Interval-valued intuitionistic fuzzy ideals of K-algebras

MUHAMMAD AKRAM

University of the Punjab
Punjab University College of Information Technology,
Old Campus, P. O. Box 54000, Lahore
PAKISTAN
m.akram@pucit.edu.pk, makrammath@yahoo.com

KARAMAT HUSSAIN DAR

Department of Mathematics, G. C. University Lahore
Lahore-54000, PAKISTAN.
prof_khdar @yahoo.com

BIAO LONG MENG

Department of Basic Courses,
Xian University of Science and Technology,
Xian 710054, P.R.CHINA.
mengbl_100@yahoo.com.cn

KAR-PING SHUM

Department of Mathematics, The University of Hong Kong
Pokfulam Road, HONG KONG.
kpshum@maths.hku.hk

Abstract: The notion of interval-valued intuitionistic fuzzy sets was first introduced by Atanassov and Gargov in 1989 as a generalization of both interval-valued fuzzy sets and intuitionistic fuzzy sets. In this paper we first apply the concept of interval-valued intuitionistic fuzzy sets to K-algebras. Then we introduce the notion of interval-valued intuitionistic fuzzy ideals (IIFIs, in short) of K-algebras and investigate some interesting properties. We characterize Artinian and Noetherian K-algebras by considering $IIFIs$ of a K-algebra $\mathcal{K}$. Characterization theorems of fully invariant and characteristic IIFIs are also discussed.

Key–Words: K-algebras; Interval-valued intuitionistic fuzzy sets; Equivalence relations; Artinian and Noetherian K-algebras; Fully invariant; Characteristic.

1 Introduction

It is known that mathematical logic is a discipline used in sciences and humanities with different point of view. Non-classical logic takes the advantage of the classical logic (two-valued logic) to handle information with various facts of uncertainty. The non-classical logic has become a formal and useful tool for computer science to deal with fuzzy information and uncertain information. The notion of logical algebras: BCK-algebras [21] was initiated by Imai and Iséki in 1966 as a generalization of both classical and non-classical positional calculus. In the same year, Iséki introduced BCI-algebras [22] as a super class of the class of BCK-algebras. In 1983, Hu and Li introduced BCH-algebras [20]. They demonstrated that the class of BCI-algebras is a proper subclass of the class of BCH-algebras.

Dar and Akram [11] introduced a new kind of logical algebra: K-algebra $(G, \cdot, \odot, e)$. A K-algebra is an algebra built on a group $(G, \cdot, e)$ with identity e by adjoining an induced binary operation $\odot$ on $(G, \cdot, e)$ which is attached to an abstract algebraic system $(G, \cdot, \odot, e)$ [11]. It is known that this system is non-commutative and non-associative with a right identity e. It was proved in [11] that a K-algebra on an abelian group is equivalent to a p-semisimple BCI-algebra. For the sake of convenience, a K-*algebra* built on a group G was re-named as a $K(G)$-algebra [12]. The $K(G)$-algebras have been characterized by their left and right mappings [12]. Dar and Akram further proved in [14] that a class of K-algebras is a super class of the class of non-classical $BCH/BCI/BCK$-algebras [20, 21, 22] and the class of B-algebras [29] when the group is abelian and non-abelian, respectively.

Interval-valued fuzzy sets were first introduced by Zadeh [31] as a generalization of fuzzy sets. An interval-valued fuzzy set is a fuzzy set whose membership function is many-valued and forms an interval in the membership scale. This idea gives the simplest method to capture the imprecision of the membership grades for a fuzzy set. Thus, interval-valued fuzzy sets provide a more adequate description of uncertainty than the traditional fuzzy sets. It is therefore important to use interval-valued fuzzy sets in applications. One of the main applications is in fuzzy control and the most computationally intensive part of fuzzy control is defuzzification. Since the transition of interval-valued fuzzy sets usually increases the amount of computations, it is vitally important to design some faster algorithms for the necessarily defuzzification. On the other hand, Atanassov [5] introduced the notion of intuitionistic fuzzy sets as an extension of fuzzy set in which not only a membership degree is given, but also a non-membership degree is involved. Considering the increasing interest in intuitionistic fuzzy sets, it is useful to determine the position of intuitionistic fuzzy sets in a frame of different theories of imprecision.

With the above background, Atanassov and Gargov [8] introduced the notion of interval-valued intuitionistic fuzzy sets which is a common generalization of intuitionistic fuzzy sets and interval-valued fuzzy sets. The fuzzy structures of K-algebras was introduced in [1]. Since then, the concepts and results of K-algebras have been broadened to the fuzzy setting frames (see, for example, [2-4, 10, 23]). In this paper we first apply the concept of interval-valued intuitionistic fuzzy sets to K-algebras. Then we introduce the notion of interval-valued intuitionistic fuzzy ideals (IIFIs, in short) of K-algebras and investigate some interesting properties. We characterize Artinian and Noetherian K-algebras by considering $IIFIs$ of a K-algebra $\mathcal{K}$. Characterization theorems of fully invariant and characteristic IIFIs are also discussed.

2 Preliminaries

In this section we cite some definitions and properties that are necessary for this paper.

Definition 1 [11] A K-*algebra* $\mathcal{K} = (G, \cdot, \odot, e)$ is an algebra of type (2, 2, 0) defined on a group $(G, \cdot, e)$ in which each non-identity element is not of order 2 and observes the following $\odot$-axioms:

(K1) $(x \odot y) \odot (x \odot z)$=$(x \odot ((e \odot z) \odot (e \odot y))) \odot x$,

(K2) $x \odot (x \odot y)$ = $(x \odot (e \odot y)) \odot x$,

(K3) $x \odot x = e$,

(K4) $x \odot e = x$,

(K5) $e \odot x = x^{-1}$.

for all x, y, $z \in G$.

If the group $(G, \cdot, e)$ is abelian , then the above axioms $(K1)$ and $(K2)$ can be replaced by:

$(\overline{K1})$ $(x \odot y) \odot (x \odot z) = z \odot y$.

$(\overline{K2})$ $x \odot (x \odot y) = y$.

A nonempty subset H of a K-algebra $\mathcal{K}$ is called a *subalgebra* [11] of the K-algebra $\mathcal{K}$ if $a \odot b \in H$ for all a, $b \in H$. Note that every subalgebra of a K-algebra $\mathcal{K}$ contains the identity e of the group $(G, \cdot, e)$. A mapping $f : \mathcal{K}_1 = (G_1, \cdot, \odot, e_1) \to \mathcal{K}_2 = (G_2, \cdot, \odot, e_2)$ of K-algebras is called a *homomorphism* [14] if $f(x \odot y) = f(x) \odot f(y)$ for all $x, y \in \mathcal{K}_1$. We note that if f is a homomorphism, then $f(e) = e$.

Definition 2 [1] A nonempty subset I of a K-algebra $\mathcal{K}$ is called an *ideal* of $\mathcal{K}$ if it satisfies:

(i) $e \in I$,

(ii) $x \odot y \in I$, $y \odot (y \odot x) \in I \Rightarrow x \in I$ for all x, $y \in G$.

Let μ be a *fuzzy set* on G, i.e., a map $\mu : G \to [0, 1]$.

Definition 3 [28] A fuzzy set μ in a group G is called a *fuzzy subgroup*‘ of G if it satisfies:

- $(\forall x, y \in G)$ $(\mu(xy) \geq \min\{\mu(x), \mu(y)\})$.
- $(\forall x \in G)$ $(\mu(x^{-1}) \geq \mu(x))$.

Lemma 4 [28] Let μ be a fuzzy subgroup of a group G. Then $\mu(x^{-1}) = \mu(x)$ and $\mu(x) \leq \mu(e)$ for all $x \in G$, where e is the identity element of G.

Lemma 5 [28] A fuzzy set μ in a group G is a fuzzy subgroup of G if and only if it satisfies:

$$(\forall x, y \in G)(\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y)\}).$$

Definition 6 [1] A fuzzy set μ in a K-algebra $\mathcal{K}$ is called a *fuzzy subalgebra* of $\mathcal{K}$ if it satisfies:

- $(\forall x, y \in G)\ (\mu(x \odot y) \geq \min\{\mu(x), \mu(y)\})$.

Note that every fuzzy subalgebra μ of a K-algebra $\mathcal{K}$ satisfies the following inequality:

$$(\forall x \in G)\,(\mu(e) \geq \mu(x)).$$

Proposition 7 [1] Let $\mathcal{K} = (G, \cdot, \odot, e)$ be a K-algebra in which the operation "$\odot$" is induced by the group operation. Then every fuzzy subgroup of $(G, \cdot, e)$ is a fuzzy subalgebra of $\mathcal{K}$ and vice versa.

Definition 8 [1] A fuzzy ideal of a K-algebra $\mathcal{K}$ is a mapping $\mu : G \to [0, 1]$ such that

(i) $(\forall x \in G)\ (\mu(e) \geq \mu(x)$,

(ii) $(\forall x, y \in G)\ (\mu(x) \geq \min\{\mu(x \odot y), \mu(y \odot (y \odot x))\})$.

