigma_k^i \sigma_l^j}{i!j!} + \sum_{\substack{k,l,m=1\\k\ne l\ne m}}^{4}\sum_{i,j,r=1} A_k^i A_l^j A_m^r \frac{\sigma_k^i\sigma_l^j\sigma_m^r}{i!j!r!} + \sum_{\substack{k,l,m,n=1\\k\ne l\ne m\ne n}}^{4}\sum_{i,j,r,s=1} A_k^i A_l^j A_m^r A_n^s \frac{\sigma_k^i\sigma_l^j\sigma_m^r\sigma_n^s}{i!j!r!s!}, \tag{34}$$

and introducing in (31) it results straightforwardly that the functions $A_n(t)$, $n = 1,2,3,4$ , i.e. vibrations of permanent form at $t \to \infty$ satisfying (32) has the form

$$A_n(t) = \sum_{k,\eta=0}\{(\mu t)^{k+1}\tilde{A}_{nk}(\eta)\Phi_k(\mu t,\eta) + (\mu t)^k \tilde{B}_{nk}(\eta)\psi_k(\mu t,\eta)\}, \tag{35}$$

and $k = 0,1,2..k_{\max}$, $\eta = 0,1,2,3,....$ In (35) the functions $\Phi_k(\mu t,\eta)$ and $\psi_k(\mu t,\eta)$ are given by

$$\Phi_k(\mu t,\eta) = \sum_{m=k+1}(\mu t)^{m-k-1}A_m^{(k)}(\eta),$$

$$\psi_k(\mu t,\eta) = \sum_{m=k+1}(\mu t)^{m-k-1}B_m^{(k)}(\eta), \tag{36}$$

and $\mu = \mu(\eta)$

$$\mu = \sum_{j=1}^{4}\alpha_j\lambda_j, \tag{37}$$

with $\sum_{l=1}^{4}\alpha_j = \eta + 1$ , $\alpha_j \ge 0$, $j = 1,2,3,4$ .

The quantities $\lambda_j$ in (37) are

$$\lambda_1 = p_1, \lambda_2 = -p_1, \lambda_3 = p_2, \lambda_4 = -p_2,$$

where $p_j$ , $j = 1,2,3,4$ are the roots of the characteristic equation $(\lambda^4 + p\lambda^2 + \Delta) = 0$, with

$$\mathrm{p} = \alpha\xi + \tilde{\alpha}\xi + \beta w + w, \quad \Delta = \alpha\xi\beta w + \tilde{\alpha}\xi w + w^2\beta,$$

The unknowns $\tilde{A}_{nk}$ , $B_{nk}$ are expressed

$$\tilde{A}_{nk}(\eta) = C_k(\eta)B_{nk}(\eta), \tilde{B}_{nk}(\eta) = C_k(\eta)C_{nk}(\eta),$$

where the constants $C_k(\eta)$ verify the recurrence relation

$$C_k(\eta) = \frac{\sqrt{k^2+\eta^2}}{k(2k+1)}C_{k-1}(\eta), \quad C_0(\eta) = \frac{p|\Gamma(1+i\eta)|}{2!}.$$

$\Gamma$ is the gamma function and

$$|\frac{\Gamma(1+i\eta)}{\Gamma(2)}|^2 = \prod_{n=0}[1 + \frac{\eta^2}{(2+n)^2}]^{-1}.$$

The constants $A_m^{(k)}$, $B_m^{(k)}$ are related by the relation $B_m^{(k)}(\eta) = m\Delta A_m^{(k)}(\eta)$, where $A_m^{(k)}(\eta)$ are given by

$$A_{k+1}^{(k)} = 1, \quad A_{k+2}^{(k)} = \frac{\eta}{k+1}$$
$$(m+k)(m-k-1)A_m^{(k)} = 2\eta A_{m-1}^{(k)} - A_{m-2}^{(k)}, \tag{38}$$
$$m > k+2$$

The constants $B_{nk}(\eta)$, $C_{nk}(\eta)$ depend on initial conditions and on coefficients from (1). From (38) and $(36)_2$ we obtain

$$\psi_k(\mu t,\eta)=\sum_{m=k+1}(\mu t)^{m-k-1}B_m^{(k)}(\eta)\,.$$

The function $\psi(\mu t,\eta)$ is linked to the $\Phi(\mu t,\eta)$ by

$$\Psi(\mu t,\eta)=\mu\frac{d}{dt}\Phi(\mu t,\eta)=\mu\dot{\Phi}(\mu t,\eta)\,.$$

