Bifuzzy ideals of K-algebras

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Abstract: In this paper we introduce the notion of bifuzzy ideals of K-algebras and investigate some interesting properties. Then we study the homomorphisms between the ideals of K-algebras and their relationship between the domains and the co-domains of the bifuzzy ideals under these homomorphisms. Finally the Cartesian product of bifuzzy ideals is discussed.

Key–Words: Bifuzzy ideals; Characteristic; Equivalence relations; Homomorphisms; Cartesian product.

1 Introduction

The study of BCK-algebras [22] was initiated by Imai and Iséki in 1966 as a generalization of set-theoretic difference and propositional calculus i.d. classical and non-classical calculus. In the same year, Iséki introduced BCI-algebras [23] as a super class of the class of BCK-algebras. In 1983, Hu and Li introduced BCH-algebras [21]. They demonstrated that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. Non-classical logic has become a considerable formal tool for computer science and computational intelligence to deal with fuzzy information and uncertain information. In particular, $BCK/BCI/BCH$-algebras are non-classical logic algebras and they are algebraic formulations of BCK-system, BCI-system and BCH-system in combinatory logic.

The notion of a K-algebra $(G, \cdot, \odot, e)$ was first introduced by Dar and Akram in [10]. A K-algebra was built on a group $(G, \cdot, e)$ with identity element e, by adjoining the induced binary operation $\odot$ on $(G, \cdot, e)$. It is attached to an abstract K-algebra $(G, \cdot, \odot, e)$, which is non-commutative and non-associative with right identity element e. It is proved in [3, 10] that a K-algebra on an abelian group is equivalent to a p-semisimple BCI-algebra. For the convenience of study, authors renamed *a K-algebra built on a group G* as a $K(G)$-algebra [11]. The $K(G)$-algebra has been characterized by using its left and right mappings in [11]. Recently, Dar and Akram [13] have further proved that the class of $K(G)$-algebras is a generalized class of B-algebras [31] when $(G, \cdot, e)$ is a non-abelian group, and they also proved that the $K(G)$-algebra is a generalized class of the class of $BCH/BCI/BCK$-algebras [21, 22, 23] when $(G, \cdot, e)$ is an abelian group.

The concept of a fuzzy set was introduced by Zadeh [34], and it is now a rigorous area of research with manifold applications ranging from engineering and computer science to medical diagnosis and social behavior studies. Fuzzy set theory has pervaded almost all fields of study and its applications have percolated down to consumers goods level. Apart from this, it is being applied on major scale in industries through intelligent robots for machine-building such as car, engines, turbines, ships. In 1983, Atanassov introduced notion of intuitionistic fuzzy sets as a generalization of fuzzy sets. In 1995, Gerstenkorn and Mańko [20] re-named the intuitionistic fuzzy sets as bifuzzy sets. The elements of the bifuzzy sets are featured by an additional degree which is called the degree of uncertainty. Bifuzzy sets have also been defined by Takeuti and Titanti in [33]. Takeuti and Titanti considered bifuzzy logic in the narrow sense and derived a set theory from logic which they called bifuzzy set theory. The bifuzzy sets have drawn the attention of many researchers in the last decades. This is mainly due to the fact that bifuzzy sets are consistent with human behavior, by reflecting and modeling the hesitancy present in real-life situations. This kind of fuzzy sets have gained a wide recognition as a useful tool in the modeling of some uncertain phenomena. These have numerous applications in various areas of sciences, for instance, computer science, mathematics, medicine, chemistry, economics, astronomy, etc. Akram *et al.* introduced the notions of subalgebras and fuzzy (maximal) ideals of K-algebras in [1], and further was studied in [3, 4, 5, 9, 24]. In this paper we introduce the notion of bifuzzy ideals of K-algebras and investigate some of their properties. We study the homomorphisms between the ideals of K-algebras and their relationship between the domains and the co-domains of the bifuzzy ideals under these homomorphisms. Finally the Cartesian product of bifuzzy ideals is discussed. The definitions and terminologies that we used in this paper are standard. For other notations and terminologies not mentioned in this paper, the readers are refereed to [6, 7, 8, 34, 35].

2 Preliminaries

In this section we review some elementary aspects that are necessary for this paper.
A K-algebra $\mathcal{K} = (G, \cdot, \odot, e)$ [10] is an algebra of type (2, 2, 0) defined on a group $(G, \cdot, e)$ in which each non-identity element is not of order 2 and observes the following $\odot$-axioms:

(K1) $(x \odot y) \odot (x \odot z) = (x \odot (z^{-1} \odot y^{-1})) \odot x = (x \odot ((e \odot z) \odot (e \odot y))) \odot x$,

(K2) $x \odot (x \odot y) = (x \odot (e \odot y)) \odot x$,

(K3) $x \odot x = e$,

(K4) $x \odot e = x$,

(K5) $e \odot x = x^{-1}$

for all x, y, $z \in G$.
If the group $(G, \cdot, e)$ is abelian , then the above axioms $(K1)$ and $(K2)$ can be replaced by:

$(\overline{K1})$ $(x \odot y) \odot (x \odot z) = z \odot y$.

$(\overline{K2})$ $x \odot (x \odot y) = y$.

A nonempty subset H of a K-algebra $\mathcal{K}$ is called a *subalgebra* [10] of the K-algebra $\mathcal{K}$ if $a \odot b \in H$ for all a, $b \in H$. Note that every subalgebra of a K-algebra $\mathcal{K}$ contains the identity e of the group $(G, \cdot, e)$. A mapping $f : \mathcal{K}_1 = (G_1, \cdot, \odot, e_1) \to \mathcal{K}_2 = (G_2, \cdot, \odot, e_2)$ of K-algebras is called a *homomorphism* [12] if $f(x \odot y) = f(x) \odot f(y)$ for all $x, y \in \mathcal{K}_1$. We note that if f is a homomorphism, then $f(e) = e$. A nonempty subset I of a K-algebra $\mathcal{K}$ is called an *ideal* [1] of $\mathcal{K}$ if it satisfies:

(i) $e \in I$,

(ii) $x \odot y \in I$, $y \odot (y \odot x) \in I \Rightarrow x \in I$ for all x, $y \in G$.