Lemma 9 [1] μ is a fuzzy ideal of a K-algebra $\mathcal{K}$ if and only if μ is a fuzzy normal subgroup of G.

By an interval number D on [0, 1], we mean an interval $[a^-, a^+]$, where $0 \leq a^- \leq a^+ \leq 1$. The set of all closed subintervals of [0, 1] is denoted by $D[0, 1]$.

For interval numbers $D1 = [a_1^-, b_1^+]$ and $D2 = [a_2^-, b_2^+] \in D[0, 1]$, we define

$$\min(D1, D2) = \min([a_1^-, b_1^+], [a_2^-, b_2^+])$$
$$= [\min\{a_1^-, a_2^-\}, \min\{b_1^+, b_2^+\}],$$
$$\max(D1, D2) = \max([a_1^-, b_1^+], [a_2^-, b_2^+])$$
$$= [\max\{a_1^-, a_2^-\}, \max\{b_1^+, b_2^+\}],$$
$$D1 + D2 = [\min\{a_1^-, a_2^-\}, \max\{b_1^+, b_2^+\}].$$

and

- $D1 \leq D2 \Longleftrightarrow a_1^- \leq a_2^-$ and $b_1^+ \leq b_2^+$,
- $D1 = D2 \Longleftrightarrow a_1^- = a_2^-$ and $b_1^+ = b_2^+$,
- $D1 < D2 \Longleftrightarrow D1 \leq D2$ and $D1 \neq D2$,
- $mD = m[a_1^-, b_1^+] = [ma_1^-, mb_1^+]$, where $0 \leq m \leq 1$.

Obviously, $(D[0, 1], \leq, \vee, \wedge)$ forms a complete lattice with $[0, 0]$ as its least element and $[1, 1]$ as its greatest element. If $G(\neq \emptyset)$ be a given set , then an interval-valued fuzzy set B on G is defined by

$$B = \{(x, [\mu_B^-(x), \mu_B^+(x)]) : x \in G\}$$

, where $\mu_B^-(x)$ and $\mu_B^+(x)$ are fuzzy sets of G such that $\mu_B^-(x) \leq \mu_B^+(x)$, for all $x \in G$. If $\widetilde{\mu}_B(x) = [\mu_B^-(x), \mu_B^+(x)]$, then

$$B = \{(x, \widetilde{\mu}_B(x)) : x \in G\},$$

where $\widetilde{\mu}_B : G \to D[0, 1]$.
We now assume that $\widetilde{\mu}_A(x) = [\mu_A^-(x), \mu_A^+(x)]$ satisfies the following condition: $[\mu_A^-(x), \mu_A^+(x)] < [0.5, 0.5]$ or $[0.5, 0.5] \leq [\mu_A^-(x), \mu_A^+(x)]$ for all x.
For a nonempty set G, we call a mapping $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A) : G \to D[0, 1] \times D[0, 1]$ an *interval-valued intuitionistic fuzzy set* in G if $\mu_A^+(x) + \lambda_A^+(x) \leq 1$ and $\mu_A^-(x) + \lambda_A^-(x) \leq 1$ for all $x \in G$, where the mappings $\widetilde{\mu}_A(x) = [\mu_A^-(x), \mu_A^+(x)] : G \to D[0, 1]$ and $\widetilde{\lambda}_A(x) = [\lambda_A^-(x), \lambda_A^+(x)] : G \to D[0, 1]$ are the *degree of membership* functions and the *degree of non-membership functions*, respectively.

We adopt the following symbols and terminology.

(i) The symbol $\widetilde{0}$ is used to denote the *interval-valued fuzzy empty set* and $\widetilde{1}$ to denote the *interval-valued fuzzy universal set* in a set G, and we define $\widetilde{0}(x) = [0, 0]$ and $\widetilde{1}(x) = [1, 1]$ for all $x \in G$.

(ii) The symbol $\widehat{0}$ is used to denote the *interval-valued intuitionistic fuzzy empty set* and $\widehat{1}$ is used to denote the *interval-valued intuitionistic fuzzy universal set* in a given set G, and we define $\widehat{0}(x) = (\widetilde{0}, \widetilde{1}) = ([0, 0], [1, 1])$ and $\widehat{1}(x) = (\widetilde{1}, \widetilde{0}) = ([1, 1], [0, 0])$ for all $x \in G$.

(iii) We write $\widetilde{t} = [t_1, t_2]$, $\widetilde{s} = [s_1, s_2]$, $\widetilde{s_1} = [s_3, s_4]$ and $\widetilde{t_1} = [t_3, t_4] \in D[0, 1]$.

3 Interval-valued intuitionistic fuzzy ideals

Definition 10 An interval-valued intuitionistic fuzzy set $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$ in a K-algebra $\mathcal{K}$ is called an *interval-valued intuitionistic fuzzy ideal* ($IIFI$, in short) if it satisfies the following conditions:

(1) $\widetilde{\mu}_A(e) \geq \widetilde{\mu}_A(x)$,

(2) $\widetilde{\lambda}_A(e) \leq \widetilde{\lambda}_A(x)$,

(3) $\widetilde{\mu}_A(x) \geq \min\{\widetilde{\mu}_A(x \odot y), \widetilde{\mu}_A(y \odot (y \odot x))\}$,

(4) $\widetilde{\lambda}_A(x) \leq \max\{\widetilde{\lambda}_A(x \odot y), \widetilde{\lambda}_A(y \odot (y \odot x))\}$

for all $x, y \in G$.

We now give an example of an $IIFI$ of a K-algebra $\mathcal{K}$.

Example 11 Consider the K-algebra $\mathcal{K}$=$(G, \cdot, \odot, e)$ on the Dihedral group $G = \{< a, b >: a^4 = e = b^2 = (ab)^2\}$, where $u = a^2, v = a^3, x = ab, y = a^2b$, $z = a^3b$, and $\odot$ is given by the following Cayley table:

$\odot$	e	a	u	v	b	x	y	z
e	e	v	u	a	b	x	y	z
a	a	e	v	u	x	y	z	b
u	u	a	e	v	y	z	b	x
v	v	u	a	e	z	b	x	y
b	b	x	y	z	e	v	u	a
x	x	y	z	b	a	e	v	u
y	y	z	b	x	u	a	e	v
z	z	b	x	y	v	u	a	e

We define an interval-valued intuitionistic fuzzy set $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A) : G \rightarrow D[0, 1] \times D[0, 1]$ by

$$\widetilde{\mu}_A(x) := \begin{cases} [0.4, 0.5] & \text{if } x = e, \\ [0.2, 0.3] & \text{if } x \neq e, \end{cases}$$

$$\widetilde{\lambda}_A(x) := \begin{cases} [0.06, 0.4] & \text{if } x = e, \\ [0.07, 0.2] & \text{if } x \neq e. \end{cases}$$

By routine computations, $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$ can be verified to be an $IIFI$ of $\mathcal{K}$.

The following Propositions are obvious.

Proposition 12 If $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$ is an $IIFI$ of $\mathcal{K}$, then the level subsets $U(\widetilde{\mu}_A; \widetilde{s})$ and $L(\widetilde{\lambda}_A; \widetilde{s})$ are ideals of $\mathcal{K}$ for every $\widetilde{s} \in \mathrm{Im}(\widetilde{\mu}_A) \cap \mathrm{Im}(\widetilde{\lambda}_A) \subseteq D[0, 1]$, where $\mathrm{Im}(\widetilde{\mu}_A)$ and $\mathrm{Im}(\widetilde{\lambda}_A)$ are sets of values of $\widetilde{\mu}_A$ and $\widetilde{\lambda}_A$, respectively.

Proposition 13 If all nonempty level subsets $U(\widetilde{\mu}_A; \widetilde{s})$ and $L(\widetilde{\lambda}_A; \widetilde{s})$ of an interval-valued intuitionistic fuzzy set $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$ are ideals of $\mathcal{K}$, then A is an $IIFI$ of $\mathcal{K}$.

Definition 14 Let $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$ be an interval-valued intuitionistic fuzzy set on G. Pick $\widetilde{s}, \widetilde{t} \in D[0, 1]$ such that $\widetilde{s} + \widetilde{t} \leq [1, 1]$. Then the set

$$G_A^{(\widetilde{s},\widetilde{t})} := \{x \in G \mid \widetilde{s} \leq \widetilde{\mu}_A(x),\ \widetilde{\lambda}_A(x) \leq \widetilde{t}\}$$

is called an $(\widetilde{s}, \widetilde{t})$-*level subset* of A.

The set of all $(\widetilde{s}, \widetilde{t}) \in \mathrm{Im}(\widetilde{\mu}_A) \times \mathrm{Im}(\widetilde{\lambda}_A)$ such that $\widetilde{s} + \widetilde{t} \leq [1, 1]$ is called the *image of* $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$. Obviously, $G_A^{(\widetilde{s},\widetilde{t})} = U(\widetilde{\mu}_A, \widetilde{s}) \cap L(\widetilde{\lambda}_A, \widetilde{t})$.