By noting

$$F_k(\mu t,\eta)=C_k(\eta)\,(\mu t)^{k+1}\Phi_k(\mu t,\eta)\,,$$

the functions (35) become

$$A_n(t)=\sum_{\eta,k=0}\{B_{nk}(\eta)F_k(\mu t,\eta)+C_{nk}(\eta)\dot{F}_k(\mu t,\eta)\}\,.$$

We have

$$F_0(\mu t,0)=\sin\mu t,\quad \dot{F}_0(\mu t,0)=\cos\mu t\,.$$

The solution (34) is reduced to

$$v=\exp(\sigma_1A_1+\sigma_2A_2+\sigma_3A_3+\sigma_4A_4)\,.$$

Finally, the LEM representation of the solution of the nonlinear simplified system of equations (1) and (10) is found to

$$z_n(t)=A_n(t)=\sum_{k,\eta=0}\{B_{nk}(\eta)F_k(\mu t,\eta)+C_{nk}(\eta)\dot{F}_k(\mu t,\eta)\}. \qquad (39)$$

The functions $F_k(\mu t,\eta)$ have the form of Coulomb wave functions [18]. If $v(t,\sigma)$ is a solution bounded at $t\to\infty$ them $\frac{\partial v}{\partial t}$ is also bounded at $t\to\infty$.

We summarize the result in the following result:

*The bounded solutions* (39) (*LEM representations) of the simplified system of equations* (1) and (10) *are describable as a linear superposition of Coulomb vibrations* $F_k(\mu t,\eta)$.

# 4 The cnoidal method

The investigation is not limited to the analysis of the simplified set of equations (1) and (10). The analysis of the LEM solutions of this system is designed in an attempt to establish some qualitative conclusions about the solutions of the exact set of equations. So, we find some interesting cases that uncouple the equations (4) and reduce them to the Weierstrass equations of fifth order, with solutions describable as a linear and nonlinear superposition of cnoidal waves.

We consider the simple case in the modified version ($\alpha=\tilde{\alpha}$)

$$\ddot{\theta}_1+w\sin\theta_1+\alpha\cos\theta_1=0\,,$$
$$\ddot{\theta}_2+\beta w\sin\theta_2-\alpha\cos\theta_2=0\,, \qquad (40)$$

and associated initial conditions

$$\theta_1(0)=\theta_1^0,\ \ \theta_2(0)=\theta_2^0,\ \dot{\theta}_1(0)=\theta_{p1}^0,\ \ \dot{\theta}_2(0)=\theta_{p2}^0, \qquad (41)$$

Multiplying the first equation by $2\dot{\theta}_1$, and the second one by $2\dot{\theta}_2$, and integrating we obtai

$$\dot{\theta}_1^2=2w\cos\theta_1-2\alpha\sin\theta_1+C_1\,,$$
$$\dot{\theta}_2^2=2w\beta\cos\theta_2+2\alpha\sin\theta_2+C_2\,, \qquad (42)$$

with $C_i$, $i=1,2$ integration constants.

Approximating the trigonometric functions by polynomials of five-order we have $\dot{\theta}_i^2=P_i(\theta_i)$, $i=1,2$, where $P_i(\theta_i)$ are polynomials of fifth-order in $\theta_i$, $i=1,2$

$$P_i(\theta_i)=a_{0i}+a_{1i}\theta_i+a_{2i}\theta_i^2+a_{3i}\theta_i^3+a_{4i}\theta_i^4+a_{5i}\theta_i^5\,,$$

with

$$a_{01}=2w+C_1\,,\ a_{02}=2\beta w+C_2,$$
$$a_{11}=-2\alpha\,,\ aA_{12}=2\alpha,$$
$$a_{21}=2w\tilde{c}\,,\ a_{22}=2\beta w\tilde{c},$$
$$a_{31}=-2\alpha\tilde{a}\,,\ a_{32}=2\alpha\tilde{a},$$
$$a_{41}=2w\tilde{d}\,,\ a_{42}=2w\tilde{d}\beta,$$
$$a_{51}=-2\alpha\tilde{b}\,,\ a_{52}=2\alpha\tilde{b},$$

where, for sake of simplicity, we have taken $-2w=C_1$, $-2\beta w=C_2$ and $a_{11}=-a_{12}=2\alpha\neq0$.

