A note on the modified Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems

Shi-Liang Wu, Ting-Zhu Huang
School of Applied Mathematics,
University of Electronic Science and Technology of China,
Chengdu, Sichuan, 610054,
People's Republic of China
wushiliang1999@126.com or tingzhuhuang@126.com http://www.uestc.edu.cn/web3

Abstract: Comparing the lopsided Hermitian/skew-Hermitian splitting (LHSS) method and Hermitian/skew-Hermitian splitting (HSS) method, a new criterion for choosing the above two methods is presented, which is better than that of Li, Huang and Liu [Modified Hermitian and skew-Hermitian splitting methods for non-Hermitian positive-definite linear systems, *Numer. Lin. Alg. Appl.*, 14 (2007): 217-235].

Key-Words: non-Hermitian matrix; splitting; skew-Hermitian matrix; Hermitian matrix; iteration

1 Introduction

The solutions of many problems in scientific computing are eventually turned into the solutions of the large linear systems, that is,

$$Ax = b \tag{1}$$

where $A \in C^{n\times n}$ is a large sparse non-Hermitian positive definite matrix, and $x, b \in C^n$. To solve (1) iteratively, the efficient splitting of the coefficient matrix A are usually required. For example, the classic Jacobi and Gauss-Seidel iteration split the matrix A into its diagonal and off-diagonal parts. One can see [3, 5-8] for a comprehensive survey. Recently, a Hermitian and skew-Hermitian splitting [1] gains people's attention, that is,

$$A = H + S$$

where $H = 2^{-1}(A + A^*)$ and $S = 2^{-1}(A - A^*)$.

In [1], Bai, Golub and Ng developed the HSS iteration method by the above splitting of A for non-Hermitian positive definite system:

Given an initial guess $x^{(0)}$, for $k = 0,1,2,\dots$, until $x^{(k)}$ converges, compute

$$\begin{cases} (\alpha I + H)x^{(k+2^{-1})} = (\alpha I - S)x^{(k)} + b, \\ (\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+2^{-1})} + b, \end{cases} \tag{2}$$

where $H = 2^{-1}(A + A^*)$ and $S = 2^{-1}(A - A^*)$ are the Hermitian and skew-Hermitian parts of A, respectively, and α is a given positive constant.

In matrix-vector form, the HSS iteration method (2) can be equivalently rewritten as

$$x^{(k+1)} = M(\alpha)x^{(k)} + G(\alpha)b, \ k = 0,1,2,\dots$$

where

$$\begin{cases} M(\alpha) = (\alpha I + S)^{-1}(\alpha I - H)(\alpha I + H)^{-1}(\alpha I - S), \\ G(\alpha) = 2\alpha(\alpha I + S)^{-1}(\alpha I + H)^{-1}. \end{cases}$$

Here, $M(\alpha)$ is the iteration matrix of the HSS iteration method. To describe the convergence property of the HSS iteration, the following theorem was established in [1].

Theorem 1 Let $A \in C^{n\times n}$ be a positive definite matrix, $H = 2^{-1}(A + A^*)$ and $S = 2^{-1}(A - A^*)$ be, respectively, its Hermitian and skew-Hermitian parts, and α be a positive constant. Then the spectral radius $\rho(M(\alpha))$ of the iteration matrix $M(\alpha)$ of the HSS method is bounded by

$$\gamma(\alpha) = \max_{\lambda_i \in \lambda(H)} \left| \frac{\alpha - \lambda_i}{\alpha + \lambda_i} \right|$$

with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ being the eigenvalues of H. Thus, it holds that

$$\rho(M(\alpha)) \le \gamma(\alpha) < 1 \text{ for all } \alpha > 0.$$

The optimal parameter α is

$$\overline{\alpha} = \arg\min_{\alpha} \left\{ \max_{\lambda_n \le \lambda \le \lambda_1} \left| \frac{\alpha - \lambda}{\alpha + \lambda} \right| \right\} = \sqrt{\lambda_1 \lambda_n}$$

and

$$\gamma\left(\overline{\alpha}\right) = \frac{\sqrt{\lambda_1} - \sqrt{\lambda_n}}{\sqrt{\lambda_1} + \sqrt{\lambda_n}}. \tag{3}$$

Recently, Li, Huang and Liu [2] presented the lopsided Hermitian and skew-Hermitian splitting

(LHSS) iteration method based on the following splitting:

$$A = H + S = (\alpha I + S) - (\alpha I - H).$$

The LHSS iteration method: Given an initial guess $x^{(0)}$, for $k = 0,1,2,\dots$, until $x^{(k)}$ converges, compute

$$\begin{cases} Hx^{(k+2^{-1})} = -Sx^{(k)} + b, \\ (\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+2^{-1})} + b, \end{cases} \quad (4)$$

where α is a given non-zero positive constant.

Note that when the matrix A is positive definite, H must be a positive definite matrix with S being skew-Hermitian and

$$A + A^* = H + H^*.$$

Since S is skew-Hermitian, it is not difficult to see that $\alpha I + S$ is also nonsingular. The above LHSS iteration method (4) can be equivalently transformed into the following matrix-vector form:

$$x^{(k+1)} = \mathbf{M}(\alpha)x^{(k)} + \mathbf{G}(\alpha)b,\ k = 0,1,2,\dots \quad (5)$$

where

$$\begin{cases} \mathbf{M}(\alpha) = (\alpha I + S)^{-1}(\alpha I - H)H^{-1}(-S), \\ \mathbf{G}(\alpha) = \alpha(\alpha I + S)^{-1}H^{-1}. \end{cases}$$

Here, $\mathbf{M}(\alpha)$ is the iteration matrix of the LHSS method. In fact, (5) may also result from the splitting

$$A = M - N$$

of the coefficient matrix A, with

$$\begin{cases} M = \alpha^{-1}H(\alpha I + S), \\ N = \alpha^{-1}(\alpha I - H)(-S). \end{cases}$$

The following theorem established in [2] describes the convergence property of the LHSS iteration.