Theorem 15 An interval-valued intuitionistic fuzzy set A=$(\widetilde{\mu}_A, \widetilde{\lambda}_A)$ of $\mathcal{K}$ is an $IIFI$ of $\mathcal{K}$ if and only if $G_A^{(\widetilde{s},\widetilde{t})}$ is an ideal of $\mathcal{K}$ for every $(\widetilde{s}, \widetilde{t}) \in \mathrm{Im}(\widetilde{\mu}_A) \times \mathrm{Im}(\widetilde{\lambda}_A)$ with $\widetilde{s} + \widetilde{t} \leq [1, 1]$.

Proof: We only need to prove the necessity because the sufficiency is trivial. Assume that $G_A^{(\widetilde{s},\widetilde{t})}$ is an ideal of $\mathcal{K}$ and $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$ an interval-valued intuitionistic fuzzy set on $\mathcal{K}$. It is easy to see that $\widetilde{\mu}_A(e) \geq \widetilde{\mu}_A(x)$ and $\widetilde{\lambda}_A(e) \leq \widetilde{\lambda}_A(x)$. Consider $x, y \in G$ such that $A(x) = (\widetilde{s}, \widetilde{t})$ and $A(y) = (\widetilde{s_1}, \widetilde{t_1})$, that is, $\widetilde{\mu}_A(x) = \widetilde{s}$, $\widetilde{\lambda}_A(x) = \widetilde{t}$, $\widetilde{\mu}_A(y) = \widetilde{s_1}$ and $\widetilde{\lambda}_A(y) = \widetilde{t_1}$. Without loss of generality, we may assume that $(\widetilde{s}, \widetilde{t}) \leq (\widetilde{s_1}, \widetilde{t_1})$, i.e., $\widetilde{s} \leq \widetilde{s_1}$ and $\widetilde{t_1} \leq \widetilde{t}$. Then $G_A^{(\widetilde{s_1},\widetilde{t_1})} \subseteq G_A^{(\widetilde{s},\widetilde{t})}$, i.e., $x, y \in G_A^{(\widetilde{s},\widetilde{t})}$. This implies that $x \odot y, y \odot (y \odot x) \in G_A^{(\widetilde{s},\widetilde{t})}$ since $G_A^{(\widetilde{s},\widetilde{t})}$ is an ideal of $\mathcal{K}$. Hence, we deduce that

$$\widetilde{\mu}_A(x) \geq \widetilde{s} = \min\{\widetilde{\mu}_A(x \odot y), \widetilde{\mu}_A(y \odot (y \odot x))\},$$

$$\widetilde{\lambda}_A(x) \leq \widetilde{t} = \min\{\widetilde{\lambda}_A(x \odot y), \widetilde{\lambda}_A(y \odot (y \odot x))\}.$$

This shows that $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$ is an $IIFI$ of $\mathcal{K}$. □

Theorem 16 Let $A = (\widetilde{\alpha}_A, \widetilde{\lambda}_A)$ and $B = (\widetilde{\beta}_B, \widetilde{\nu}_B)$ be $IIFI$s of $\mathcal{K}$. Then the functions $\widetilde{\alpha}_A \wedge \widetilde{\beta}_B : G \rightarrow D[0, 1]$ and $\widetilde{\lambda}_A \vee \widetilde{\nu}_B : G \rightarrow D[0, 1]$ defined by

$$(\forall x \in G)\,((\widetilde{\alpha}_A \wedge \widetilde{\beta}_B)(x) = \min\{\widetilde{\alpha}_A(x), \widetilde{\beta}_B(x)\}),$$

$$(\forall x \in G)\,((\widetilde{\lambda}_A \vee \widetilde{\nu}_B)(x) = \max\{\widetilde{\lambda}_A(x), \widetilde{\nu}_B(x)\})$$

are $IIFI$s of $\mathcal{K}$.

Proof: For every $x \in G$, we have

$$\begin{aligned}(\widetilde{\alpha}_A \wedge \widetilde{\beta}_B)(e) &= \min\{\widetilde{\alpha}_A(e), \widetilde{\beta}_B(e)\} \\ &\geq \min\{\widetilde{\alpha}_A(x), \widetilde{\beta}_B(x)\} \\ &= (\widetilde{\alpha}_A \wedge \widetilde{\beta}_B)(x),\end{aligned}$$

$$\begin{aligned}(\widetilde{\lambda}_A \vee \widetilde{\nu}_B)(e) &= \max\{\widetilde{\lambda}_A(e), \widetilde{\nu}_B(e)\} \\ &\leq \max\{\widetilde{\lambda}_A(x), \widetilde{\nu}_B(x)\} \\ &= (\widetilde{\lambda}_A \vee \widetilde{\nu}_B)(x).\end{aligned}$$

Hence, for $x, y \in G$, we deduce that

$$\begin{aligned}(\widetilde{\alpha}_A \wedge \widetilde{\beta}_B)(x) &= \min\{\widetilde{\alpha}_A(x), \widetilde{\beta}_B(x)\} \\ &\geq \min\{\min\{\widetilde{\alpha}_A(x \odot y) \\ &\quad , \widetilde{\alpha}_A(y \odot (y \odot x))\} \\ &\quad , \min\{\widetilde{\beta}_B(x \odot y) \\ &\quad , \widetilde{\beta}_B(y \odot (y \odot x))\}\} \\ &= \min\{\min\{\widetilde{\alpha}_A(x \odot y), \widetilde{\beta}_B(x \odot y)\} \\ &\quad , \min\{\widetilde{\alpha}_A(y \odot (y \odot x)) \\ &\quad , \widetilde{\beta}_B(y \odot (y \odot x))\}\} \\ &= \min\{(\widetilde{\alpha}_A \wedge \widetilde{\beta}_B)(x \odot y) \\ &\quad , (\widetilde{\alpha}_A \wedge \widetilde{\beta}_B)(y \odot (y \odot x))\},\end{aligned}$$

$$\begin{aligned}(\widetilde{\lambda}_A \vee \widetilde{\nu}_B)(x) &= \max\{\widetilde{\lambda}_A(x), \widetilde{\nu}_B(x)\} \\ &\leq \max\{\max\{\widetilde{\lambda}_A(x \odot y) \\ &\quad , \widetilde{\lambda}_A(y \odot (y \odot x))\} \\ &\quad , \max\{\widetilde{\nu}_B(x \odot y), \widetilde{\nu}_B(y \odot (y \odot x))\}\} \\ &= \max\{\max\{\widetilde{\lambda}_A(x \odot y), \widetilde{\nu}_B(x \odot y)\} \\ &\quad , \max\{\widetilde{\lambda}_A(y \odot (y \odot x)) \\ &\quad , \widetilde{\nu}_B(y \odot (y \odot x))\}\} \\ &= \max\{(\widetilde{\lambda}_A \wedge \widetilde{\nu}_B)(x \odot y) \\ &\quad , (\widetilde{\lambda}_A \wedge \widetilde{\nu}_B)(y \odot (y \odot x))\}.\end{aligned}$$

This shows that $\widetilde{\alpha}_A \wedge \widetilde{\beta}_B$ and $\widetilde{\lambda}_A \vee \widetilde{\nu}_B$ are $IIFI$s of $\mathcal{K}$. □

Notation 17 Denote the family of all $IIFI$s of $\mathcal{K}$ by $IIFI(\mathcal{K})$. For any $\widetilde{t} \in D[0,1]$, define on $IIFI(\mathcal{K})$ two binary relations,say $\mathcal{U}^{\widetilde{t}}$ and $\mathcal{L}^{\widetilde{t}}$ by :

$$(A, B) \in \mathcal{U}^{\widetilde{t}} \longleftrightarrow U(\widetilde{\mu}_A; \widetilde{t}) = U(\widetilde{\mu}_B; \widetilde{t})$$

and

$$(A, B) \in \mathcal{L}^{\widetilde{t}} \longleftrightarrow L(\widetilde{\lambda}_A; \widetilde{t}) = L(\widetilde{\lambda}_B; \widetilde{t}),$$

respectively, where $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$, $B = (\widetilde{\mu}_B, \widetilde{\lambda}_B)$. Obviously, $\mathcal{U}^{\widetilde{t}}$ and $\mathcal{L}^{\widetilde{t}}$ are equivalent relations on $IIFI(\mathcal{K})$. For any $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A) \in IIFI(\mathcal{K})$, denote the system of all equivalence classes of A modulo $\mathcal{U}^{\widetilde{t}}$ (resp. $\mathcal{L}^{\widetilde{t}}$) by $IIFI(\mathcal{K})/\mathcal{U}^{\widetilde{t}}$ (resp. $IIFI(\mathcal{K})/\mathcal{L}^{\widetilde{t}}$) and for any element A in $IIFI(\mathcal{K})$, write $[A]_{\mathcal{U}^{\widetilde{t}}}$ (resp. $[A]_{\mathcal{L}^{\widetilde{t}}}$).
Then

$$IIFI(\mathcal{K})/\mathcal{U}^{\widetilde{t}} = \{ [A]_{\mathcal{U}^{\widetilde{t}}} \mid A = (\widetilde{\mu}_A, \widetilde{\lambda}_A) \in SIFI(\mathcal{K}) \}$$

(resp. $IIFI(\mathcal{K})/\mathcal{L}^{\widetilde{t}} = \{ [A]_{\mathcal{L}^{\widetilde{t}}} \mid A = (\widetilde{\mu}_A, \widetilde{\lambda}_A) \in IIFI(\mathcal{K}) \}$) .