Let μ be a *fuzzy set* on G, i.e., a map $\mu : G \to [0, 1]$. A fuzzy ideal [1] of a K-algebra $\mathcal{K}$ is a mapping $\mu : G \to [0, 1]$ such that

(i) $(\forall x \in G)\ (\mu(e) \geq \mu(x))$,

(ii) $(\forall x, y \in G)\ (\mu(x) \geq \min\{\mu(x \odot y), \mu(y \odot (y \odot x))\})$.

Definition 1 A mapping $A = (\mu_A, \lambda_A) : G \to [0, 1] \times [0, 1]$ is called *bifuzzy set* in G if $\mu_A(x) + \lambda_A(x) \leq 1$, for all $x \in G$, where the mappings $\mu_A : G \to [0, 1]$ and $\lambda_A : G \to [0, 1]$ denote the *degree of membership* (namely $\mu_A(x)$) and the *degree of non-membership* (namely $\lambda_A(x)$) of each element $x \in G$ to A respectively.
In particular, we use $0_\sim$ and $1_\sim$ to denote the *bifuzzy empty set* and the *bifuzzy whole set* in a set G such that $0_\sim(x) = (0, 1)$ and $1_\sim(x) = (1, 0)$ for each $x \in G$, respectively.

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Definition 2 A bifuzzy set $A = (\mu_A, \lambda_A) : G \to [0, 1] \times [0, 1]$ is called a *bifuzzy ideal* of K-algebra $\mathcal{K}$ if the following conditions hold:

(i) $(\forall x \in G)\ (\mu_A(e) \geq \mu_A(x), \lambda_A(e) \leq \lambda_A(x))$,

(ii) $(\forall x, y \in G)\ (\mu_A(x) \geq \min\{\mu_A(x \odot y), \mu_A(y \odot (y \odot x))\})$,

(iii) $(\forall x, y \in G)\ (\lambda_A(x) \leq \max\{\lambda_A(x \odot y), \lambda_A(y \odot (y \odot x))\})$.

Example 3 Consider the K-algebra $\mathcal{K} = (G, \cdot, \odot, e)$ on the cyclic group $G = \{e, a, b, c, d, f\}$, where $a = a$, $b = a^2$, $c = a^3$, $d = a^4$, $f = a^5$ and $\odot$ is given by the following Cayley's table:

⊙	e	a	b	c	d	f
e	e	f	d	c	b	a
a	a	e	f	d	c	b
b	b	a	e	f	d	c
c	c	b	a	e	f	d
d	d	c	b	a	e	f
f	f	d	c	b	a	e

We define a bifuzzy set $A = (\mu_A, \lambda_A) : G \to [0, 1] \times [0, 1]$ by $\mu_A(e) = 0.56$, $\mu_A(x) = 0.03$ and $\lambda_A(e) = 0.06$, $\lambda_A(x) = 0.63$ for all $x \neq e$ in G, By routine computations, we can easily verify that bifuzzy set A is a bifuzzy ideal of K-algebra $\mathcal{K}$.

The proofs of the following propositions are obvious and hence omitted.

Proposition 4 *If $A = (\mu_A, \lambda_A)$ is a bifuzzy ideal of a K-algebra $\mathcal{K}$, then the level subsets $U(\mu_A, \alpha) = \{x \in G \,|\, \mu_A(x) \geq \alpha\}$ and $L(\lambda_A, \alpha) = \{x \in G \,|\, \lambda_A(x) \leq \alpha\}$ are ideals of $\mathcal{K}$ for every $\alpha \in \mathrm{Im}(\mu_A) \cap \mathrm{Im}(\lambda_A) \subseteq [0, 1]$, where $\mathrm{Im}(\mu_A)$ and $\mathrm{Im}(\lambda_A)$ are sets of values of μ_A and λ_A, respectively.*

Proposition 5 Let $A = (\mu_A, \lambda_A)$ be a bifuzzy ideal of a K-algebra $\mathcal{K}$ and let $x \in G$. Then $\mu_A(x) = t$, $\lambda_A(x) = s$ if and only if $x \in U(\mu_A, t)$, $x \notin U(\mu_A, s)$ and $x \in L(\lambda_A, s)$, $x \notin L(\lambda_A, t)$, for all $s > t$.

Definition 6 Let $A = (\mu_A, \lambda_A)$ be a bifuzzy set on G and let $(\alpha, \beta) \in [0, 1] \times [0, 1]$ with $\alpha + \beta \leq 1$. Then
(i) the set $G_A^{(\alpha,\beta)} := \{x \in G \mid \alpha \leq \mu_A(x),\ \lambda_A(x) \leq \beta\}$ is called an (α, β)-*level subset* of A. The set of all $(\alpha, \beta) \in \mathrm{Im}(\mu_A) \times \mathrm{Im}(\lambda_A)$ such that $\alpha + \beta \leq 1$ is called the *image of* $A = (\mu_A, \lambda_A)$.
(ii) the set $G_A^{(\alpha,\beta)} := \{x \in G \mid \alpha < \mu_A(x),\ \lambda_A(x) < \beta\}$ is called a strong (α, β)-*level subset* of A.