Theorem 2 Let A, H and S be defined as those in Theorem 1 and α be a non-zero constant. Then the spectral radius $\rho(\mathbf{M}(\alpha))$ of the iteration matrix $\mathbf{M}(\alpha)$ of the LHSS iteration is bounded by

$$\delta(\alpha) = \frac{\sigma_{\max}}{\sqrt{\alpha^2 + \sigma_{\max}^2}} \max_{\lambda_i \in \lambda(H)} \left| \frac{\alpha - \lambda_i}{\lambda_i} \right|,$$

where $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ are the eigenvalues of H and $\sigma_{\max}$ is the maximum singular value of S. Moreover, we have

(i) If $\sigma_{\max} \le \lambda_n$, when

$$\alpha > 0 \quad \text{or} \quad \alpha < \frac{2\lambda_1\sigma_{\max}^2}{\sigma_{\max}^2 - \lambda_1^2},$$

the bound of $\delta(\alpha) < 1$, i.e., the LHSS iteration converges;

(ii) If $\lambda_n < \sigma_{\max} < \lambda_1$, when

$$0 < \alpha < \frac{2\lambda_n\sigma_{\max}^2}{\sigma_{\max}^2 - \lambda_n^2} \quad \text{or} \quad \alpha < \frac{2\lambda_1\sigma_{\max}^2}{\sigma_{\max}^2 - \lambda_1^2},$$

the bound of $\delta(\alpha) < 1$, i.e., the LHSS iteration converges;

(iii) If $\sigma_{\max} \ge \lambda_1$, when

$$0 < \alpha < \frac{2\lambda_n\sigma_{\max}^2}{\sigma_{\max}^2 - \lambda_n^2}$$

the bound of $\delta(\alpha) < 1$, i.e., the LHSS iteration converges.

The optimal parameter α is obtained at

$$\alpha^* = \frac{2\lambda_1\lambda_n}{\lambda_1 + \lambda_n}$$

and

$$\delta(\alpha^*) = \frac{(\lambda_1 - \lambda_n)\sigma_{\max}}{\sqrt{4\lambda_1^2\lambda_n^2 + \sigma_{\max}^2(\lambda_1 - \lambda_n)^2}}. \quad (6)$$

Theorem 2 shows that if the coefficient matrix A is positive definite the LHSS iteration (5) converges to the unique solution of the linear systems (1) for a loose restriction on the choice of α. Moreover, the upper bound of the contraction factor of the LHSS iteration is dependent on the choice of α, the spectrum of the Hermitian part H and the maximum singular value of the skew-Hermitian part S, that is, $\sigma_{\max}$, but is independent of the rest singular values of S as well as the eigenvectors of the matrices A, H and S.

Remark 1 Since S is the skew-Hermitian matrix, then the eigenvalues of S are complex number without the real parts. Let $\mu(S) = \{i\mu_1, i\mu_2, \dots i\mu_n\}$ $\mu_i \in R$ $(i = 1,\dots,n)$ and let $\mu_{\max} = \max\{|\mu_1|,\dots,|\mu_n|\}$. It is easy to know that $\sigma_{\max} = \mu_{\max}$. That is to say, $\sigma_{\max}$ of Theorem 2 can be replaced by $\mu_{\max}$, which shows that the upper bound of the contraction factor of the LHSS iteration is dependent on the choice of α, the spectrum of the Hermitian part H and the maximum module of the eigenvalues of the skew-Hermitian part S, but is independent of the module of the rest eigenvalue of S as well as the eigenvectors of the matrices A, H and S.

2 Main results

We now give the following main result, which is a new criterion for choosing between the two methods.

Theorem 3 Let λ_1, λ_n, α^* and $\delta(\alpha^*)$ be defined as those in Theorem 2, respectively. Let $\overline{\alpha}$ and $\gamma\left(\overline{\alpha}\right)$ be defined as those in Theorem 1, respectively.

Case 1: If $\lambda_1 = \lambda_n$, then $\delta(\alpha^*) = \gamma\left(\overline{\alpha}\right)$;

Case 2: If $\lambda_1 \neq \lambda_n$ and

$$\frac{\lambda_1^2 \lambda_n^2}{(\sqrt{\lambda_1} + \sqrt{\lambda_n})^2 \sqrt{\lambda_1 \lambda_n}} < \sigma_{\max}^2,$$

then the following inequality holds:

$$\gamma\left(\overline{\alpha}\right) < \delta(\alpha^*).$$

Proof. It is easy to know that Case 1 holds from (3) and (6). We need only to prove that Case 2 holds.

Supposing $\gamma\left(\overline{\alpha}\right) < \delta(\alpha^*)$. From $\lambda_1 > \lambda_n$, we have

$$\frac{\sqrt{\lambda_1} - \sqrt{\lambda_n}}{\sqrt{\lambda_1} + \sqrt{\lambda_n}} < \frac{(\lambda_1 - \lambda_n)\sigma_{\max}}{\sqrt{4\lambda_1^2 \lambda_n^2 + \sigma_{\max}^2 (\lambda_1 - \lambda_n)^2}},$$

which implies

$$\frac{1}{\sqrt{\lambda_1} + \sqrt{\lambda_n}} < \frac{(\sqrt{\lambda_1} + \sqrt{\lambda_n})\sigma_{\max}}{\sqrt{4\lambda_1^2 \lambda_n^2 + \sigma_{\max}^2 (\lambda_1 - \lambda_n)^2}}.$$

Applying squaring operation on both sides and combining the coefficients of $\sigma_{\max}$, we get

$$4\lambda_1^2 \lambda_n^2 < [(\sqrt{\lambda_1} + \sqrt{\lambda_n})^4 - (\lambda_1 - \lambda_n)^2]\sigma_{\max}^2.$$

By the simple manipulations, we obtain

$$\lambda_1^2 \lambda_n^2 < \sqrt{\lambda_1 \lambda_n} (\sqrt{\lambda_1} + \sqrt{\lambda_n})^2 \sigma_{\max}^2,$$

which completes the proof.

Apparently, we have the following theorem.