Definition 18 Let $I(\mathcal{K})$ be the family of all ideals of $\mathcal{K}$ and $\widetilde{t} \in D[0,1]$. Define two mappings $f_{\widetilde{t}}$ and $g_{\widetilde{t}}$ from $IIFI(\mathcal{K})$ to $I(\mathcal{K}) \cup \{\emptyset\}$ by $f_{\widetilde{t}}(A) = U(\widetilde{\mu}_A; \widetilde{t}), g_{\widetilde{t}}(A) = L(\widetilde{\lambda}_A; \widetilde{t})$ for all $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A) \in IIFI(\mathcal{K})$. Then $f_{\widetilde{t}}$ and $g_{\widetilde{t}}$ are well-defined $IIFI-$ functions.

Theorem 19 For any $\widetilde{t} \in D(0,1)$, the $IIFI$-functions $f_{\widetilde{t}}$ and $g_{\widetilde{t}}$ are surjective from $IIFI(\mathcal{K})$ to $I(\mathcal{K}) \cup \{\emptyset\}$.

Proof: Let $\widetilde{t} \in D(0,1)$. Then, it is clear that $\widehat{0} = ([0,0],[1,1])$ is an $IIFI(\mathcal{K})$, where $[0,0]$ and $[1,1]$ are interval-valued fuzzy sets in $\mathcal{K}$ defined by $[0,0](x) = [0,0]$ and $[1,1](x) = [1,1]$, for all $x \in \mathcal{K}$. Obviously $f_t(\widehat{0}) = U([0,0]; \widetilde{t}) = \emptyset = L([1,1]; \widetilde{t}) = g_{\widetilde{t}}(\widehat{0})$. Let $\emptyset \neq B \in I(\mathcal{K})$. For $B = (\chi_B, \overline{\chi_B}) \in IIFI(\mathcal{K})$, we have $f_{\widetilde{t}}(\widetilde{B}) = U(\chi_B; \widetilde{t}) = B$ and $g_{\widetilde{t}}(\widetilde{B}) = L(\overline{\chi_B}; \widetilde{t}) = B$. Hence $f_{\widetilde{t}}$ and $g_{\widetilde{t}}$ are surjective. □

Theorem 20 The quotient sets $IIFI(\mathcal{K})/\mathcal{U}^{\widetilde{t}}$ and $IIFI(\mathcal{K})/\mathcal{L}^{\widetilde{t}}$ are equipotent to $I(\mathcal{K}) \cup \{\emptyset\}$ for every $\widetilde{t} \in D(0,1)$.

Proof: For $\widetilde{t} \in D(0,1)$, let $f^*_{\widetilde{t}}$ (resp. $g^*_{\widetilde{t}}$) be a mapping from $IIFI(\mathcal{K})/\mathcal{U}^{\widetilde{t}}$ (resp. $IIFI(\mathcal{K})/\mathcal{L}^{\widetilde{t}}$) to $I(\mathcal{K}) \cup \{\emptyset\}$ defined by $f^*_{\widetilde{t}}([A]_{\mathcal{U}^{\widetilde{t}}}) = f_{\widetilde{t}}(A)$ (resp. $g^*_{\widetilde{t}}([A]_{\mathcal{L}^{\widetilde{t}}}) = g_{\widetilde{t}}(A)$) for all $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A) \in IIFI(\mathcal{K})$. If $U(\widetilde{\mu}_A; \widetilde{t}) = U(\widetilde{\mu}_B; \widetilde{t})$ and $L(\widetilde{\lambda}_A; \widetilde{t}) = L(\widetilde{\beta}_B; \widetilde{t})$ for $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$ and $B = (\widetilde{\mu}_B, \widetilde{\beta}_B)$ in $IIFI(\mathcal{K})$, then $(A,B) \in \mathcal{U}^{\widetilde{t}}$ and $(A,B) \in \mathcal{L}^{\widetilde{t}}$. Thus $[\widetilde{A}]_{\mathcal{U}^{\widetilde{t}}} = [\widetilde{B}]_{\mathcal{U}^{\widetilde{t}}}$ and $[\widetilde{A}]_{\mathcal{L}^{\widetilde{t}}} = [\widetilde{B}]_{\mathcal{L}^{\widetilde{t}}}$. This proves that the mappings $f^*_{\widetilde{t}}$ and $g^*_{\widetilde{t}}$ are injective. Now let $\emptyset \neq D \in I(\mathcal{K})$. For $D = (\chi_D, \overline{\chi_D}) \in IIFI(\mathcal{K})$, we have

$$f^*_{\widetilde{t}}([\widetilde{D}]_{\mathcal{U}^{\widetilde{t}}}) = f_{\widetilde{t}}(\widetilde{D}) = U(\chi_D; \widetilde{t}) = D$$

and

$$g^*_{\widetilde{t}}([\widetilde{D}]_{\mathcal{L}^{\widetilde{t}}}) = g_{\widetilde{t}}(\widetilde{D}) = L(\overline{\chi_D}; \widetilde{t}) = D.$$

Finally, for $\widetilde{0}$, we get

$$f^*_{\widetilde{t}}([\widetilde{0}]_{\mathcal{U}^{\widetilde{t}}}) = f_{\widetilde{t}}(\widetilde{0}) = U([0,0]; \widetilde{t}) = \emptyset$$

and

$$g^*_{\widetilde{t}}([\widetilde{0}]_{\mathcal{L}^{\widetilde{t}}}) = g_{\widetilde{t}}(\widetilde{0}) = L([1,1]; \widetilde{t}) = \emptyset.$$

This shows that $f^*_{\widetilde{t}}$ and $g^*_{\widetilde{t}}$ are surjective. □

Definition 21 For any $\widetilde{t} \in D[0,1]$, we define another relation $\mathcal{R}^{\widetilde{t}}$ on $IIFI(\mathcal{K})$ by:

$$(A,B) \in \mathcal{R}^{\widetilde{t}} \longleftrightarrow G_A^{(\widetilde{t},\widetilde{t})} = G_B^{(\widetilde{t},\widetilde{t})},$$

where $G_A^{(\widetilde{t},\widetilde{t})} = U(\widetilde{\mu}_A;\widetilde{t}) \cap L(\widetilde{\lambda}_A;\widetilde{t})$ and $R_B^{(\widetilde{t},\widetilde{t})} = U(\widetilde{\mu}_B;\widetilde{t}) \cap L(\widetilde{\lambda}_B;\widetilde{t})$.

The relation $\mathcal{R}^t$ is clearly an equivalent relation on $IIFI(\mathcal{K})$.

Theorem 22 For any $\widetilde{t} \in D(0,1)$ the mapping $\varphi_{\widetilde{t}} : IIFI(\mathcal{K}) \to I(\mathcal{K}) \cup \{\emptyset\}$ defined by $\varphi_{\widetilde{t}}(A) = G_A^{(\widetilde{t},\widetilde{t})}$ is surjective.

Proof: Let $\widetilde{t} \in D(0,1)$. Then $\varphi_{\widetilde{t}}(\widehat{0}) = G_A^{(\widetilde{t},\widetilde{t})} = U([0,0];\widetilde{t}) \cap L([1,1];\widetilde{t}) = \emptyset$. For any $H \in IIFI(\mathcal{K})$, there exists $\widetilde{H} = (\chi_H, \overline{\chi_H}) \in IIFI(\mathcal{K})$ such that $\varphi_{\widetilde{t}}(\widetilde{H}) = G_A^{(\widetilde{t},\widetilde{t})} = U(\chi_H;\widetilde{t}) \cap L(\overline{\chi_H};\widetilde{t}) = H$. Hence $\varphi_{\widetilde{t}}$ is surjective. □

Theorem 23 For any $\widetilde{t} \in D(0,1)$, the quotient set $IIFI(\mathcal{K})/\mathcal{R}^{\widetilde{t}}$ is equipotent to $I(\mathcal{K}) \cup \{\emptyset\}$.