The equations (44) are similar with the equation

$$\dot{\theta}^2=A_1\theta+A_2\theta^2+A_3\theta^3+A_4\theta^4+A_5\theta^5\,, \qquad (43)$$

where

$$A_1=\frac{1}{2}a_1,\ \ A_2=a_2,\ \ A_3=\frac{3}{2}a_{,3}\ \ A_4=2a_4,\ \ A_5=\frac{5}{2}a_5.$$

This equation admits a particular solution expressed as an elliptic Weierstrass function that is reduced, in this case, to the cnoidal function $cn$ [18]

$$\wp(t+\delta';g_2,g_3)=e_2-(e_2-e_3)\mathrm{cn}^2(\sqrt{e_1-e_3}\,t+\delta'), \qquad (44)$$

where $\delta'$ is an arbitrary constant, $e_1, e_2, e_3$ are the real roots of the equation $4y^3 - g_2 y - g_3 = 0$ with $e_1 > e_2 > e_3$ and $g_2, g_3$ the invariants expressed in term of the parameters of the exact system of equations (9) and (10). As we know, a linear superposition of cnoidal functions (44) is also a solution for (43).

To see the form of the nonlinear superposition term we assume the solution of (43) in the Krishnan form [19]

$$\theta_{\text{int}}(t) = \frac{\lambda\,\wp(t)}{1 + \mu\,\wp(t)} \quad , \tag{45}$$

where $\wp(t)$ is the Weierstrass elliptic function satisfying the differential equation [18]

$$\wp'^2 = 4\wp^3 - g_2\wp - g_1 \ , \tag{46}$$

with the invariants $g_2$ and $g_1$ real in the pendulum case, and satisfying $g_2^3 - 27g_3^2 > 0$ , $\lambda$ and $\mu$ are arbitrary constants.

Substituting (45) into (43) we obtain four equations for the unknowns $\lambda$ , $\mu$ , $g_2$ and $g_1$

$$\begin{aligned} &-2\lambda\mu^2 = \\ &A_1\mu^4 + A_2\lambda\mu^3 + A_3\lambda^2\mu^2 + A_4\lambda^3\mu \ + A_5\lambda^4, \end{aligned} \tag{47}$$

$$4\lambda\mu = 4A_1\mu^3 + 3A_2\lambda\mu^2 + 2A_3\lambda^2\mu + A_4\lambda^3 , \tag{48}$$

$$6\lambda + \frac{3}{2}\lambda\mu^2 g_2 = 6A_1\mu^2 + 3A_2\lambda\mu + A_3\lambda^2 , \tag{49}$$

$$\lambda\mu g_2 + 2\lambda\mu^2 g_3 = A_1\mu + A_2\lambda \ . \tag{50}$$

From (47), (48) we have

$$\begin{aligned} &6A_1\mu^4 + 5A_2\lambda\mu^3 + 4A_3\lambda^2\mu^2 + \\ &3A_4\lambda^3\mu \ + 2A_5\lambda^4 = 0. \end{aligned} \tag{51}$$

Let us consider the special case where (54) is reducible to

$$(R\lambda + S\mu)^4 = 0 , \tag{52}$$

$$\mu = -\left(\frac{A_5}{3A_1}\right)^{1/4} \lambda , \tag{53}$$

with

$$R = (2A_5)^{1/4} \ , \ S = (6A_1)^{1/4} ,$$

$$A_2 = \frac{4}{5}RS^3 \ , \ A_3 = \frac{3}{2}R^2S^2 , \ A_4 = \frac{4}{3}R^3S .$$

We observe that both quantities $R$ and $S$ in (52) are both or real or imaginary. In the last case this equation leads to $i(R'\lambda + S'\mu)^4 = 0$ with $R'$ and $S'$ real quantities.We calculate

$A_1 = \frac{1}{2}a_1 = -\alpha \neq 0$ for the first equation (42) and $A_1 = \frac{1}{2}a_1 = \alpha \neq 0$ for the second equation (42), with $\alpha > 0$. In both cases we have $\frac{A_5}{3A_1} = \frac{5}{3}\tilde{b} > 0$. Then

$$\mu = -\left(\frac{5\tilde{b}}{3}\right)^{1/4} \lambda \ . \tag{54}$$

Using (53) we can obtain a unique constant $\lambda$ from (47) and (48)

$$\lambda = -30(3A_1A_5)^{-3/2} \ . \tag{55}$$

The constant $A_5$ is $A_5 = \frac{5}{2}a_5 = -5\alpha\tilde{b}$ or $A_5 = \frac{5}{2}a_5 = 5\alpha\tilde{b}$ with $\tilde{b} > 0$. From (55) we have for the both situations

$$\lambda = -30(15\alpha^2\tilde{b})^{-3/2} \ \text{with} \ \alpha^2\tilde{b} > 0 . \tag{56}$$

From (54) and (56) we obtain

$$\mu = 30\left(\frac{5\tilde{b}}{3}\right)^{1/4}(15\alpha^2\tilde{b})^{-3/2} . \tag{57}$$