Note that

$$\begin{aligned} G_A^{(\alpha,\beta)} &= \{x \in G \mid \mu_A(x) \geq \alpha,\ \lambda_A(x) \leq \beta\} \\ &= \{x \in G \mid \mu_A(x) \geq \alpha\} \cap \{x \in G \mid \lambda_A(x) \leq \beta\} \\ &= U(\mu_A, \alpha) \cap L(\lambda_A, \beta). \end{aligned}$$

Theorem 7 A bifuzzy set $A = (\mu_A, \lambda_A)$ of $\mathcal{K}$ is a bifuzzy ideal of $\mathcal{K}$ if and only if $G_A^{(\alpha,\beta)}$ is an ideal of $\mathcal{K}$ for every $(\alpha, \beta) \in \mathrm{Im}(\mu_A) \times \mathrm{Im}(\lambda_A)$ with $\alpha + \beta \leq 1$.

Proof: If $A = (\mu_A, \lambda_A)$ is a bifuzzy ideal of $\mathcal{K}$, then according to Proposition 4, all nonempty level subsets $U(\mu_A, \alpha)$ and $L(\lambda_A, \beta)$ are ideals of $\mathcal{K}$. So, $G_A^{(\alpha,\beta)} = U(\mu_A, \alpha) \cap L(\lambda_A, \beta)$ is an ideal of $\mathcal{K}$.
Conversely, let $G_A^{(\alpha,\beta)}$ be an ideal of $\mathcal{K}$ and let $A = (\mu_A, \lambda_A)$ be a bifuzzy set on $\mathcal{K}$. Condition (i) of Definition 2 is obvious. Consider $x, y \in G$ such that $A(x \odot y) = (\alpha_1, \beta_1)$ and $A(y \odot (y \odot x)) = (\alpha_2, \beta_2)$, that is, $\mu_A(x \odot y) = \alpha_1$, $\lambda_A(x \odot y) = \beta_1$, $\mu_A(y \odot (y \odot x)) = \alpha_2$ and $\lambda_A(y \odot (y \odot x)) = \beta_2$. Without loss of generality we can assume that $(\alpha_1, \beta_1) \leq (\alpha_2, \beta_2)$, i.e., $\alpha_1 \leq \alpha_2$ and $\beta_2 \leq \beta_1$. Then $G_A^{(\alpha_2,\beta_2)} \subseteq G_A^{(\alpha_1,\beta_1)}$, i.e., $x, y \in G_A^{(\alpha_1,\beta_1)}$, which implies $x \in G_A^{(\alpha_1,\beta_1)}$ because $G_A^{(\alpha_1,\beta_1)}$ is an ideal of $\mathcal{K}$. Thus

$$\mu_A(x) \geq \alpha_1 = \min\{\mu_A(x \odot y), \mu_A(y \odot (y \odot x))\},$$

$$\lambda_A(x) \leq \beta_1 = \max\{\lambda_A(x \odot y), \lambda_A(y \odot (y \odot x))\}.$$

Hence $A = (\mu_A, \lambda_A)$ is a bifuzzy ideal of $\mathcal{K}$. □

Theorem 8 *Let $A = (\mu_A, \lambda_A)$ be a bifuzzy ideal of $\mathcal{K}$ and (α_1, β_1), $(\alpha_2, \beta_2) \in \mathrm{Im}(\mu_A) \times \mathrm{Im}(\lambda_A)$ with $\alpha_i + \beta_i \leq 1$ for $i = 1, 2$. Then $G_A^{(\alpha_1,\beta_1)} = G_A^{(\alpha_2,\beta_2)}$ if and only if $(\alpha_1, \beta_1) = (\alpha_2, \beta_2)$.*

Proof: If $(\alpha_1, \beta_1) = (\alpha_2, \beta_2)$, then clearly $G_A^{(\alpha_1,\beta_1)} = G_A^{(\alpha_2,\beta_2)}$. Assume that $G_A^{(\alpha_1,\beta_1)} = G_A^{(\alpha_2,\beta_2)}$. Since $(\alpha_1, \beta_1) \in \mathrm{Im}(\mu_A) \times \mathrm{Im}(\lambda_A)$, there exists $x \in G$ such that $\mu_A(x) = \alpha_1$ and $\lambda_A(x) = \beta_1$. It follows that $x \in G_A^{(\alpha_1,\beta_1)} = G_A^{(\alpha_2,\beta_2)}$ so that $\alpha_1 = \mu_A(x) \geq \alpha_2$ and $\beta_1 = \lambda_A(x) \leq \beta_2$. Similarly, we have $\alpha_1 \leq \alpha_2$ and $\beta_1 \geq \beta_2$. Hence $(\alpha_1, \beta_1) = (\alpha_2, \beta_2)$. □

Theorem 9 *Let $G_0 \subset G_1 \subset G_2 \subset \ldots G_n = G$ be a chain of ideals of a K- algebra $\mathcal{K}$. Then there exists a bifuzzy ideal $A = (\mu_A, \lambda_A)$ of $\mathcal{K}$ for which level subsets $U(\mu_A, \alpha)$ and $L(\lambda_A, \beta)$ coincide with this chain.*

Proof: Let $\{\alpha_k \,|\, k = 0, 1, \ldots, n\}$ and $\{\beta_k \,|\, k = 0, 1, \ldots, n\}$ be finite decreasing and increasing sequences in $[0, 1]$ such that $\alpha_i + \beta_i \leq 1$, for $i = 0, 1, \ldots, n$. Let $A = (\mu_A, \lambda_A)$ be a bifuzzy set in $\mathcal{K}$ defined by $\mu_A(G_0) = \alpha_0$, $\lambda_A(G_0) = \beta_0$, $\mu_A(G_k \setminus G_{k-1}) = \alpha_k$ and $\lambda_A(G_k \setminus G_{k-1}) = \beta_k$ for $0 < k \leq n$. Let $x, y \in G$. If $x, y \in G_k \setminus G_{k-1}$, then $x \in G_k$ and

$$\mu_A(x) \geq \alpha_k = \min\{\mu_A(x \odot y), \mu_A(y \odot (y \odot x))\},$$

$$\lambda_A(x) \leq \beta_k = \max\{\lambda_A(x \odot y), \lambda_A(y \odot (y \odot x))\}.$$