Proof: Let $\widetilde{t} \in D(0,1)$ and let $\varphi_{\widetilde{t}}^* : IIFI(\mathcal{K})/\mathcal{R}^{\widetilde{t}} \to I(\mathcal{K}) \cup \{\emptyset\}$ be a mapping defined by $\varphi_{\widetilde{t}}^*([\widetilde{A}]_{\mathcal{R}^{\widetilde{t}}}) = \varphi_{\widetilde{t}}(\widetilde{A})$ for all $[\widetilde{A}]_{\mathcal{R}^{\widetilde{t}}} \in IIFI(\mathcal{K})/\mathcal{R}^{\widetilde{t}}$. If $\varphi_{\widetilde{t}}^*([\widetilde{A}]_{\mathcal{R}^{\widetilde{t}}}) = \varphi_{\widetilde{t}}^*([\widetilde{B}]_{\mathcal{R}^{\widetilde{t}}})$ for any $[\widetilde{A}]_{\mathcal{R}^{\widetilde{t}}}$, $[\widetilde{B}]_{G^{\widetilde{t}}} \in IIFI(\mathcal{K})/\mathcal{R}^{\widetilde{t}}$, then $G_A^{(\widetilde{t},\widetilde{t})} = G_B^{(\widetilde{t},\widetilde{t})}$, i.e., $(\widetilde{A},\widetilde{B}) \in \mathcal{R}^{\widetilde{t}}$. It follows that $[A]_{\mathcal{R}^{\widetilde{t}}} = [\widetilde{B}]_{\mathcal{R}^{\widetilde{t}}}$ and so $\varphi_{\widetilde{t}}^*$ is injective. Moreover, $\varphi_{\widetilde{t}}^*([\widetilde{0}]_{\mathcal{R}^{\widetilde{t}}}) = \varphi_t(\widetilde{0}) = G_{\widetilde{0}}^{(\widetilde{t},\widetilde{t})} = \emptyset$. For any $H \in I(\mathcal{K})$ we have $\widetilde{H} = (\chi_H, \overline{\chi_H}) \in IIFI(\mathcal{K})$ and

$$\begin{aligned}
\varphi_{\widetilde{t}}^*([\widetilde{H}]_{\mathcal{R}^{\widetilde{t}}}) &= \varphi_{\widetilde{t}}(\widetilde{H}) \\
&= G_{\widetilde{H}}(\widetilde{t},\widetilde{t}) = U(\chi_H;\widetilde{t}) \cap L(\overline{\chi_H};\widetilde{t}) \\
&= \widetilde{H}.
\end{aligned}$$

This shows that $\varphi_{\widetilde{t}}^*$ is surjective. □

4 Artinian and Noetherian K-algebras

Definition 24 A K-algebra $\mathcal{K}$ is said to be *Noetherian* if every ideal of $\mathcal{K}$ is finitely generated. We say that $\mathcal{K}$ satisfies the *ascending chain condition* on $IIFI$s if for every ascending sequence $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ of ideals of $\mathcal{K}$, there exists a natural number n such that $I_n = I_k$ for all $n \geq k$. We call $\mathcal{K}$ *satisfies the Interval-valued intuitionistic fuzzy ascending chain condition* if for every ascending sequence $A_1 \subseteq A_2 \subseteq \cdots$ of $IIFI$s in $\mathcal{K}$, there exists a natural number n such that $\widetilde{\mu}_n = \widetilde{\mu}_k$, for all $n \geq k$.

The following lemma is obvious.

Lemma 25 Let $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$ be an $IIFI$ of $\mathcal{K}$ and let $\widetilde{s}$, $\widetilde{t} \in Im(\widetilde{\mu}_A)$, $\widetilde{s}$, $\widetilde{t} \in Im(\widetilde{\lambda}_A)$. Then $U(\widetilde{\mu}_A;\widetilde{s}) = U(\widetilde{\mu}_A;\widetilde{t}) \Longleftrightarrow \widetilde{s} = \widetilde{t}$, and $L(\widetilde{\lambda}_A;\widetilde{s}) = L(\widetilde{\lambda}_A;\widetilde{t}) \Longleftrightarrow \widetilde{s} = \widetilde{t}$.

Theorem 26 Let $\mathcal{K}$ be a K-algebra. Then every $IIFI$ of $\mathcal{K}$ has finite number of values if and only if $\mathcal{K}$ is Artinian.

Proof: Suppose that every $IIFI$ of $\mathcal{K}$ has finite number of values but $\mathcal{K}$ is not Artinian. Then there exists a strictly descending chain

$$G = A_0 \supset A_1 \supset A_2 \supset \cdots$$

of ideals of $\mathcal{K}$. Conditions (1) and (2) of Definition 3 hold obviously. In order to prove (3), we consider an interval-valued intuitionistic fuzzy set $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$ given by

$$\widetilde{\mu}_A(x) := \begin{cases} [\frac{1}{n+1}, \frac{n}{n+1}] & \text{if } x \in A_n \setminus A_{n+1}, \quad n = 0, 1, \ldots \\ [1,1] & \text{if } x \in \bigcap\limits_{n=0}^{\infty} A_n, \end{cases}$$

$$\widetilde{\lambda}_A(x) := [1,1] - \widetilde{\mu}_A(x).$$

Let $x, y \in \mathcal{K}$. Then $x \odot y$, $y \odot (y \odot x) \in A_n \setminus A_{n+1}$ for some $n = 0, 1, 2, \cdots$, and either $x \odot y \notin A_{n+1}$ or $y \odot (y \odot x) \notin A_{n+1}$. We now let $x, y \in U_n \setminus A_{n+1}$, for $k \leq n$. Then, it follows that

$$\begin{aligned}
\widetilde{\mu}_A(x) &= [\frac{1}{n+1}, \frac{n}{n+1}] \geq [\frac{1}{k+1}, \frac{k}{k+1}] \\
&\geq \min(\widetilde{\mu}_A(x \odot y), \widetilde{\mu}_A(y \odot (y \odot x))).
\end{aligned}$$

Thus $\widetilde{\mu}_A$ is an $IIFI$ of $\mathcal{K}$ and $\widetilde{\mu}_A$ has infinite number of different values. In a similar way, $\widetilde{\lambda}_A$ is also an $IIFI$ of $\mathcal{K}$ and $\widetilde{\lambda}_A$ has infinite number of different values. Hence $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$ is an $IIFI$ of $\mathcal{K}$ and A has infinite number of different values. This contradiction proves that the $K-$algebra $\mathcal{K}$ is Artinian .
Conversely, if the K-algebra $\mathcal{K}$ is Artinian then we can let $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$ be an $IIFI$ of $\mathcal{K}$. Suppose that

$Im(\widetilde{\mu}_A)$ is infinite. Then, we can observe that every subset of $D[0,1]$ contains either a strictly ascending sequence or a strictly descending sequence.
Now, let $[s_1,t_1] < [s_2,t_2] < [s_3,t_3] < \cdots$ be a strictly ascending sequence in $Im(\widetilde{\mu}_A)$. Then

$$U(\widetilde{\mu}_A;[s_1,t_1]) \supset U(\widetilde{\mu}_A;[s_2,t_2]) \supset U(\widetilde{\mu}_A;[s_3,t_3]) \supset \cdots$$

is strictly descending chain of ideals of $\mathcal{K}$. Since $\mathcal{K}$ is Artinian, there exists a natural number i such that $U(\widetilde{\mu}_A;[s_i,t_i]) = U(\widetilde{\mu}_A;[s_{i+n},t_{i+n}])$ for all $n \geq 1$. Since $[s_i,t_i] \in Im(\widetilde{\mu}_A)$ for all i, by applying Lemma 25, we have $s_i = s_{i+n}$, $t_i = t_{i+n}$, for all $n \geq 1$. This is a contradiction since s_i, t_i are distinct. On the other hand, if $[s_1,t_1] > [s_2,t_2] > [s_3,t_3] > \cdots$ is a strictly descending sequence in $Im(\widetilde{\mu}_A)$, then

$$U(\widetilde{\mu}_A;[s_1,t_1]) \subset U(\widetilde{\mu}_A;[s_2,t_2]) \subset U(\widetilde{\mu}_A;[s_3,t_3]) \subset \cdots$$

is an ascending chain of ideals of $\mathcal{K}$. Since $\mathcal{K}$ is Artinian, there exists a natural number j such that $U(\widetilde{\mu}_A;[s_j,t_j]) = U(\widetilde{\mu}_A;[s_{j+n},t_{j+n}])$ for all $n \geq 1$. Since $[s_j,t_j] \in Im(\widetilde{\mu}_A)$ for all j, by Lemma 25 $s_j = s_{j+n}$,$t_j = t_{j+n}$ for all $n \geq 1$, which is again a contradiction since s_j, t_j are distinct. This shows that $Im(\widetilde{\mu}_A)$ is finite. For $\mathrm{Im}(\widetilde{\lambda}_A)$, the proof is similar.
□

The proof of the following theorem is routine and we hence omit the proof.

Theorem 27 Let a K-algebra $\mathcal{K}$ be Artinian. If $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$ is an $IIFI$ of $\mathcal{K}$, then $|U_{\widetilde{\mu}_A}| = |Im(\widetilde{\mu}_A)|$ and $|L_{\widetilde{\lambda}_A}| = |Im(\widetilde{\lambda}_A)|$, where $U_{\widetilde{\mu}_A}$ and $L_{\widetilde{\lambda}_A}$ are families of all level ideals of $\mathcal{K}$ with respect to $\widetilde{\mu}_A$ and $\widetilde{\lambda}_A$, respectively. □

Theorem 28 Let a K-algebra $\mathcal{K}$ be Artinian. If $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$ and $B = (\widetilde{\nu}_B, \widetilde{\eta}_B)$ are $IIFIs$ of $\mathcal{K}$, then the following statements hold:

(i) $|U_{\widetilde{\mu}_A}| = |U_{\widetilde{\nu}_B}|$ and $Im(\widetilde{\mu}_A)$=$Im(\widetilde{\nu}_B)$ if and only if $\widetilde{\mu}_A$= $\widetilde{\nu}_B$,

(ii) $|L_{\widetilde{\lambda}_A}| = |L_{\widetilde{\eta}_B}|$ and $Im(\widetilde{\lambda}_A)$=$Im(\widetilde{\eta}_B)$ if and only if $\widetilde{\lambda}_A$= $\widetilde{\eta}_B$.