Theorem 4 Under conditions of Theorem 3, If $\lambda_1 \neq \lambda_n$ and

$$\sigma_{\max}^2 < \frac{\lambda_1^2 \lambda_n^2}{(\sqrt{\lambda_1} + \sqrt{\lambda_n})^2 \sqrt{\lambda_1 \lambda_n}},$$

then $\gamma\left(\overline{\alpha}\right) < \delta(\alpha^*)$ holds.

Remark 2 It is easy to know that $\sigma_{\max}$ of Theorem 3 and 4 can be replaced by $\mu_{\max}$. Moreover, Theorem 2.5 in [2] is invalid when $\lambda_1 = \lambda_n$. In fact, in this case, $\delta(\alpha^*) = \gamma\left(\overline{\alpha}\right)$.

Corollary 1 Under conditions of Theorem 3, if $\lambda_1 \neq \lambda_n$ and $2^{-1}\lambda_1\lambda_2 < \sigma_{\max}^2$, then $\gamma\left(\overline{\alpha}\right) < \delta(\alpha^*)$.

3 Two examples

Example 1 We consider the three-dimensional convection-diffusion equation

$$-(u_{xx} + u_{yy} + u_{zz}) + q(u_x + u_y + u_z) = f(x, y, z)$$

on the unit cube $\Omega = [0,1]\times[0,1]\times[0,1]$, with constant coefficient q and subject to Dirichlet-type boundary conditions. Discretizing this equation with seven-point finite difference and assuming the numbers (n) of grid points in all three directions are the same, we define $h = 1/(1+n)$ as the step size, $r = qh/2 (q > 0)$ is the mesh Reynolds number. From [1], when the problem size becomes reasonably large, i.e., h is reasonably small, we know for the centred difference scheme that

$$\lambda_1 = 6(1 + \cos \pi h) \approx 12,$$
$$\lambda_n = 6(1 - \cos \pi h) \approx 3\pi^2 h^2,$$
$$\sigma_{\max} = 6r \cos \pi h \approx 3qh.$$

From Theorem 3, we know that if

$$\frac{8\pi^3 h}{(2 + \pi h)^2} \leq \frac{8\pi^3 h}{8\pi h} = \pi^2 < q^2, \qquad (7)$$

that is, $q > \pi$, then the HSS method may be a good choice.

From Theorem 4, we find that if

$$q^2 < \frac{8\pi^3 h}{9} \leq \frac{8\pi^3 h}{(2 + \pi h)^2}, \qquad (8)$$

the LHSS method may be a good choice.

In the sequel, we investigate the spectral radius of the matrix $M(\alpha)$ and $\mathbf{M}(\alpha)$ with the different value of q. All the matrices tested are 512×512 unless otherwise mentioned in Example 1, i.e., $n = 8$. From the above discussing, we know that if $n = 8$, then

$$\frac{8\pi^3 h}{9} \doteq 3.0623$$

To confirm the results of (7) and (8), here we test q is equal to 1, 1.5, 50 and 100, respectively. In figures (1)-(4), we show that the spectral radius $\rho(M(\alpha))$ and $\rho(\mathbf{M}(\alpha))$ of the iteration matrices of both LHSS and HSS method with the defferent value of α, and the bound $\delta(\alpha)$.