For $i > j$, if $x \in G_i \setminus G_{i-1}$ and $y \in G_j \setminus G_{j-1}$, then $\mu_A(x \odot y) = \alpha_i = \mu_A(y \odot (y \odot x))$, $\lambda_A(x \odot y) = \beta_j = \lambda_A(y \odot (y \odot x))$ and $x \in G_i$. Thus

$$\mu_A(x) \geq \alpha_i = \min\{\mu_A(x \odot y), \mu_A(y \odot (y \odot x))\},$$

$$\lambda_A(x) \leq \beta_j = \max\{\lambda_A(x \odot y), \lambda_A(y \odot (y \odot x))\}.$$

So, $A = (\mu_A, \lambda_A)$ is a bifuzzy ideal of a K-algebra $\mathcal{K}$ and all its nonempty level subsets are ideals. Since $\mathrm{Im}(\mu_A) = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$, $\mathrm{Im}(\lambda_A) = \{\beta_0, \beta_1, \ldots, \beta_n\}$, level subsets of A form chains:

$$U(\mu_A, \alpha_0) \subset U(\mu_A, \alpha_1) \subset \ldots \subset U(\mu_A, \alpha_n) = G$$

and

$$L(\lambda_A, \beta_0) \subset L(\lambda_A, \beta_1) \subset \ldots \subset L(\lambda_A, \beta_n) = G,$$

respectively. Indeed,

$$U(\mu_A, \alpha_0) = \{x \in \mathcal{K} \,|\, \mu_A(x) \geq \alpha_0\} = G_0,$$

$$L(\lambda_A, \beta_0) = \{x \in \mathcal{K} \,|\, \lambda_A(x) \leq \beta_0\} = G_0.$$

We now prove that

$$U(\mu_A, \alpha_k) = G_k = L(\lambda_A, \beta_k) \text{ for } 0 < k \leq n.$$

Clearly, $G_k \subseteq U(\mu_k, \alpha_k)$ and $G_k \subseteq L(\lambda_A, \beta_k)$. If $x \in U(\mu_A, \alpha_k)$, then $\mu_A(x) \geq \alpha_k$ and so $x \notin G_i$ for $i > k$. Hence

$$\mu_A(x) \in \{\alpha_0, \alpha_1, \ldots, \alpha_k\},$$

which implies $x \in G_i$ for some $i \leq k$. Since $G_i \subseteq G_k$, it follows that $x \in G_k$. Consequently, $U(\mu_A, \alpha_k) = G_k$ for some $0 < k \leq n$. Now if $y \in L(\lambda_A, \beta_k)$, then $\lambda_A(x) \leq \beta_k$ and so $y \notin G_i$ for $j \leq k$. Thus

$$\lambda_A(x) \in \{\beta_0, \beta_1, \ldots, \beta_k\},$$

which implies $x \in G_j$ for some $j \leq k$. Since $G_j \subseteq G_k$, it follows that $y \in G_k$. Consequently, $L(\lambda_A, \beta_k) = G_k$ for some $0 < k \leq n$. This completes the proof. □

Definition 10 Let $I(\mathcal{K})$ denote the family of all ideals of $\mathcal{K}$ and let $BF(\mathcal{K})$ denote the family of all bifuzzy ideals of $\mathcal{K}$. For any $t \in [0,1]$, we define relation $\mathcal{R}^t$ on $BF(\mathcal{K})$ as follows:

$$(A, B) \in \mathcal{R}^t \;\longleftrightarrow\; G_A^{(t,t)} = G_B^{(t,t)}$$

for any $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$. Then the relation $\mathcal{R}^t$ is an equivalence relation on $BF(\mathcal{K})$.

Theorem 11 For any $t \in (0, 0.5]$ the map $\varphi_t : BF(\mathcal{K}) \to BF(\mathcal{K}) \cup \{\emptyset\}$ defined by $\varphi_t(A) = G_A^{(t,t)}$ is surjective.

Proof: Let $t \in (0, 0.5]$. Then $\varphi_t(0_\sim) = G_A^{(t,t)} = U(0;t) \cap L(1;t) = \emptyset$. For any $H \in BF(\mathcal{K})$, there exists $H_\sim = (\chi_H, \overline{\chi}_H) \in BF(\mathcal{K})$ such that $\varphi_t(H_\sim) = G_A^{(t,t)} = U(\chi_H; t) \cap L(\overline{\chi}_H; t) = H$. So, φ_t is surjective. □

Theorem 12 *For any $t \in (0, 0.5]$, the quotient set $BF(\mathcal{K})/\mathcal{R}^t$ is equipotent to $I(\mathcal{K}) \cup \{\emptyset\}$.*

Proof: Let $t \in (0, 0.5]$ and let $\varphi_t^* : BF(\mathcal{K})/\mathcal{R}^t \to I(\mathcal{K}) \cup \{\emptyset\}$ be a map defined by $\varphi_t^*([A]_{\mathcal{R}^t}) = \varphi_t(A)$ for all $[A]_{\mathcal{R}^t} \in BF(\mathcal{K})/\mathcal{R}^t$. If $\varphi_t^*([A]_{\mathcal{R}^t}) = \varphi_t^*([B]_{\mathcal{R}^t})$ for any $[A]_{\mathcal{R}^t}$, $[B]_{R^t} \in IF(\mathcal{K})/\mathcal{R}^t$, then $G_A^{(t,t)} = G_B^{(t,t)}$, i.e., $(A, B) \in \mathcal{R}^t$. It follows that $[A]_{\mathcal{R}^t} = [B]_{\mathcal{R}^t}$ so that φ_t^* is injective. Moreover $\varphi_t^*([0_\sim]_{\mathcal{R}^t}) = \varphi_t(0_\sim) = \mathcal{K}_{0_\sim}^{(t,t)} = \emptyset$. For any $H \in I(\mathcal{K})$ we have $H_\sim = (\chi_H, \overline{\chi}_H) \in IF(\mathcal{K})$ and

$$\varphi_t^*([H_\sim]_{\mathcal{R}^t}) = \varphi_t(H_\sim) = G_{H_\sim}^{(t,t)} = U(\chi_H; t) \cap L(\overline{\chi}_H; t) = H.$$