Proof: (i) If $\widetilde{\mu}_A = \widetilde{\nu}_B$, then $U_{\widetilde{\mu}_A} = U_{\widetilde{\nu}_B}$ and $Im(\widetilde{\mu}_A) = Im(\widetilde{\nu}_B)$. Now we suppose that $U_{\widetilde{\mu}_A} = U_{\widetilde{\nu}_B}$ and $Im(\widetilde{\mu}_A) = Im(\widetilde{\nu}_B)$. By Theorems 26 and 27, $Im(\widetilde{\mu}_A) = Im(\widetilde{\nu}_B)$ are finite and $|U_{\widetilde{\mu}_A}| = |Im(\widetilde{\mu}_A)|$ and $|U_{\widetilde{\nu}_B}| = |Im(\widetilde{\nu}_B)|$. Let

$$Im(\widetilde{\mu}_A) = \{\widetilde{t_1}, \widetilde{t_2}, \cdots, \widetilde{t_n}\}$$

and

$$Im(\widetilde{\nu}_B) = \{\widetilde{s_1}, \widetilde{s_2}, \cdots, \widetilde{s_n}\},$$

where $\widetilde{t_1} < \widetilde{t_2} < \cdots < \widetilde{t_n}$ and $\widetilde{s_1} < \widetilde{s_2} < \cdots < \widetilde{s_n}$. This shows that $\widetilde{t_i} = \widetilde{s_i}$ for all i. We now prove that $U(\widetilde{\mu}_A;\widetilde{t_i}) = U(\widetilde{\nu}_B;\widetilde{t_i})$ for all i. Note that $U(\widetilde{\mu}_A;\widetilde{t_1}) = \mathcal{K} = U(\widetilde{\nu}_B;\widetilde{t_1})$. Consider $U(\widetilde{\mu}_A;\widetilde{t_2})$ and $U(\widetilde{\nu}_B;\widetilde{t_2})$. Suppose that $U(\widetilde{\mu}_A;\widetilde{t_2}) \neq U(\widetilde{\nu}_B;\widetilde{t_2})$. Then $U(\widetilde{\mu}_A;\widetilde{t_2}) = U(\widetilde{\nu}_B;\widetilde{t_k})$ for some $k > 2$ and $U(\widetilde{\nu}_B;\widetilde{t_2}) = U(\widetilde{\mu}_A;\widetilde{t_j})$ for some $j > 2$. If there exist $x \in G$ such that $\widetilde{\mu}_A(x) = \widetilde{t_2}$, then

$$\widetilde{\mu}_A(x) < \widetilde{t_j} \;\; \forall \, j > 2. \qquad (1)$$

Since $U(\widetilde{\mu}_A;\widetilde{t_2}) = U(\widetilde{\nu}_B;\widetilde{t_k})$, $x \in U(\widetilde{\nu}_B;\widetilde{t_k})$. This implies that $\widetilde{\nu}_B(x) \geq \widetilde{t_k} > \widetilde{t_2}$, $k > 2$. Thus $x \in U(\widetilde{\nu}_B;\widetilde{t_2})$. Since $U(\widetilde{\nu}_B;\widetilde{t_2}) = U(\widetilde{\mu}_A;t_j)$, $x \in U(\widetilde{\mu}_A;\widetilde{t_j})$. Hence

$$\widetilde{\mu}_A(x) \geq \widetilde{t_j} \, for \, some \, j > 2. \qquad (2)$$

Clearly, (1) and (2) contradict each other. Hence $U(\widetilde{\mu}_A;\widetilde{t_2}) = U(\widetilde{\nu}_B;\widetilde{t_2})$. Continuing in this way, we eventually obtain $U(\widetilde{\mu}_A;\widetilde{t_i}) = U(\widetilde{\nu}_B;\widetilde{t_i})$, for all i.
Now let $x \in G$. Suppose that $\widetilde{\mu}_A(x) = \widetilde{t_i}$ for some i. Then $x \notin U(\widetilde{\mu}_A;\widetilde{t_j})$ for all $i+1 \leq j \leq n$. This implies that $x \notin U(\widetilde{\nu}_B;\widetilde{t_j})$ for all $i+1 \leq j \leq n$. But then we have $\widetilde{\nu}_B(x) < \widetilde{t_j}$ for all $i+1 \leq j \leq n$. Suppose that $\widetilde{\nu}_B(x) = \widetilde{t_m}$ for some $i \leq m \leq i$. If $i \neq m$, then $x \in U(\widetilde{\nu}_B;\widetilde{t_i})$. On the other hand, since $\widetilde{\mu}_A(x) = \widetilde{t_i}$, $x \in U(\widetilde{\mu}_A;\widetilde{t_i}) = U(\widetilde{\nu}_B;\widetilde{t_i})$. Thus we arrive a contradiction. Hence $i = m$ and $\widetilde{\mu}_A(x) = \widetilde{t_i} = \widetilde{\nu}_B(x)$. Consequently $\widetilde{\mu}_A = \widetilde{\nu}_B$.

(ii) The proof is similar and is omitted. □

We now characterize the Noetherian K-algebra in the following theorem .

Theorem 29 A K-algebra $\mathcal{K}$ is Noetherian if and only if the set of values of $IIFI$s of $\mathcal{K}$ are well ordered subsets of $D[0,1]$.

Proof: Suppose that $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$ is an $IIFI$ of $\mathcal{K}$ whose set of values is not a well ordered subset of $D[0,1]$. Then there exists a strictly decreasing sequence $[s_n,t_n]$ such that $\widetilde{\mu}_A(x_n) = [s_n,t_n]$. Denote by U_n the set $\{x \in G \mid \widetilde{\mu}_A(x) \geq [s_n,t_n]\}$. Then

$$U_1 \subset U_2 \subset U_3 \cdots$$

is a strictly ascending chain of ideals of $\mathcal{K}$. This clearly contradicts that $\mathcal{K}$ is Noetherian. Hence, $\mathrm{Im}(\widetilde{\mu}_A)$ must be a well-ordered subset of $D[0,1]$. Similarly, for $\mathrm{Im}(\widetilde{\lambda}_A)$.
Conversely, assume that the set of values of an $IIFI$ of $\mathcal{K}$ is a well ordered subset of $D[0,1]$ and $\mathcal{K}$ is not Noetherian K-algebra. Then there exists a strictly ascending chain

$$U_1 \subset U_2 \subset U_3 \cdots \qquad (*)$$

of ideals of $\mathcal{K}$. Define an interval-valued intuitionistic fuzzy set $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$ on $\mathcal{K}$ by putting

$$\widetilde{\mu}_A(x) := \begin{cases} [\frac{1}{k+1}, \frac{1}{k}] & \text{for } x \in A_k \backslash A_{k-1}, \\ [0,0] & \text{for } x \notin \bigcup_{k=1}^{\infty} A_k, \end{cases}$$

$$\widetilde{\lambda}_A(x) := [1,1] - \widetilde{\mu}_A(x),$$

Then, we can easily prove that $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$ is an $IIFI$ of $\mathcal{K}$. Since the ascending chain $(*)$ is not terminating, A has a strictly descending sequence of values, this contradicts that the value set of an $IIFI$ is well ordered. Consequently, $\mathcal{K}$ is Noetherian. □

Finally, we state a theorem of Noetherian $K-$ algebra. Since the proof is straightforward, we omit the proof.

Theorem 30 If $\mathcal{K}$ is a Noetherian K-algebra, then every $IIFI$ of $\mathcal{K}$ is finite valued.

5 Fully invariant and characteristic IIFIs

Definition 31 An ideal F of a K-algebra $\mathcal{K}$ is said to be a *fully invariant* ideal if $f(F) \subseteq F$ for all $f \in \mathrm{End}(\mathcal{K})$, where End($\mathcal{K}$) is the set of all endomorphisms of $\mathcal{K}$. An $IIFI$ $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$ of L is called a *fully invariant* if $\widetilde{\mu}^f_A(x) = \widetilde{\mu}_A(f(x)) \leq \widetilde{\mu}_A(x)$ and $\widetilde{\lambda}^f_A(x) = \widetilde{\lambda}_A(f(x)) \leq \widetilde{\lambda}_A(x)$ for all $x \in G$ and $f \in \mathrm{End}(\mathcal{K})$.