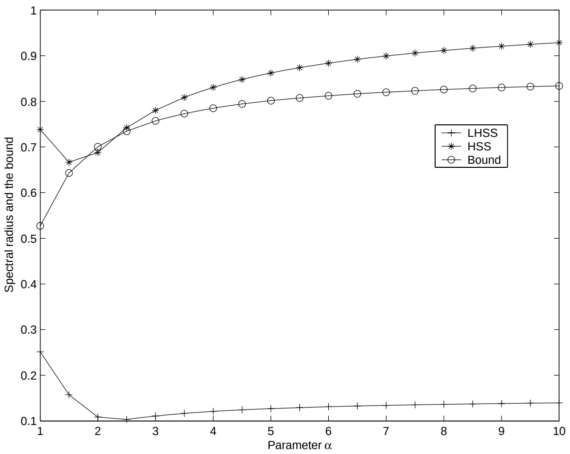

Fig 1. The spectral radius and bound with q=1

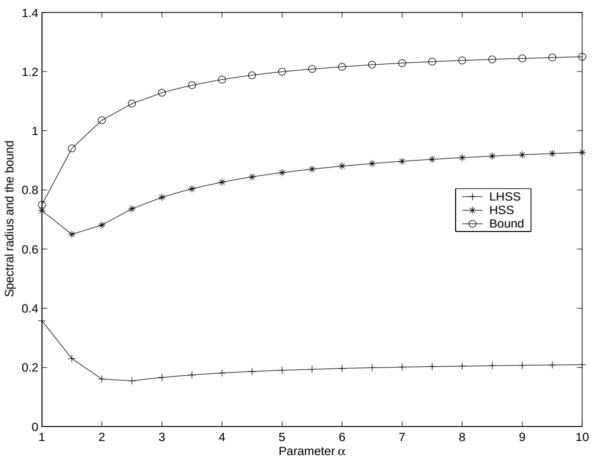

Fig 2. The spectral radius and bound with q=1.5

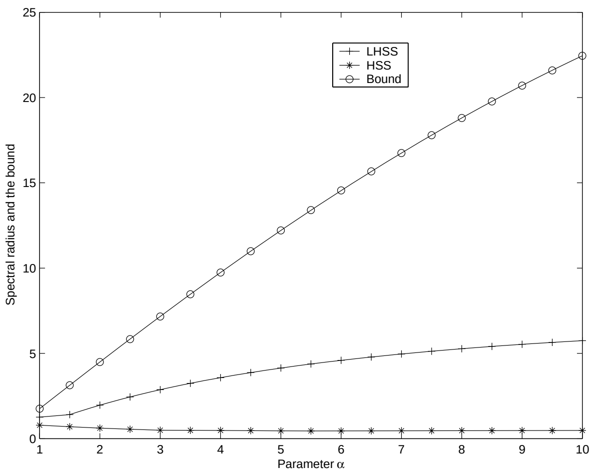

Fig 3. The spectral radius and bound with q=50

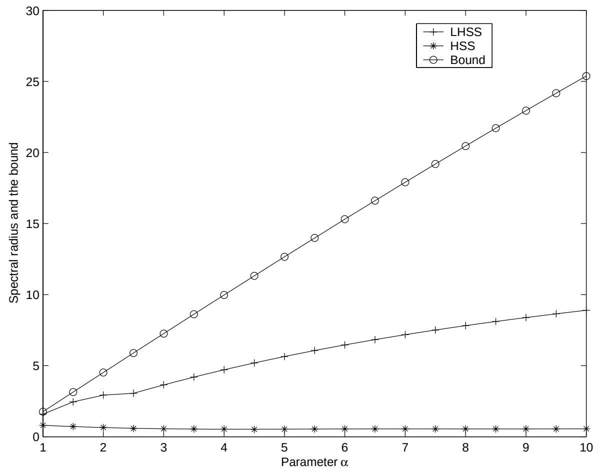

Fig 4. The spectral radius and bound with q=100

From figures (1)-(4), it is not difficult to find that the spectral radius of the iteration matrix LHSS method is much smaller than that of the HSS method with q be 1 and 1.5, on the other hand, the spectral radius of the HSS method seems to be better than that of LHSS method when q are equal to 50 and 100. In the meanwhile, here the distribution of the eigenvalues of the iteration matrix is depicted in figures (5)-(8) with α=1, which correspond to the different value of q.

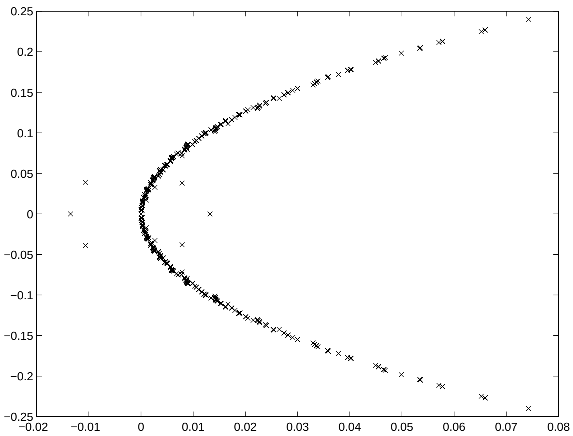

Fig 5. The distribution of eigenvalues with q=1

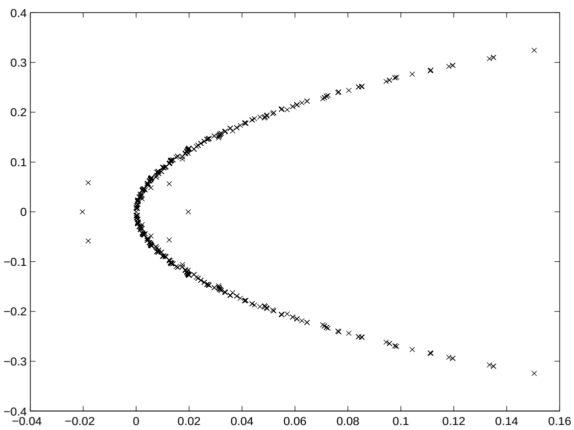

Fig 6. The distribution of eigenvalues with q=1.5

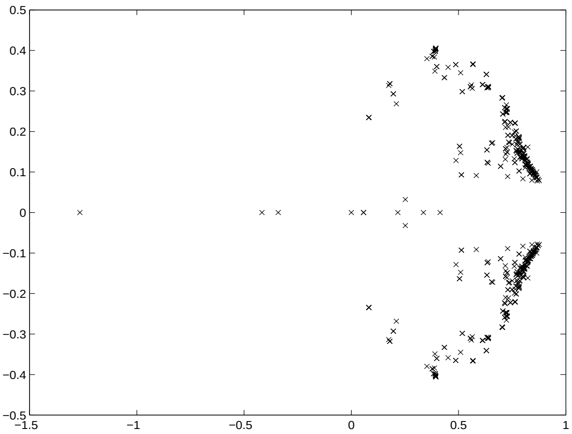

Fig 7. The distribution of eigenvalues with q=50

From figures (5)-(8), we find that the distribution of the eigenvalues of the iteration matrix becomes clustered when q becomes large. It is not difficult to find that the spectral radius becomes large with the increasing of q (the matrix becomes dominant).

Next, we study the HSS and LHSS iteation method. We try to use the HSS and LHSS iteration method to solve the systems of linear equation (1), which raises from the disccretized three dimension convection-diffusion equations.

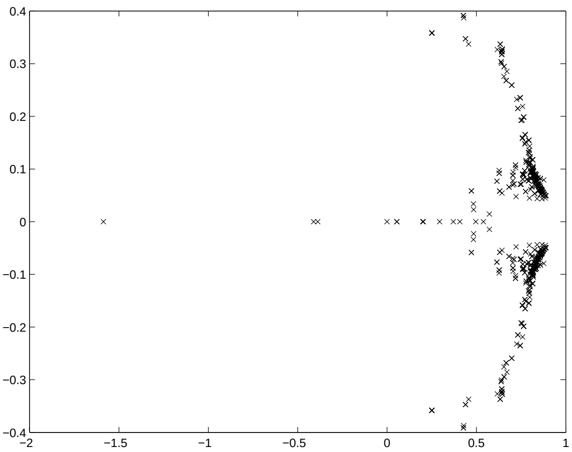

Fig 8. The distribution of eigenvalues with q=100

For the simplicity, we set up the tested problem so that the right hand side function is equal to 1 throughout the unit cube domain. All tests are started from the zero vector, performed in Matlab 6.5, and terminated when the current iterate satisfies

$$\frac{\|r_k\|_2}{\|r_0\|_2} < 10^{-6},$$

where $r_0 = b - Ax_0$ and r_k is the residual of the k-th HSS and LHSS iteration.

Some results are presented to illustrate the behavior of the convergence of the HSS and LHSS method, respectively, which are listed in Tables 1 – 4, corresponding to Figures (9)-(12). The purpose of these experiments is just to investigate the influence of the eigenvalue distribution on the convergence behavior of HSS and LHSS method, respectively.