This proves that φ_t^* is surjective. □

Using the same method as in the proofs of Theorems 4.6 and 4.7 in [18] we can prove the following two theorems.

Theorem 13 Let $\{C_\alpha \mid \alpha \in \Lambda \subseteq [0, \frac{1}{2}]\}$ be a collection of ideals of a K-algebra $\mathcal{K}$ such that $G = \bigcup_{\alpha \in \Lambda} C_\alpha$, and for every $\alpha, \beta \in \Lambda$, $\alpha < \beta$ if and only if $C_\beta \subset C_\alpha$. Then a bifuzzy set $A = (\mu_A, \lambda_A)$ defined by

$$\mu_A(x) = \sup\{\alpha \in \Lambda \mid x \in C_\alpha\}$$

and

$$\lambda_A(x) = \inf\{\alpha \in \Lambda \mid x \in C_\alpha\}$$

is a bifuzzy ideal of $\mathcal{K}$.

Theorem 14 If $A = (\mu_A, \lambda_A)$ is a bifuzzy ideal of a K- algebra $\mathcal{K}$, then

$$\mu_A(x) = \sup\{\alpha \in [0,1] \mid x \in U(\mu_A, \alpha)\},$$

$$\lambda_A(x) = \inf\{\beta \in [0,1] \mid x \in L(\lambda_A, \beta)\}$$

for every $x \in G$.

Theorem 15 *The family of bifuzzy ideals of $\mathcal{K}$ forms a complete distributive lattice under the ordering of bifuzzy set inclusion $\subset$.*

Proof: $\{A_i \,|\, i \in I\}$ is a family of bifuzzy ideals of $\mathcal{K}$. Since $[0,1]$ is a completely distributive lattice with respect to the usual ordering in $[0,1]$, it is sufficient to show that $\bigcap A_i = (\bigwedge \mu_{A_i}, \bigvee \lambda_{A_i})$ is a bifuzzy ideal of $\mathcal{K}$. For any $x \in G$,

$$(\bigvee_{i \in I} \mu_{A_i})(e) = \sup_{i \in I} \mu_{A_i}(e) \geq \sup_{i \in I} \mu_{A_i}(x) = (\bigvee_{i \in I} \mu_{A_i})(x),$$

and

$$(\bigwedge_{i \in I} \lambda_{A_i})(e) = \inf_{i \in I} \lambda_{A_i}(e) \leq \inf_{i \in I} \lambda_{A_i}(x) = (\bigwedge_{i \in I} \lambda_{A_i})(x).$$

Let $x, y \in G$. Then

$$\begin{aligned}
&(\bigvee \mu_{A_i})(x) = \sup\{\mu_{A_i}(x) | i \in I\} \\
&\geq \sup\{\max(\mu_{A_i}(x \odot y), \\
&\mu_{A_i}(y \odot (y \odot x))) | i \in I\} \\
&= \max(\sup\{\mu_{A_i}(x \odot y) | i \in I\}, \\
&\sup\{\mu_{A_i}(y \odot (y \odot x)) | i \in I\}) \\
&= \max((\bigvee \mu_{A_i})(x \odot y), (\bigvee \mu_{A_i})(y \odot (y \odot x))),
\end{aligned}$$

$$\begin{aligned}
&(\bigwedge \lambda_{A_i})(x) = \inf\{\lambda_{A_i}(x) | i \in I\} \\
&\leq \inf\{\min(\lambda_{A_i}(x \odot y), \\
&\lambda_{A_i}(y \odot (y \odot x))) | i \in I\} \\
&= \min(\inf\{\lambda_{A_i}(x \odot y) | i \in I\}, \\
&\inf\{\lambda_{A_i}(y \odot (y \odot x)) | i \in I\}) \\
&= \min((\bigwedge \lambda_{A_i})(x \odot y), \\
&(\bigwedge \lambda_{A_i})(y \odot (y \odot x))).
\end{aligned}$$

Hence $\bigcap A_i = (\bigwedge \mu_{A_i}, \bigvee \lambda_{A_i})$ is a bifuzzy ideal of $\mathcal{K}$ □

Theorem 16 The family of all bifuzzy ideals of $\mathcal{K}$ is bounded.

Proof: Obviously, $0_\sim$ and $1_\sim$ are bifuzzy ideals of $\mathcal{K}$. Moreover, $0_\sim \subset A \subset 1_\sim$ for every bifuzzy ideal of $\mathcal{K}$. Hence the family of all bifuzzy ideals of $\mathcal{K}$ is bounded. □

Theorem 17 (a) *The family of all bifuzzy ideals of $\mathcal{K}$ is not complementary.*

(b) *The family of all bifuzzy ideals of $\mathcal{K}$ has no atoms.*

(c) *The family of all bifuzzy ideals of $\mathcal{K}$ has no dual atoms.*

Proof:

(a) We define a mapping $A = (\mu_A, \lambda_A) : G \rightarrow [0,1] \times [0,1]$ as follows $A(x) = (\frac{1}{2}, \frac{1}{2})$ for all $x \in G$. Then clearly A, A^c are bifuzzy ideals of $\mathcal{K}$. But $A \cup A^c \neq 1_\sim$ and $A \cap A^c \neq 0_\sim$. Thus A has no complement in bifuzzy ideals of $\mathcal{K}$. Hence bifuzzy ideals of $\mathcal{K}$ is not completed.