Theorem 32 If $\{A_i \,|\, i \in I\}$ is a family of $IIFI$ fully invariant ideals of $\mathcal{K}$, then $\bigcap_{i\in I} A_i = (\bigwedge_{i\in I} \widetilde{\mu}_{A_i}, \bigvee_{i\in I} \widetilde{\lambda}_{A_i})$ is an interval-valued intuitionistic fully invariant ideal of $\mathcal{K}$, where

$$\bigwedge_{i\in I} \widetilde{\mu}_{A_i}(x) = \inf\{\widetilde{\mu}_{A_i}(x) \,|\, i \in I,\, x \in L\},$$

$$\bigvee_{i\in I} \widetilde{\lambda}_{A_i}(x) = \sup\{\widetilde{\lambda}_{A_i}(x) \,|\, i \in I,\, x \in L\}.$$

Proof: It can be easily seen that $\bigcap A_i = (\bigwedge \widetilde{\mu}_{A_i}, \bigvee \widetilde{\lambda}_{A_i})$ is an $IIFI$ of $\mathcal{K}$. Let $x \in G$ and $f \in \mathrm{End}(\mathcal{K})$. Then

$$\begin{aligned}(\textstyle\bigwedge_{i\in I} \widetilde{\mu}_{A_i})^f(x) &= (\textstyle\bigwedge_{i\in I} \widetilde{\mu}_{A_i})(f(x)) \\ &= \inf\{\widetilde{\mu}_{A_i}(f(x)) \,|\, i \in I\} \\ &\leq \inf\{\widetilde{\mu}_{A_i}(x) \,|\, i \in I\} \\ &= (\textstyle\bigwedge_{i\in I} \widetilde{\mu}_{A_i})(x),\end{aligned}$$

$$\begin{aligned}(\textstyle\bigvee_{i\in I} \widetilde{\lambda}_{A_i})^f(x) &= (\textstyle\bigvee_{i\in I} \widetilde{\lambda}_{A_i})(f(x)) \\ &= \sup\{\widetilde{\lambda}_{A_i}(f(x)) \,|\, i \in I\} \\ &\leq \sup\{\widetilde{\lambda}_{A_i}(x) \,|\, i \in I\} \\ &= (\textstyle\bigvee_{i\in I} \widetilde{\lambda}_{A_i})(x).\end{aligned}$$

Hence $\bigcap_{i\in I} A_i = (\bigwedge_{i\in I} \widetilde{\mu}_{A_i}, \bigvee_{i\in I} \widetilde{\lambda}_{A_i})$ is an interval-valued intuitionistic fully invariant ideal of $\mathcal{K}$. □

Theorem 33 Let H be a nonempty subset of a algebra $\mathcal{K}$ and $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$ an $IIFIs$ defined by

$$\widetilde{\mu}_A(x) = \begin{cases} [s_2, t_2] & \text{if } x \in H, \\ [s_1, t_1] & \text{otherwise}, \end{cases}$$

$$\widetilde{\lambda}_A(x) = \begin{cases} [\alpha_2, \beta_2] & \text{if } x \in H, \\ [\alpha_1, \beta_1] & \text{otherwise}, \end{cases}$$

where $[0,0] \leq [s_1, t_1] < [s_2, t_2] \leq [1,1]$, $[0,0] \leq [\alpha_2, \beta_2] < [\alpha_1, \beta_1] \leq [1,1]$, $[0,0] \leq [s_i, t_i] + [\alpha_i, \beta_i] \leq [1,1]$ for $i = 1,2$. If H is an interval-valued intuitionistic fully invariant ideal of $\mathcal{K}$, then $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$ is an interval-valued intuitionistic fully invariant ideal of $\mathcal{K}$.

Proof: We can easily see that $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$ is an $IIFI$ of $\mathcal{K}$. Let $x \in G$ and $f \in \mathrm{End}(\mathcal{K})$. If $x \in H$, then $f(x) \in f(H) \subseteq H$. Thus, we have

$$\widetilde{\mu}^f_A(x) = \widetilde{\mu}_A(f(x)) \leq \widetilde{\mu}_A(x) = [s_2, t_2],$$

$$\widetilde{\lambda}^f_A(x) = \widetilde{\lambda}_A(f(x)) \leq \widetilde{\lambda}_A(x) = [\alpha_2, \beta_2].$$

For if otherwise, then we have

$$\widetilde{\mu}^f_A(x) = \widetilde{\mu}_A(f(x)) \leq \widetilde{\mu}_A(x) = [s_1, t_1],$$

$$\widetilde{\lambda}^f_A(x) = \widetilde{\lambda}_A(f(x)) \leq \widetilde{\lambda}_A(x) = [\alpha_1, \beta_1].$$

Thus, we have verified that $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$ is an interval-valued intuitionistic fully invariant ideal of $\mathcal{K}$. □

Definition 34 An $IIFI$ $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$ of $\mathcal{K}$ *has the same type* as an $IIFI$ $B = (\widetilde{\mu}_B, \widetilde{\lambda}_B)$ of $\mathcal{K}$ if there exists $f \in \mathrm{End}(\mathcal{K})$ such that $A = B \circ f$, i.e., $\widetilde{\mu}_A(x) \geq \widetilde{\mu}_B(f(x))$, $\widetilde{\lambda}_A(x) \geq \widetilde{\lambda}_B(f(x))$ for all $x \in G$.

Theorem 35 $IIFIs$ of $\mathcal{K}$ have same type if and only if they are isomorphic.

Proof: We only need to prove the necessity part because the sufficiency part is obvious. If an $IIFI$ $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$ of $\mathcal{K}$ has the same type as $B = (\widetilde{\mu}_B, \widetilde{\lambda}_B)$, then there exists $\varphi \in \mathrm{End}(\mathcal{K})$ such that

$$\widetilde{\mu}_A(x) \geq \widetilde{\mu}_B(\varphi(x)),\ \widetilde{\lambda}_A(x) \geq \widetilde{\lambda}_B(\varphi(x))\ \ \forall x \in G.$$

Let $f : A(\mathcal{K}) \to B(\mathcal{K})$ be a mapping defined by $f(A(x)) = B(\varphi(x))$ for all $x \in G$, that is,

$$f(\widetilde{\mu}_A(x)) = \widetilde{\mu}_B(\varphi(x)),\ f(\widetilde{\lambda}_A(x)) = \widetilde{\lambda}_B(\varphi(x))\ \forall x \in G.$$

Then, it is clear that f is a surjective homomorphism. Also, f is injective because $f(\widetilde{\mu}_A(x)) = f(\widetilde{\mu}_A(y))$ for all $x, y \in G$ implies $\widetilde{\mu}_B(\varphi(x)) = \widetilde{\mu}_B(\varphi(y))$. Whence $\widetilde{\mu}_A(x) = \widetilde{\mu}_B(y)$. Likewise, from $f(\widetilde{\lambda}_A(x)) = f(\widetilde{\lambda}_A(y))$ we conclude $\widetilde{\lambda}_A(x) = \widetilde{\lambda}_B(y)$ for all $x \in G$. Hence $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$ is isomorphic to $B = (\widetilde{\mu}_B, \widetilde{\lambda}_B)$. This completes the proof. □

Definition 36 An ideal C of $\mathcal{K}$ is said to be *characteristic* if $f(C) = C$ for all $f \in \mathrm{Aut}(\mathcal{K})$, where Aut($\mathcal{K}$) is the set of all automorphisms of $\mathcal{K}$. An $IIFI$ $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$ of $\mathcal{K}$ is called a *characteristic* if $\widetilde{\mu}_A(f(x)) = \widetilde{\mu}_A(x)$ and $\widetilde{\lambda}_A(f(x)) = \widetilde{\lambda}_A(x)$ for all $x \in G$ and $f \in \mathrm{Aut}(\mathcal{K})$.

Lemma 37 Let $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$ be an $IIFI$ of $\mathcal{K}$ and let $x \in G$. Then $\widetilde{\mu}_A(x) = \widetilde{t}$, $\widetilde{\lambda}_A(x) = \widetilde{s}$ if and only if $x \in U(\widetilde{\mu}_A; \widetilde{t})$, $x \notin U(\widetilde{\mu}_A; \widetilde{s})$ and $x \in L(\widetilde{\lambda}_A, \widetilde{s})$, $x \notin L(\widetilde{\lambda}_A, \widetilde{t})$ for all $\widetilde{s} > \widetilde{t}$.

Proof: Straightforward. □

Theorem 38 An $IIFI$ is characteristic if and only if each its level set is a characteristic ideal.