The unknowns $g_2$ and $g_3$ are computable from (52) and (53). For the considered pendulum we found that $\lambda$ , $\mu$ , $g_2$ and $g_1$ are always real. The nonlinear term becomes

$$\theta(t) = \frac{\lambda\,\wp(t + \delta'; g_2, g_3)}{1 + \mu\,\wp(t + \delta'; g_2, g_3)} ,$$

where $\lambda$ and $\mu$ are given by (55) and (57), and $\delta'$ is an integration constant of the equation (46). The exact periodical solutions are obtained by using (44) where $\delta'$ is an arbitrary real constant, and $e_1, e_2, e_3$ the real roots of $4y^3 - g_2 y - g_3 = 0$ with $e_1 > e_2 > e_3$.

The nonlinear interaction solution of (46) is a bounded periodical function

$$\theta_{\text{int}}(t) = \frac{\lambda[e_2 - (e_2 - e_3)\text{cn}^2(\sqrt{e_1 - e_3}\,t + \delta')]}{1 + \mu[e_2 - (e_2 - e_3)\text{cn}^2(\sqrt{e_1 - e_3}\,t + \delta')]} . \tag{58}$$

The modulus $m$ of the Jacobean elliptic function is $m = \frac{e_2 - e_3}{e_1 - e_3}$ .The solitary vibration is a vibration with the period ($m$=1 or $e_1 = e_2$). In this case the solution is

$$\theta_{\text{int}}(t) = \frac{\lambda[e_1 - (e_1 - e_3)\operatorname{sech}^2(\sqrt{e_1 - e_3}t + \delta')]}{1 + \mu[e_1 - (e_1 - e_3)\operatorname{sech}^2(\sqrt{e_1 - e_3}t + \delta')]} .$$

Now we return to the exact system of equations (5) and (7) (or (9) and (10)). We begin by taking the solutions $z_k(t)$, $k = 1,2$ of the forms [1], [11]

$$z_k(t) = 2\frac{\partial^2}{\partial t^2}\log\Theta_n^{(k)}(\eta) , \qquad (59)$$

where $\Theta$ - function is given by

$$\Theta_n^{(k)}(\eta) = \sum_{\substack{M_i^{(k)}=-\infty \\ 1\le i\le n}}^{\infty} \exp(\sum_{j=1}^{n} \mathrm{i}M_j^{(k)}\eta_j + \frac{1}{2}\sum_{i,j=1}^{n} M_i^{(k)}B_{ij}^{(k)}M_j^{(k)})$$

and

$$\eta = [\eta_1\eta_2...\eta_n] \text{ with } \eta_j = -\omega_j t + \beta_j ,\ 1 \le j \le n .$$

Here $n$ is the finite number of degrees of freedom for a particular solution of the problem, $\omega_j$ the frequencies and $\beta_j$ the phases.

The solution (59) can be written in the form

$$z(t) = 2\frac{\partial^2}{\partial t^2}\log\Theta_n(\eta) = z_{cn}(\eta) + z_{\text{int}}(\eta) ,$$

where we have omitted for simplicity the index $k$ .The first term $z_{cn}$ *represents a linear superposition of cnoidal waves* and it is given by

$$z_{cn}(\eta) = 2\frac{\partial^2}{\partial t^2}\log G\ (\eta) ,$$

$$G(\eta) = \sum_{M} \exp(\mathrm{i}M\eta + \frac{1}{2}M^T DM) ,$$

and the second term $z_{\text{int}}$, *a nonlinear superposition of the cnoidal functions* and it is given by

$$z_{\text{int}}(\eta) = 2\frac{\partial^2}{\partial t^2}\log(1 + \frac{F\ (\eta, C)}{G\ (\eta)}) ,$$

$$F\ (\eta, C) = \sum_{M^\alpha} C\exp(\mathrm{i}M\eta + \frac{1}{2}M^T DM) ,$$

$$C = \exp(\frac{1}{2}M^T OM) - 1 .$$

The interaction matrix $B$ is composed by a diagonal matrix $D$ and an off-diagonal matrix $O$, $B = D + O$. The first term $\theta_{cn}$ has an explicit form as

$$z_{cn}(t) = 2\sum_{m=1}^{n} A_m \operatorname{cn}^2(C_m\eta_m) ,$$

with $A_m, C_m$ unknown constants. The second term $z_{\text{int}}$ has an explicit form as

$$\theta_{\text{int}}(t) = \frac{\sum_{m=1}^{n} E_m \operatorname{cn}^2(C_m\eta_m)}{1 + \sum_{m=1}^{n} D_m \operatorname{cn}^2(C_m\eta_m)} ,$$

with $E_m, D_m$ unknown constants. The parameters $\omega_j, \beta_j, B_{ij}, A_m, C_m, E_m, D_m$, $1 \le j, m \le n$, can be analytically determined by an identification procedure.