(b) Suppose that A is a bifuzzy ideal of $\mathcal{K}$ with $A \neq 0_\sim$. We define a mapping $B = (\mu_B, \lambda_B) : G \rightarrow [0,1] \times [0,1]$ as follows $\mu_B(x) = \frac{1}{2}\mu_A(x)$ and $\lambda_B(x) = 1 - \frac{1}{2}\lambda_A(x)$ for all $x \in G$. Then clearly B is a bifuzzy ideal of $\mathcal{K}$. Moreover $0_\sim \subset B \subset A$. Hence the family of all bifuzzy ideals of $\mathcal{K}$ has no atoms.

(c) Suppose that A is a bifuzzy ideal of $\mathcal{K}$ with $A \neq 1_\sim$. We define a mapping $B = (\mu_B, \lambda_B) : G \rightarrow [0,1] \times [0,1]$ as follows $\mu_B(x) = \frac{1}{2} + \frac{1}{2}\mu_A(x)$ and $\lambda_B(x) = \frac{1}{2} - \frac{1}{2}\lambda_A(x)$ for all $x \in G$. Then clearly $A \subset B \subset 1_\sim$. Condition (i) of Definition 2 is obvious. we now prove condition (ii) of Definition 2. Let $x, y \in G$. Then

$$\begin{aligned} \mu_B(x) &= \frac{1}{2} + \frac{1}{2}\mu_A(x) \\ &\geq \frac{1}{2} + \frac{1}{2}\min(\mu_A(x \odot y) \\ &, \mu_A(y \odot (y \odot x))) \\ &= \min(\frac{1}{2} + \frac{1}{2}\mu_A(x \odot y) \\ &, \frac{1}{2} + \frac{1}{2}\mu_A(y \odot (y \odot x))) \\ &= \min(\mu_B(x \odot y), \mu_B(y \odot (y \odot x))) \end{aligned}$$

$$\begin{aligned} \lambda_B(x) &= \frac{1}{2} - \frac{1}{2}\lambda_A(x) \\ &\leq \frac{1}{2} - \frac{1}{2}\max(\lambda_A(x \odot y) \\ &, \lambda_A(y \odot (y \odot x))) \\ &= \max(\frac{1}{2} - \frac{1}{2}\lambda_A(x \odot y) \\ &, \frac{1}{2} - \frac{1}{2}\lambda_A(y \odot (y \odot x))) \\ &= \max(\lambda_B(x \odot y), \lambda_B(y \odot (y \odot x))). \end{aligned}$$

Thus B is a bifuzzy ideal $\mathcal{K}$. The family of all bifuzzy ideals of $\mathcal{K}$ has no dual atoms. □

4 Homomorphisms and bifuzzy ideals

Definition 18 Let $\mathcal{K}_1 = (G_1, \cdot, \odot, e_1)$ and $\mathcal{K}_2 = (G_2, \cdot, \odot, e_2)$ be two K-algebras and let f be a function from $\mathcal{K}_1$ into $\mathcal{K}_2$. If $B = (\mu_B, \lambda_B)$ is a bifuzzy set in $\mathcal{K}_2$, then the *preimage* of $B = (\mu_B, \lambda_B)$ under f is the bifuzzy set in $\mathcal{K}_1$ defined by $f^{-1}(\mu_B)(x) = \mu_B(f(x))$ and $f^{-1}(\lambda_B)(x) = \lambda_B(f(x))$ for all $x \in G_1$.

Theorem 19 Let $f : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ be an epimorphism of K-algebras. If $B = (\mu_B, \lambda_B)$ is a bifuzzy ideal in $\mathcal{K}_2$, then $f^{-1}(B)$ is a bifuzzy ideal in $\mathcal{K}_1$.

Proof: It is easy to see that $f^{-1}(\mu_B)(e) \geq f^{-1}(\mu_B)(x)$, $f^{-1}(\lambda_B)(e) \leq f^{-1}(\lambda_B)(x)$ for all $x \in G_1$. Let $x, y \in G_1$, then

$$\begin{aligned} f^{-1}(\mu_B)(x) &= \mu_B(f(x)) \\ &\geq \min(\mu_B(f(x \odot y), f(y \odot (y \odot x))) \\ &= \min(\mu_B(f(x \odot y)), \mu_B(f(y \odot (y \odot x)))) \\ &= \min(f^{-1}(\mu_B)(x \odot y) \\ &, f^{-1}(\mu_B)(y \odot (y \odot x))), \end{aligned}$$

$$\begin{aligned} f^{-1}(\lambda_B)(x) &= \lambda_B(f(x)) \\ &\leq \max(\lambda_B(f(x \odot y), f(y \odot (y \odot x))) \\ &= \max(\lambda_B(f(x \odot y)), \lambda_B(f(y \odot (y \odot x)))) \\ &= \max(f^{-1}(\lambda_B)(x \odot y) \\ &, f^{-1}(\lambda_B)(y \odot (y \odot x))). \end{aligned}$$

Hence $f^{-1}(B)$ is a bifuzzy ideal in $\mathcal{K}_1$. □

Theorem 20 Let $f : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ be an epimorphism of K-algebras. If $B = (\mu_B, \lambda_B)$ is a bifuzzy ideal of $\mathcal{K}_2$ and $A = (\mu_A, \lambda_A)$ is the pre-image of B under f. Then A is a bifuzzy ideal of $\mathcal{K}_1$.