Proof: Let an $IIFI$ $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$ be characteristic, $\widetilde{t} \in Im(\widetilde{\mu}_A)$, $f \in \mathrm{Aut}(\mathcal{K})$, $x \in U(\widetilde{\mu}_A; \widetilde{t})$. Then $\widetilde{\mu}_A(f(x)) = \widetilde{\mu}_A(x) \geq \widetilde{t}]$, which means that $f(x) \in U(\widetilde{\mu}_A; \widetilde{t})$. Thus $f(U(\widetilde{\mu}_A; \widetilde{t})) \subseteq U(\widetilde{\mu}_A; \widetilde{t})$. Since for each $x \in U(\widetilde{\mu}_A; \widetilde{t})$ there exists $y \in G$ such that $f(y) = x$ we have $\widetilde{\mu}_A(y) = \widetilde{\mu}_A(f(y)) = \widetilde{\mu}_A(x) \geq \widetilde{t}$, whence we conclude $y \in U(\widetilde{\mu}_A; \widetilde{t})$. Consequently $x = f(y) \in f(U(\widetilde{\mu}_A; \widetilde{t}))$. Hence $f(U(\widetilde{\mu}_A; \widetilde{t}) = U(\widetilde{\mu}_A; \widetilde{t})$. Similarly, $f(L(\widetilde{\lambda}_A; \widetilde{s})) = L(\widetilde{\lambda}_A; \widetilde{s})$. This proves that $U(\widetilde{\mu}_A; \widetilde{t})$ and $L(\widetilde{\lambda}_A; \widetilde{s})$ are characteristic.
Conversely, if all levels of $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$ are characteristic ideals of $\mathcal{K}$, then for $x \in G$, $f \in \mathrm{Aut}(\mathcal{K})$ and $\widetilde{\mu}_A(x) = \widetilde{t} < \widetilde{s} = \widetilde{\lambda}_A(x)$, by Lemma 37, we have $x \in U(\widetilde{\mu}_A; \widetilde{t})$, $x \notin U(\widetilde{\mu}_A; \widetilde{s})$ and $x \in L(\widetilde{\lambda}_A; \widetilde{s})$, $x \notin L(\widetilde{\lambda}_A; \widetilde{t})$. Thus $f(x) \in f(U(\widetilde{\mu}_A; \widetilde{t})) = U(\widetilde{\mu}_A; \widetilde{t})$ and $f(x) \in f(L(\widetilde{\lambda}_A; \widetilde{s})) = L(\widetilde{\lambda}_A; \widetilde{s})$, i.e., $\widetilde{\mu}_A(f(x)) \geq \widetilde{t}$ and $\widetilde{\lambda}_A(f(x)) \leq \widetilde{s}$. For $\widetilde{\mu}_A(f(x)) = \widetilde{t}_1 > \widetilde{t}$, $\widetilde{\lambda}_A(f(x)) = \widetilde{s}_1 < \widetilde{s}$ we have $f(x) \in U(\widetilde{\mu}_A; \widetilde{t}_1) = f(U(\widetilde{\mu}_A; \widetilde{t}_1)$, $f(x) \in L(\widetilde{\lambda}_A, \widetilde{s}_1) = f(L(\widetilde{\lambda}_A; \widetilde{s}_1))$, whence $x \in U(\widetilde{\mu}_A; \widetilde{t}_1)$, $x \in L(\widetilde{\mu}_A; \widetilde{s}_1)$. This is a contradiction. Thus $\widetilde{\mu}_A(f(x)) = \widetilde{\mu}_A(x)$ and $\widetilde{\lambda}_A(f(x)) = \widetilde{\lambda}_A(x)$. So, $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$ is characteristic. □

As a consequence of the above results we obtain the following theorem.

Theorem 39 If $A = (\widetilde{\mu}_A, \widetilde{\lambda}_A)$ is a fully invariant $IIFI$, then it is characteristic.

6 Conclusions

In the present paper, we have presented some properties of interval-valued intuitionistic fuzzy ideals of K-algebras. It is clear that the most of these results can be simply extended to interval-valued intuitionistic (T, S)-fuzzy ideals, where S and T are given imaginable triangular norms. In our opinion the future study of (interval-valued intuitionistic) fuzzy ideals of K-algebras can be connected with (1) investigating (α, β)- fuzzy ideals; (2) finding interval-valued intuitionistic fuzzy sets and triangular norms. The obtained results can be used in various fields such as artificial intelligence, signal processing, multiagent systems, pattern recognition, robotics, computer networks, genetic algorithm, neural networks, expert systems, decision making, automata theory and medical diagnosis.

References:

[1] M. Akram, K. H. Dar, Y. B. Jun and E. H. Roh, Fuzzy structures of $K(G)$-algebra, *South-*

east Asian Bulletin of Mathematics Vol. 31, Issue 4, 2007, pp. 625-637.

[2] M. Akram and K. H. Dar, Fuzzy ideals of K-algebras, *Annals of University of Craiova, Math. Comp. Sci. Ser* Vol. 34, 2007, pp. 3-13.

[3] M. Akram, Bifuzzy ideals of K-algebras, *WSEAS Transactions on Mathematics* Vol.7, Issue 5, 2008, pp. 313-322.

[4] M. Akram, $(\in, \in \vee q)-$ fuzzy ideals of K-algebras, *Ars Combinatoria* Vol. 89, 2008, pp. 191-204.

[5] K. T. Atanassov, Intuitionistic fuzzy sets, *VII ITKRs Session, Sofia (deposed in Central Science-Technical Library of Bulgarian Academy of Science* 1697/84), 1983 (in Bulgarian).

[6] K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems* Vol. 20, Issue 1, 1986, pp. 87-96.

[7] K. T. Atanassov, Intuitionistic Fuzzy Sets: Theory and Applications, *Studies in fuzziness and soft computing*, vol. 35, Heidelberg, New York, Physica-Verl., 1999.

[8] K. T. Atanassov and G. Gargov, *Interval valued intuitionistic fuzzy sets*, Fuzzy Sets and Systems, Vol. 31, Issue 3, 1989, pp. 343 - 349.

[9] R. Biswas, Rosenfeld's fuzzy subgroups with interval-valued membership functions, *Fuzzy sets and Systems* Vol. 63, 1994, pp.87-90.

[10] Y. U. Cho and Y. B. Jun, Fuzzy algebras on K(G)-algebras, *Journal of Applied Mathematics and Computing* Vol. 22, 2006, pp. 549-555.

[11] K. H. Dar and M. Akram, On a K-algebra built on a group, *Southeast Asian Bulletin of Mathematics* Vol. 29, Issue 1, 2005, pp. 41-49.

[12] K. H. Dar and M. Akram, Characterization of a $K(G)$-algebras by self maps, *Southeast Asian Bulletin of Mathematics* Vol. 28, Issue 4, 2004, pp. 601-610.

[13] K. H. Dar and M. Akram, On K-homomorphisms of K-algebras, *International Mathematical Forum* Vol. 2, Issue 46, 2007, pp. 2283-2293.

[14] K. H. Dar and M. Akram, On subclasses of $K(G)$-algebras, *Annals of University of Craiova, Math. Comp. Sci. Ser* Vol. 33, 2006, pp. 235-240.

[15] S. K. De, R. Biswas and A. R. Roy, An application of intuitionistic fuzzy sets in medical diagnosis , *Fuzzy Sets and Systems*, Vol. 117, Issue 2, 2001, pp. 209-213.

[16] G. Deschrijver, Arithmetric operators in interval-valued fuzzy theory, *Information Sciences* Vol. 177, 2007, pp.2906-2924.

[17] G. Deschrijver and E.E. Kerre, On the relationship between some extensions of fuzzy set theory, *Fuzzy Sets and Systems* Vol. 133, 2003, pp. 227-235.

[18] D. Dubois, S. Gottwald, P. Hajek, J. Kacprzyk and H. Prade, Terminological difficulties in fuzzy set theory$-$The case of "Intuitionistic Fuzzy Sets", *Fuzzy Sets and Systems* Vol. 156, Issue 3, 2005, pp.485-491.

[19] D. Dubois and H. Prade, Fuzzy sets and systems:Theory and Applications, *Academic Press*, 1997, USA.

[20] Q. P. Hu and X. Li , On BCH-algebras, *Math. Seminar Notes* Vol. 11, 1983, pp. 313-320.

[21] Y. Imai and K. Iséki, On axiom system of propositional calculi XIV, *Proc. Japonica Acad* Vol. 42, 1966, pp. 19-22.

[22] K. Iséki, An algebra related with a propositional calculus, *Proc. Japan Acad* Vol. 42, 1966, pp. 26-29.

[23] Y. B. Jun and C. H. Park, Fuzzy idelas of K(G)-algebras, *Honam Mathematical Journal* Vol. 28, 2006, pp. 485-497.

[24] J. Meng and Y. B. Jun, BCK-algebras, *Kyung Moon Sa Co. Seoul*, Korea, 1994.

[25] J. Meng and X. Guo, On fuzzy ideals of BCK/BCI-algebras, *Fuzzy Sets and Systems* Vol. 149, Issue 3, 2005, pp. 509-525.

[26] D. Minchev, An intuitionistic fuzzy sets application in multiagent systems of metamorphic robotic systems , *Intelligent Systems, 2002. Proceedings. 2002 First International IEEE Symposium*, Vol. 3, 2002, pp. 74 - 78.

[27] N. Kuroki, On fuzzy semigroups, *Information Sciences* Vol. 53, 1991, pp.203-236.

[28] A. Rosenfeld, Fuzzy groups, *J. Math. Anal. Appl.* Vol. 35, 1971, pp. 512-517.

[29] J. Neggers and H. S. Kim, On B-algebras, *Matematicki Vesnik* Vol. 54, 2002, pp. 21-29.

[30] L. A. Zadeh, Fuzzy sets, *Information and Control* Vol. 8, 1965, pp. 338-353.

[31] L. A. Zadeh, The concept of a lingusistic variable and its application to approximate reasoning, *Information Sciences* Vol. 8, 1975, pp. 199-249.

[32] L. A. Zadeh, Toward a generalized theory of uncertainty (GTU)-an outine, *Information Sciences* Vol. 172, 2005, pp.1-40.