α	HSS(IT)	LHSS(IT)
1	49	5
2	34	5
3	51	5
4	68	5
5	85	5
6	102	5
7	119	5
8	136	5
9	153	5
10	170	5

Table 1. Iteration number (IT) of HSS and LHSS with $q = 1$.

α	HSS(IT)	LHSS(IT)
1	42	7
2	33	6
3	50	6
4	67	6
5	83	7
6	100	7
7	117	7
8	133	7
9	150	7
10	167	7

Table 2. Iteration number (IT) of HSS and LHSS with $q = 1.5$.

α	HSS(IT)	LHSS(IT)
1	52	2098
2	27	1037
3	20	666
4	19	550
5	19	493
6	20	460
7	21	437
8	22	420
9	23	409
10	24	401

Table 3. Iteration number (IT) of HSS and LHSS with $q = 50$.

α	HSS(IT)	LHSS(IT)
1	54	1525
2	27	653
3	25	542
4	24	452
5	23	405
6	25	377

7	25	355
8	26	341
9	26	329
10	27	321

Table 4. Iteration number (IT) of HSS and LHSS with $q = 100$.

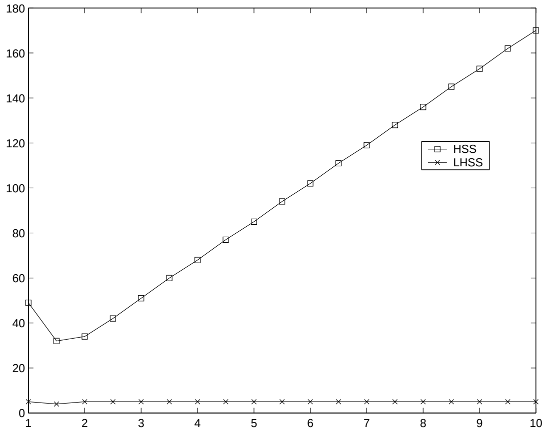

Fig 9. Iteration number with the different α , and $q=1$

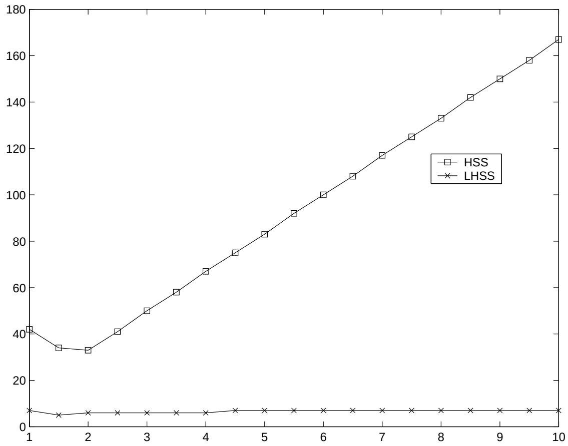

Fig 10. Iteration number with the different α , and $q=1.5$

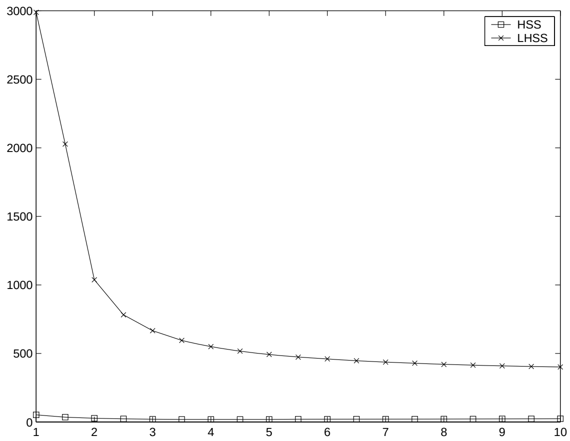

Fig 11. Iteration number with the different α , and $q=50$

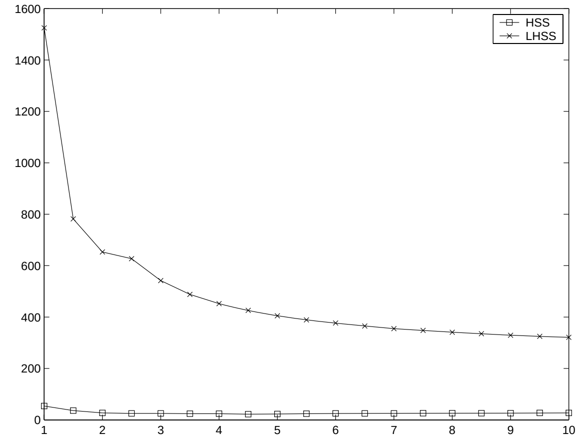

Fig 12. Iteration number with the different α , and $q=100$

From tables 1-4 and figures (9)-(12), it is shown that the LHSS performs very good for a wide range of the parameter α when q satisfies (8). In the meanwhile, when q satisfies (7), the HSS method is more efficient than LHSS method.

Example 2 We consider the two-dimensional convection-diffusion equation

$$-(u_{xx} + u_{yy}) + \tau(u_x + u_y) = g(x, y)$$

on the unit square $[0,1]\times[0,1]$, with constant coefficient τ and subject to Dirichlet-type boundary conditions. When the five-point centered finite difference discretization is used to it, it is easy to get the system of the linear system (1) with the coefficient matrix

$$A = T \otimes I + T \otimes I$$

and

$$T = \mathrm{tridiag}(-1-r, 2, -1+r),$$

where $r = \tau h/2$ is the mesh Reynolds number and $\otimes$ denotes the Kronecker product symbol, and $h = 1/(1+m)$ is used in the discretization on both disrections and the natural lexicograghic ordering is employed to the unkowns.

From [4], it is easy to get that for the centred difference scheme

$$\lambda_1 = 4(1+\cos \pi h) \approx 8,$$

$$\lambda_n = 4(1-\cos \pi h) \approx 2\pi^2 h^2,$$

$$\sigma_{\max} = 4r \cos \pi h \approx 2\tau h.$$

By simple computations, it is easy to get that if

$$\tau > \pi \tag{9}$$

then the HSS method may be a good choice. And if

$$\tau^2 < \frac{8\pi^3 h}{9} \le \frac{8\pi^3 h}{(2+\pi h)^2}, \tag{10}$$

the LHSS method may be a good choice.