So, the bounded solutions of the exact system of equations (5) and (7) (or (9) and (10)) are

$$z_k(t) = 2\sum_{m=1}^{n} A_{km}\operatorname{cn}^2(C_m\eta_m) + \frac{\sum_{m=1}^{n} E_{km}\operatorname{cn}^2(C_m\eta_m)}{1 + \sum_{m=1}^{n} D_{km}\operatorname{cn}^2(C_m\eta_m)} , \qquad (60)$$

for $k = 1,2,3,4$ .

We summarize the result in the following result: *The bounded solutions* (75) (*the cnoidal representation) of the exact system of equations* (5) *and* (6) (*or* (9) *and* (10)) *is a linear superposition of cnoidal vibrations and a nonlinear interaction among the vibrations.*

The LEM solutions (39) of the system of equations (1) and (10) and the cnoidal solutions (75) of the system of equations (5) and (7) (or (9) and (10)) are obtained by the two different methods.

These solutions appear distinct. In the case for which the simplification (8) is possible, the exact system of equations is reduced to the simplified system of equations. This case is a good test to compare these apparently distinct solutions. Even though it does not seem possible to analytically show the similarity between these solutions, it may be shown numerically, if suitable values for parameters, pertinent to the condition $\xi \le 0.3$, can be considered. The stability of pendulum motion can be easily studied through both methods, LEM and the cnoidal methods.

## 5 Examples

The theoretical results are firstly carried out as some validation examples for LEM and cnoidal representations.

We assess the efficiency of the LEM method for $\eta = 2$ and $k = 63$. For $k > 63$ the computing complicates without adding new significant terms in solutions. The cnoidal solutions for the exact set of equations are computed for $n = 3$ and $-2 \le M \le 2$.

Comparison between LEM and cnoidal solutions for $\xi \le 0.3$, and comparison with the numerical results obtained by the fourth-order Runge-Kutta scheme are performed.

The examples involve the pendulums: ($g = 10\, m/s^2$)

P1: $m = 1.5,\ \xi = 0.25,\ k = 70$ N/m),

$M_2 = 10\,\text{kg}$ , $l = 0.2\,\text{m}$ , $l_1 = l_2 = \dfrac{a}{2}$ ,

P2: ($m = 1,\ \xi = 0.5,\ k = 4 \cdot 10^4$ N/m),

$M_2 = 15\,\text{kg}$ , $l = 0.1\,\text{m}$ , $l_1 = l_2 = \dfrac{a}{2}$,

P3: (uncoupled pendulum) $M = 1\,\text{kg}$, $l =$ 0.5 m, $a =$ 0.125 m, $l_1 =$ 2 m, $k = 40$ N/m.

An interesting phenomenon is putting into evidence for the pendulum. Two kind of vibration regimes are found: an extended (phonon)- mode of vibration to both masses, and a localised (fracton) - mode of vibration to a single mass, the other mass being practically at rest.

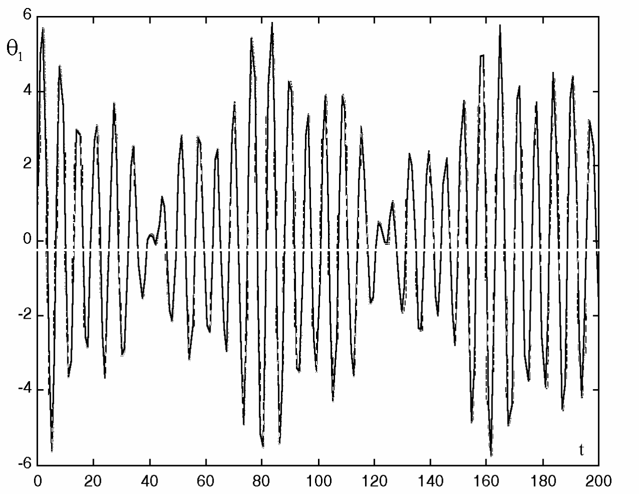


Fig. 1a The extended- mode of vibrations given by LEM solutions (continuum line) and by cnoidal solutions (dashed lines) for the solution $\theta_1(t)$ of P1 ($\theta_1(0) = 1.5$, $\theta_2(0) = 0.5$, $\dot{\theta}_1(0) = 0.05$, $\dot{\theta}_2(0) = 0.05$).