Proof: It is easy to see that $\mu_A(e) \geq \mu_A(x), \lambda_A(e) \leq \lambda_A(x)$ for all $x \in G_1$. For any $x, y \in G_1$,

$$\begin{aligned}\mu_A(x) &= \mu_B(f(x)) \\ &\geq \min(\mu_B(f(x \odot y)), \mu_B(f(y \odot (y \odot x)))) \\ &= \min(\mu_A(x \odot y), \mu_A(y \odot (y \odot x))),\end{aligned}$$

$$\begin{aligned}\lambda_A(x) &= \lambda_B(f(x)) \\ &\leq \max(\lambda_B(f(x \odot y)), \lambda_B(f(y \odot (y \odot x)))) \\ &= \max(\lambda_A(x \odot y), \lambda_A(y \odot (y \odot x))).\end{aligned}$$

Hence A is a bifuzzy ideal of $\mathcal{K}_1$. □

Definition 21 *Let a mapping $f : \mathcal{K}_1 \to \mathcal{K}_2$ from $\mathcal{K}_1$ into $\mathcal{K}_2$ of K-algebras and let $A = (\mu_A, \lambda_A)$ be a bifuzzy set of $\mathcal{K}_2$. The map $A = (\mu_A, \lambda_A)$ is called the pre-image of A under f, if $\mu_A^f(x) = \mu_A(f(x))$ and $\lambda_A^f(x) = \lambda_A(f(x))$ for all $x \in G_1$.*

Proposition 22 Let $f : \mathcal{K}_1 \to \mathcal{K}_2$ be an epimorphism of K-algebras. If $A = (\mu_A, \lambda_A)$ is a bifuzzy ideal of $\mathcal{K}_2$, then $A^f = (\mu_A^f, \lambda_A^f)$is a bifuzzy ideal of $\mathcal{K}_1$.

Proof: For any $x \in G_1$, we have $\mu_A^f(e_1) = \mu_A(f(e_1)) = \mu_A(e_2) \geq \mu_A(f(x)) = \mu_A^f(x)$, $\lambda_A^f(e_1) = \lambda_A(f(e_1)) = \lambda_A(e_2) \leq \lambda_A(f(x)) = \lambda_A^f(x)$. For any $x, y \in G_1$, since μ is a bifuzzy ideal of $\mathcal{K}_1$,

$$\begin{aligned}\mu_A^f(x) &= \mu_A(f(x)) \\ &\geq \min\{\mu_A(f(x \odot y)) \\ &, \mu(f(y) \odot f(y \odot x))\} \\ &= \min\{\mu_A(f(x \odot y)) \\ &, \mu_A(f(y \odot (y \odot x)))\} \\ &= \min\{\mu_A^f(x \odot y), \mu_A^f(y \odot (y \odot x))\},\end{aligned}$$

$$\begin{aligned}\lambda^f(x) &= \lambda(f(x)) \\ &\leq \max\{\lambda_A(f(x \odot y)), \lambda_A(f(y) \odot f(y \odot x))\} \\ &= \max\{\lambda_A(f((x \odot y)), \lambda_A(f(y \odot (y \odot x)))\} \\ &= \max\{\lambda_A^f(x \odot y), \lambda_A^f(y \odot (y \odot x))\}\end{aligned}$$

proving that $A^f = (\mu_A^f, \lambda_A^f)$ is a bifuzzy ideal of $\mathcal{K}_1$. □

Proposition 23 *Let $f : \mathcal{K}_1 \to \mathcal{K}_2$ be an epimorphism of K-algebras. If $A^f = (\mu_A^f, \lambda_A^f)$ is a bifuzzy ideal of $\mathcal{K}_2$, then $A = (\mu_A, \lambda_A)$ is a bifuzzy ideal of $\mathcal{K}_1$.*

Proof: Since there exists $x \in G_1$ such that $y = f(x)$ for any $y \in G_2$, $\mu_A(y) = \mu_A(f(x)) = \mu_A^f(x) \leq \mu_A^f(e_1) = \mu_A(f(e_1)) = \mu_A(e_2)$ and $\lambda_A(y) = \lambda_A(f(x)) = \lambda_A^f(x) \geq \lambda_A^f(e_1) = \lambda_A(f(e_1)) = \lambda_A(e_2)$.
For any $x, y \in G_2$, there exist $a, b, c \in G_1$ such that $x = f(a)$ and $y = f(b)$. It follows that

$$\begin{aligned}\mu_A(x) &= \mu_A(f(a)) \\ &= \mu_A^f(a) \\ &\geq \min\{\mu_A^f((a \odot b)), \mu^f(b \odot (b \odot a))\} \\ &= \min\{\mu_A(f(a \odot b)), \mu_A(f(b \odot (b \odot a)))\} \\ &= \min\{\mu_A((f(a) \odot f(b))) \\ &, \mu_A(f(b) \odot (f(b) \odot f(a)))\} \\ &= \min\{\mu_A(x \odot y), \mu_A(y \odot (y \odot x))\},\end{aligned}$$

$$\begin{aligned}\lambda_A(x) &= \lambda_A(f(a)) \\ &= \lambda_A^f(a) \\ &\leq \max\{\lambda_A^f((a \odot b)), \lambda^f(b \odot (b \odot a))\} \\ &= \max\{\lambda_A(f(a \odot b)), \lambda_A(f(b \odot (b \odot a)))\} \\ &= \max\{\lambda_A((f(a) \odot f(b))) \\ &, \lambda_A(f(b) \odot (f(b) \odot f(a)))\} \\ &= \max\{\lambda_A(x \odot y), \lambda_A(y \odot (y \odot x))\},\end{aligned}$$

proving that $A = (\mu_A, \lambda_A)$ is a bifuzzy ideal of $\mathcal{K}_1$. □
As a consequence of the above two propositions we obtain the following theorem.