Be similar to Example 1, we consider the spectral radius, the distribution of the eigenvalues of the iteration matrix and iteration number in Example 2 when the HSS and LHSS method are applied to solve linear systems (1), respectively.

In the sequel, we investigate the spectral radius of the matrix $M(\alpha)$ and $\mathbf{M}(\alpha)$ with the different value of τ. All the matrices tested are 100×100 unless otherwise mentioned in example 2, i.e., $\mathrm{m}=10$, From the above discussing, it easy to get that if $\mathrm{m}=10$, then

$$\frac{8\pi^3 h}{9} \doteq 2.5056$$

To explain the results of (9) and (10), here we test τ is also equal to 1, 1.5, 50 and 100, respectively. In figures (13)-(16), we show that the spectral radius $\rho(M(\alpha))$ and $\rho(\mathbf{M}(\alpha))$ of the iteration matrices of both LHSS and HSS method with the different value of τ, and the bound $\delta(\alpha)$.

From the following figures (13)-(16), it is also to find that the spectral radius of the iteration matrix HSS method is much larger than that of LHSS method with τ be 1 and 1.5, on the other hand, the spectral radius of the HSS method seems to be better than that of LHSS method when τ becomes large.

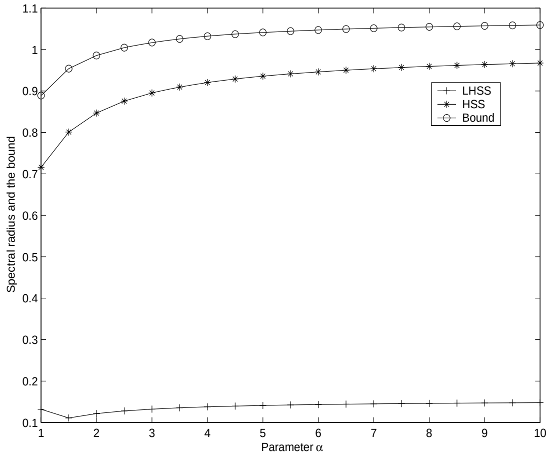

Fig 13. The spectral radius and bound with $\tau=1$

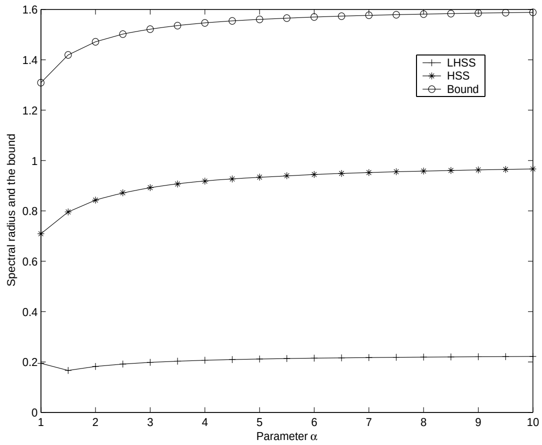

Fig 14. The spectral radius and bound with $\tau=1.5$

In the meanwhile, here figures (17)-(20) describe the distribution of the eigenvalues of the iteration matrix with $\alpha=1$, which correspond to the different value of τ.

From figures (17)-(20), it is easy to know that the distribution of the eigenvalues of the iteration matrix becomes clustered when τ becomes large.

Now we investigate the perform of the HSS and LHSS method, respectively, which is applied to solve the systems of linear equation (1) raising from the disccretized two-dimensional convection-diffusion equation.

For convenience, we set up Example 2 so that the right hand side function is also equal to 1 throughout the unit square domain. All tests are also started from the zero vector and the current iterate satisfies

$$\frac{\|r_k\|_2}{\|r_0\|_2} < 10^{-6}.$$

We present some results in Tables 5–8 to make out the behavior of the convergence of the

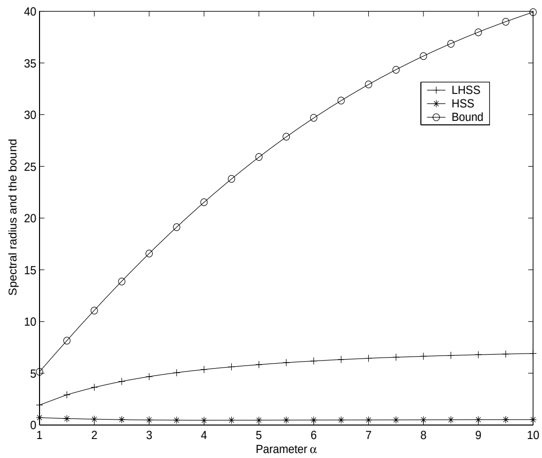

Fig 15. The spectral radius and bound with $\tau=50$

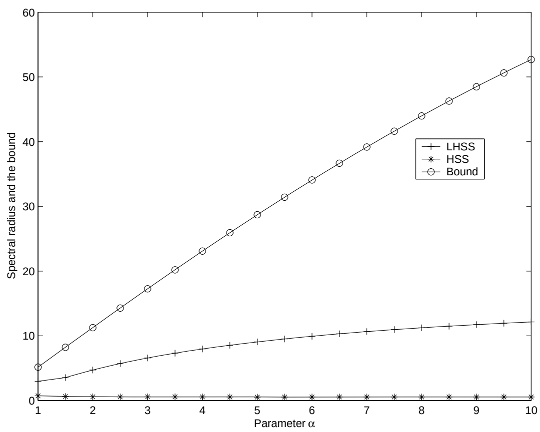

Fig 16. The spectral radius and bound with $\tau=100$

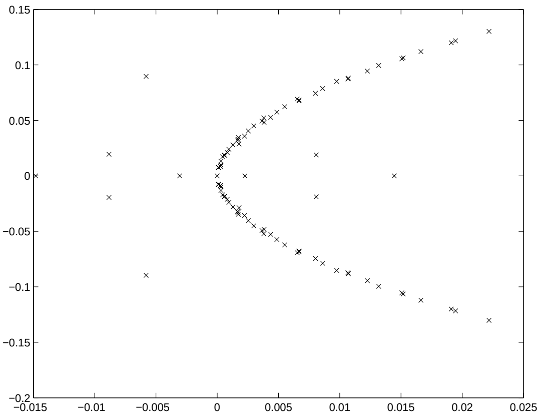

Fig 17. The distribution of eigenvalues with $\tau=1$

HSS and LHSS method, respectively. Figures (21)-(24) depict the behavior of the HSS and LHSS method with the different value of τ , which is applied to solve (1), respectively. The purpose of these experiments is just to further reflect and confirm the influence of the eigenvalue distribution on the convergence behavior of HSS and LHSS method in Example 2, respectively. The solutions of the linear systems in each iteration are computed exactly.