The first regime of vibrations is presented in Fig. 2a,b by the LEM solutions for P1 with $\theta_1(0) = 1.5$, $\theta_2(0) = 0.5$, $\dot{\theta}_1(0) = 0.05$, $\dot{\theta}_2(0) = 0.05$. Calculated by cnoidal method the solutions are practically the same. In fig. 2a dashed lines represent the cnoidal solutions, and in fig. 2b dashed lines represent the LEM solutions. Although we are unable to determine theoretically the precise connection between LEM and cnoidal solutions for $\xi \le 0.3$, we remark here similarity between them clearly depicted from graphs.

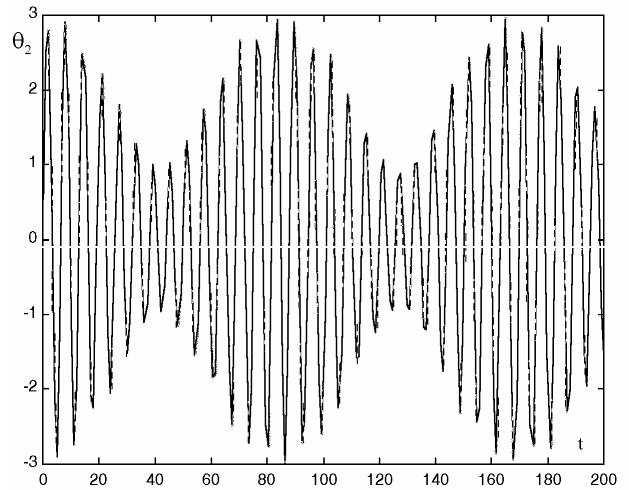


Fig. 1b The extended- mode of vibrations given by cnoidal solutions (continuum line) and by LEM solutions (dashed lines) for the solution $\theta_2(t)$ of P1 ($\theta_1(0) = 1.5$, $\theta_2(0) = 0.5$, $\dot{\theta}_1(0) = 0.05$, $\dot{\theta}_2(0) = 0.05$).

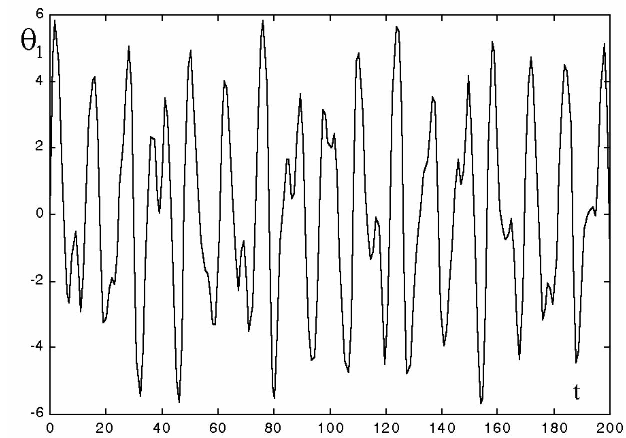


Fig. 2a The extended -mode of vibrations given by cnoidal solution $\theta_1$ (sum of cnoidal vibrations plus nonlinear interactions) for P2 ($\theta_1 = \theta_2 = 0.045,\ \ \dot{\theta}_1 = \dot{\theta}_2 = 0.02$ ).

The same regime of vibrations is given by solutions $\theta_i$, $i = 1,2$ for P2 (sum of cnoidal vibrations plus nonlinear interactions) with the initial conditions $\theta_1 = \theta_2 = 0.045,\ \ \dot{\theta}_1 = \dot{\theta}_2 = 0.02$ in fig.3a,b. The three spectral components have the moduli $m = 0.96,\ 0.68, 0.27$ for $\theta_1$, and $m = 0.87,\ 0.59, 0.31$ for $\theta_2$.

The error function $e(t)$ is defined as

$$e(t) = \sqrt{(\theta_1''(t) - \theta_1'(t))^2 + (\theta_2''(t) - \theta_2'(t))^2} \ , \qquad (61)$$

where ( $\theta_1'(t), \theta_2'(t)$ ) are solutions obtained by using a method and ( $\theta_1''(t), \theta_2''(t)$ ) solutions obtained by other method, applied to the same set of equations.