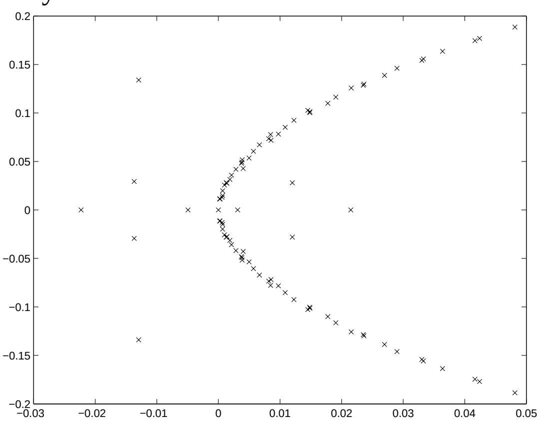

Fig 18. The distribution of eigenvalues with $\tau=1.5$

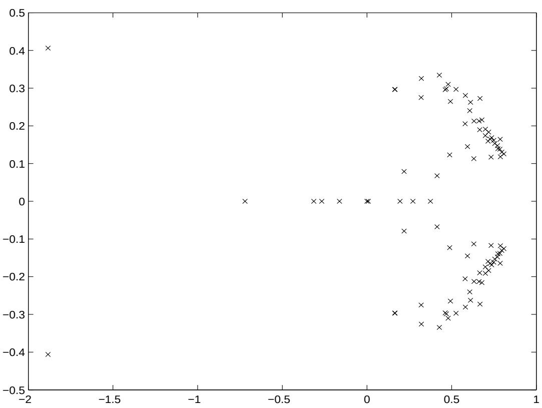

Fig 19. The distribution of eigenvalues with $\tau=50$

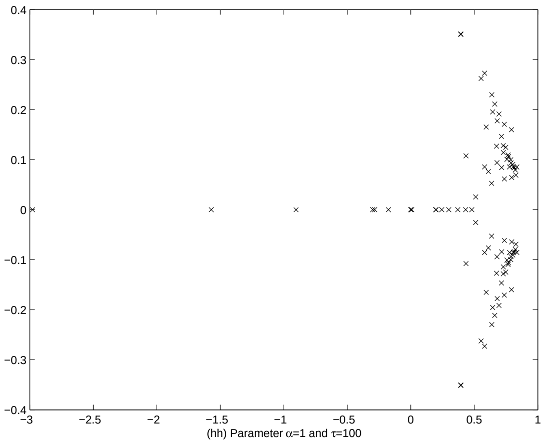

Fig 20. The distribution of eigenvalues with $\tau=100$

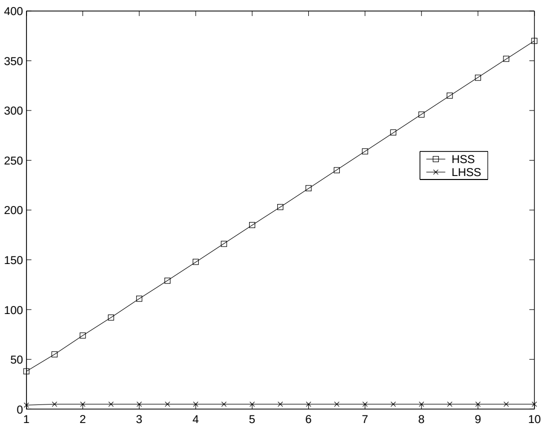

Fig 21. Iteration number with the different α , and $\tau=1$

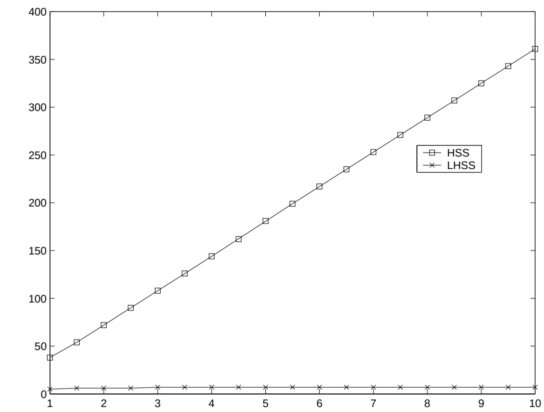

Fig 22. Iteration number with the different α , and $\tau=1.5$

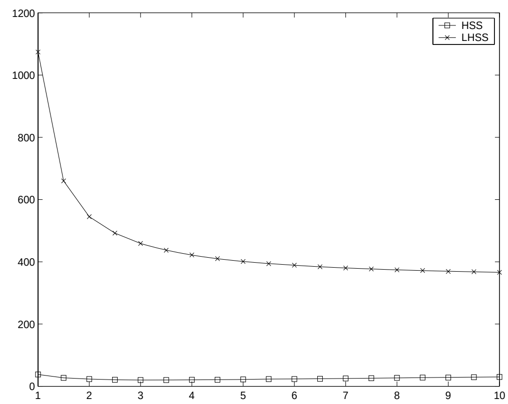

Fig 23. Iteration number with the different α , and τ=50

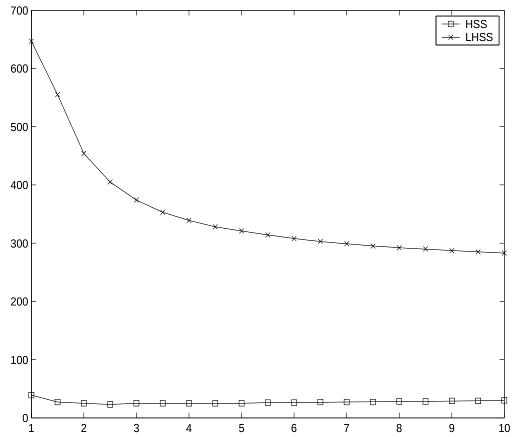

Fig 24. Iteration number with the different α , and τ=100

α	HSS(IT)	LHSS(IT)
1	38	4
2	74	5
3	111	5
4	148	5
5	185	5
6	222	5
7	259	5
8	296	5
9	333	5
10	370	5

Table 5. Iteration number (IT) of HSS and LHSS with $\tau = 1$.