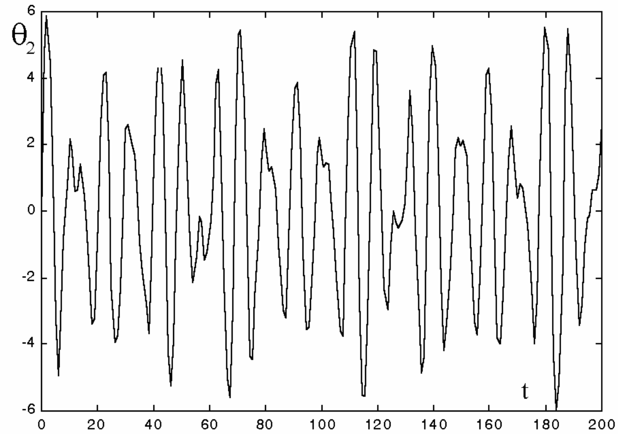

Fig.2b The extended -mode of vibrations given by cnoidal solution $\theta_2$ (sum of cnoidal vibrations plus nonlinear interactions) for P2 ( $\theta_1 = \theta_2 = 0.045$, $\dot{\theta}_1 = \dot{\theta}_2 = 0.02$ ).

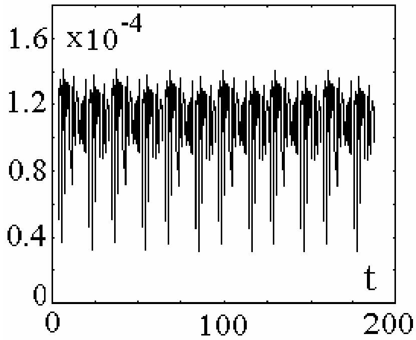

Fig. 3a The error function $e(t)$ between LEM and Runge-Kutta solutions for the pendulum P1.

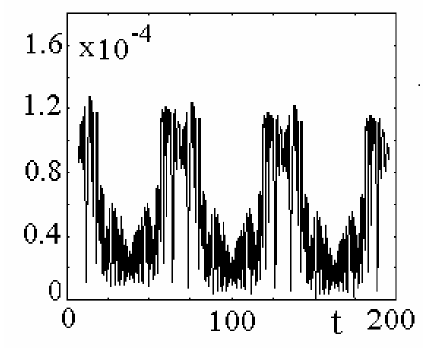

Fig. 3b The error function $e(t)$ between cnoidal and Runge-Kutta solutions for the pendulum P2.

The error function (61) between LEM and Runge-Kutta solutions for P1 is represented in fig.4a. The error function between cnoidal and Runge-Kutta solutions for P2 is represented in fig.4b.

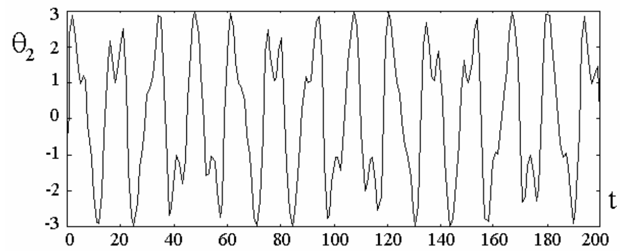

Fig. 4. The localized –mode of vibrations given by cnoidal solution $\theta_2$ ( $\theta_1 \cong 0$ ) for P1 ( $\theta_1(0) = 0.5$, $\theta_2(0) = -0.6$, $\dot{\theta}_1(0) = 0.5$, $\dot{\theta}_2(0) = 0$ ).

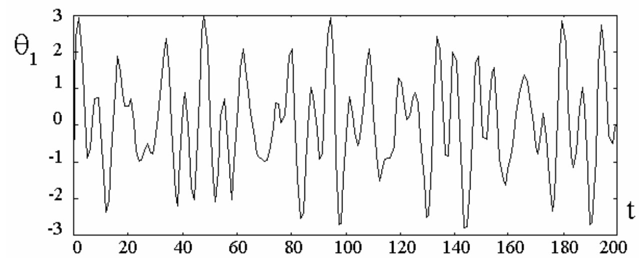

Fig.5. The localized –mode of vibrations given by cnoidal solution $\theta_1$ ( $\theta_2 \cong 0$ ) for P1 ( $\theta_1(0) = -0.6$, $\theta_2(0) = 0.5$, $\dot{\theta}_1(0) = 0.5$, $\dot{\theta}_2(0) = -0.5$ ).

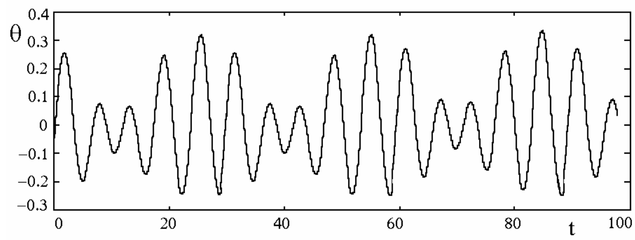

Fig.6a The first part of the solution $\theta(t)$ (superposition of two cnoidal vibrations) for P3 ( $\theta(0) = -0.05$, $\dot{\theta}(0) = 0.1$ ).

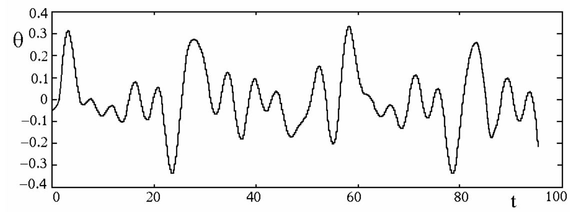

Fig.6b The second part of the solution $\theta(t)$ (nonlinear interactions between two cnoidal vibrations) for P3 ( $\theta(0) = -0.05$, $\dot{\theta}(0) = 0.1$ ).

The second regime of vibrations is represented in fig..4 for P1, with $\theta_1(0) = 0.5$, $\theta_2(0) = -0.6$, $\dot{\theta}_1(0) = 0.5$, $\dot{\theta}_2(0) = 0$. The vibrations are mostly localized on $M_2$, the $M_1$ being practically at rest.

The vibrations of $M_1$ have almost negligible amplitudes in comparison with the amplitudes of the $M_2$ vibrations. If we change the conditions as $\theta_1(0) = -0.6$, $\theta_2(0) = 0.5$, $\dot{\theta}_1(0) = 0$, $\dot{\theta}_2(0) = 0.5$, the mass $M_1$ is vibrating having the same evolution as shown in fig.5, and the mass $M_2$ is resting.

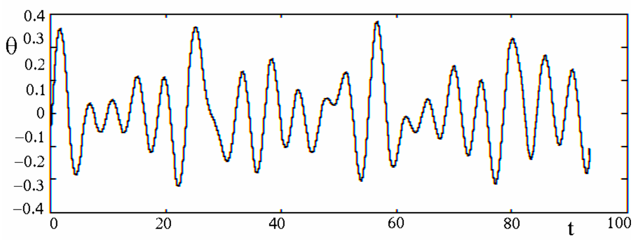


Fig.6c The final cnoidal solution $\theta(t)$ expressed as the sum between both previous parts for P3 ($\theta(0) = -0.05$, $\dot{\theta}(0) = 0.1$)

The first part of the solution $\theta(t)$ (superposition of three cnoidal vibrations), the second part of the same solution (nonlinear interactions between three cnoidal vibrations) and the final cnoidal solution expressed as the sum between them, for P3 are represented in fig.6a,b,c for $\theta(0) = -0.05$, $\dot{\theta}(0) = 0.1$. We see that the nonlinear part of this solution is not at all negligible.

## 5 Conclusions

The remarkable property – whereby the solutions of certain systems of nonlinear differential equations like (5) can be represented by a sum of a linear and a nonlinear superposition of cnoidal vibrations - is shared by a large number of nonlinear differential equations.

Two methods – the LEM and cnoidal methods- have been applied in this paper with the objective to capture and examine this property for a coupled pendulum.

The LEM representations of solutions for a simplified set of motion equations available for $\xi \le 0.3$ are describable as a superposition's of Coulomb vibrations. The LEM analysis is designed in an attempt to establish some qualitative conclusions about the solutions of the exact set of equations.

The cnoidal method is applied next to this system of equations. The cnoidal representations of solutions are described as a superposition of cnoidal vibrations and nonlinear interactions among vibrations. So, we can say that the real virtue of the cnoidal method is to give the elegant and compact expressions for the solutions in the spirit of this property.

The results of numerical computations by LEM and by cnoidal methods for the case $\xi \le 0.3$ have shown that both solutions are equivalent. The both solutions were compared to the Runge-Kutta numerical solutions.

The both methods describe successfully the stable behaviour of the coupled pendulum. The LEM method looks like partial generalisation of the linear theory and help the nonlinear analysis of dynamical systems having algebraic nonlinearities written in the form (1) and known as the Bolotin equations. These equations have the property they are reducible to the coupled and uncoupled Weierstrass equations of third, and fifth or higher order.

The cnoidal method can provide the nonlinear analysis of complex dynamical systems like (5). This method looks like a generalization of Fourier series with the cnoidal wave as the fundamental basis function, but is a completely different than an ordinary Fourier series expressed as a linear superposition of sine waves.

The analytical solutions allow the possibility of investigating in detail the effects of changing the initial conditions. For certain values of these conditions it is possible to locate two kind of vibration regimes: an extended (phonon)- mode of vibrations to both masses, and a localised (fracton) - mode of vibrations to a single mass, the other mass being practically at rest.

An advantage of the cnoidal method is that the procedure is quite elegant, straightforward, requiring only the $\Theta$ -function formulation

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