α	HSS(IT)	LHSS(IT)
1	38	5
2	72	6
3	108	7
4	144	7
5	181	7
6	217	7
7	253	7
8	289	7
9	325	7
10	361	7

Table 6. Iteration number (IT) of HSS and LHSS with $\tau = 1.5$.

α	HSS(IT)	LHSS(IT)
1	38	1074
2	23	545
3	20	459
4	21	422
5	22	401
6	23	389
7	25	380
8	27	374
9	28	369
10	30	366

Table 7. Iteration number (IT) of HSS and LHSS with $\tau = 50$.

α	HSS(IT)	LHSS(IT)
1	39	647
2	25	454
3	25	374
4	25	339
5	25	321
6	26	308
7	27	299
8	28	292
9	29	287
10	30	283

Table 8. Iteration number (IT) of HSS and LHSS with $\tau = 100$.

Remark 3 From (7)-(10), we find that if the problem size becomes more and more large, that is, h is reasonably small, q or τ must be very small when the LHSS method is used to apply. That is, Theorem 4 is too restrictive to be useful. In fact, for the case that q or τ is very small, many other efficient methods like the generalized conjugate gradient method should be much more efficient and practical than the LHSS method introduced by Li et al [2]. However, how to make the LHSS method more efficient may be further studied.

Remark 4 By observing Examples 1 and 2, q or τ may be chosen to be independent of h when the HSS method is applied to solve (1). Namely, even if the problem size becomes reasonably large, q or τ chosen is still independent of the value of h involved when the HSS method is applied to solve (1).

Remark 5 In fact, an upper bound may not truly reflect the convergence behavior of an iteration method from Theorem 3 and Theorem 4. However, by comparing the HSS method and the LHSS method, it is not difficult to find that the HSS method may be much more efficient and practical than the LHSS method. There exist two main aspects. Firstly, from Theorem 1 and Theorem 2, it is not difficult to find that the convergence domain of LHSS may be smaller than that of HSS. Secondly, from (2) and (4), under conditions of Theorem 1 and Theorem 2, we can find that LHSS may be less efficient than HSS as only a sub-system of linear equations of the coefficient matrix H is solved rather than that of the better-conditioned coefficient matrix $\alpha I + H$. Indeed, let $\lambda_{\max}(H)$ and $\lambda_{\min}(H)$, respectively, be the maximum and minimum eigenvalue of H . $\kappa(H)$ denotes the spectral condition number of H . It is easy to obtain that

$$\kappa(\alpha I + H) = \frac{\alpha + \lambda_{\max}(H)}{\alpha + \lambda_{\min}(H)} < \frac{\lambda_{\max}(H)}{\lambda_{\min}(H)} = \kappa(H)$$

with $\alpha > 0$.

4 Conclusion

In this note, we have compared the HSS and LHSS method to solve non-Hermitian positive definite linear systems. A new criterion for choosing the above two methods has been presented. Numerical tests show that the LHSS method performs very well if the Hermitian part of the coefficient matrix is dominant, however as the skew-Hermitian part becomes dominant the performance of the LHSS method becomes not as good as the HSS method. Therefore, we need to improve the LHSS method, such as introduce another parameter β in the first equation in (4), which we will study in the future.

Acknowledgements we would like to thank the anonymous referees, whose many helpful suggestions have substantially improved this paper.

This research was supported by 973 Programs (2008CB317110), NSFC (10771030), the Scientific and Technological Key Project of the Chinese Education Ministry (107098), the PhD. Programs Fund of Chinese Universities (20070614001), Sichuan Province Project for Applied Basic Research (2008JY0052) and the Project for Academic Leader and Group of UESTC.

References:

[1] Z.-Z Bai, G.H. Golub, M.K. Ng, Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems, *SIAM J. Matrix Anal. Appl.*, Vol.24, No.3, 2003, pp. 603-626.

[2] L. Li, T.Z. Huang, X.P Liu, Modified Hermitian and skew-Hermitian splitting methods for non-Hermitian positive-definite linear systems, *Numer. Lin. Alg. Appl.*, 14, 2007, pp, 217-235.

[3] Y. Saad, *Iterative Methods for Sparse Linear Systemsk*, PWS Publishing Company, Boston, 1996.

[4] Z.-Z Bai, G.H. Golub, C.K. Li, Optimal parameter in hermitian and skew-hermitian splitting method for certain two-by-two block matrices, *SIAM J. Sci. Comput.*, Vol.28, No.2, 2006, pp. 583-603.

[5] D.D. Hamide, Computers and linear algebra, *Wseas Transactions on Mathematics*, Vol.3 Issue 3, 2004,537-542.

[6] M.T. Darivishi, The best values of parameters in Accelerated successive overrelaxation methods, *Wseas Transactions on Mathematics,* Vol.3, Issue 3, 2004, pp, 705-510.

[7] Y.M. Li, A monotone iterative method for bipolar junction transistor circuit simulation, *Wseas Transactions on Mathematics*, Vol.1, Issue 4, 2002, pp, 159-164.

[8] S.J. Zhang, E. Blosch, X. Zhao, Conjugate gradient linear solvers for computational fluid dynamics, *Wseas Transactions on Mathematics,* Vol.6, Issue 1, 2004, pp, 183-190.