Theorem 24 *Let $f : \mathcal{K}_1 \to \mathcal{K}_2$ be an epimorphism of K-algebras. Then $A^f = (\mu_A^f, \lambda_A^f)$ is a bifuzzy ideal of $\mathcal{K}_1$ if and only if $A = (\mu_A, \lambda_A)$ is a bifuzzy ideal of $\mathcal{K}_2$.*

Definition 25 *An ideal C of K-algebra $\mathcal{K}$ is said to be characteristic if $f(C) = C$ for all $f \in Aut(\mathcal{K})$, where Aut($\mathcal{K}$) is the set of all automorphisms of a K-algebra $\mathcal{K}$. A bifuzzy ideal $A = (\mu_A, \lambda_A)$ of a K-algebra $\mathcal{K}$ is called characteristic if $\mu_A(f(x)) = \mu_A(x)$ and $\lambda_A(f(x)) = \lambda_A(x)$ for all $x \in G$ and $f \in Aut(\mathcal{K})$.*

Definition 26 *An ideal H of K-algebra $\mathcal{K}$ is said to be fully invariant if $f(H) \subseteq H$ for all $f \in End(\mathcal{K})$, where End($\mathcal{K}$) is the set of all endomorphisms of a K-algebra $\mathcal{K}$. A bifuzzy ideal $A = (\mu_A, \lambda_A)$ of a K-algebra $\mathcal{K}$ is called fully invariant if $\mu_A(f(x)) \leq \mu_A(x)$ and $\lambda_A(f(x)) \leq \lambda_A(x)$ for all $x \in G$ and $f \in End(\mathcal{K})$.*

Theorem 27 *A bifuzzy ideal is characteristic if and only if each its level set is a characteristic ideal.*

Proof: Let a bifuzzy ideal $A = (\mu_A, \lambda_A)$ be characteristic, $t \in Im(\mu_A)$, $f \in Aut(\mathcal{K})$, $x \in U(\mu_A; t)$. Then $\mu_A(f(x)) = \mu_A(x) \geq t$, which means that $f(x) \in U(\mu_A; t)$. Thus $f(U(\mu_A; t)) \subseteq U(\mu_A; t)$. Since for each $x \in U(\mu_A; t)$ there exists $y \in G$ such that $f(y) = x$ we have $\mu_A(y) = \mu_A(f(y)) = \mu_A(x) \geq t$, whence we conclude $y \in U(\mu_A; t)$. Consequently $x = f(y) \in f(U(\mu_A; t))$. Hence $f(U(\mu_A; t)) = U(\mu_A; t)$. Similarly, $f(L(\lambda_A; s)) = L(\lambda_A; s)$. This proves that $U(\mu_A; t)$ and $L(\lambda_A; s)$ are characteristic.
Conversely, if all levels of $A = (\mu_A, \lambda_A)$ are characteristic ideals of $\mathcal{K}$, then for $x \in G$, $f \in Aut(\mathcal{K})$ and $\mu_A(x) = t < s = \lambda_A(x)$, by Proposition 5 , we have $x \in U(\mu_A; t)$, $x \notin U(\mu_A; s)$ and $x \in L(\lambda_A; s)$, $x \notin L(\lambda_A; t)$. Thus $f(x) \in f(U(\mu_A; t)) = U(\mu_A; t)$ and $f(x) \in f(L(\lambda_A; s)) = L(\lambda_A; s)$, i.e., $\mu_A(f(x)) \geq t$ and $\lambda_A(f(x)) \leq s$. For $\mu_A(f(x)) = t_1 > t$, $\lambda_A(f(x)) = s_1 < s$ we have $f(x) \in U(\mu_A; t_1) = f(U(\mu_A; t_1))$, $f(x) \in L(\lambda_A, s_1) = f(L(\lambda_A; s_1))$, whence $x \in U(\mu_A; t_1)$, $x \in L(\mu_A; s_1)$. This is a contradiction. Thus $\mu_A(f(x)) = \mu_A(x)$ and $\lambda_A(f(x)) = \lambda_A(x)$. Hence $A = (\mu_A, \lambda_A)$ is characteristic. □

As a consequence of the above Theorem we obtain the following theorem.

Theorem 28 *If A is a fully invariant bifuzzy ideal of $\mathcal{K}$, then it is characteristic.*

Definition 29 *Let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be bifuzzy ideals of $\mathcal{K}$. Then A is said to be of the same type of B if there exists $f \in Aut(\mathcal{K})$ such that $A = B \circ f$, i.e., $\mu_A(x) = \mu_B(f(x))$, $\lambda_A(x) = \lambda_B(f(x))$ for all $x \in G$.*

Theorem 30 *Let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be two bifuzzy ideals of $\mathcal{K}$. Then A is a bifuzzy ideal having the same type of B if and only if A is isomorphic to B.*

Proof: We only need to prove the necessity part because the sufficiency part is trivial. Let $A = (\mu_A, \lambda_A)$ be a bifuzzy ideal having the same type of $B = (\mu_B, \lambda_B)$. Then there exists $\phi \in Aut(\mathcal{K})$ such that

$$\mu_A(x) = \mu_B(\phi(x)),\ \lambda_A(x) = \lambda_B(\phi(x))\ \ \forall x \in G.$$

Let $f : A(\mathcal{K}) \to B(\mathcal{K})$ be a mapping defined by $f(A(x)) = B(\phi(x))$ for all $x \in G$, that is, for all $x \in G$

$$f(\mu_A(x)) = \mu_B(\phi(x)),\ f(\lambda_A(x)) = \lambda_B(\phi(x)).$$

Then, it is clear that f is surjective. Also, f is injective because if $f(\mu_A(x)) = f(\mu_A(y))$ for all x, $y \in G$, then $\mu_B(\phi(x)) = \mu_B(\phi(y))$ and hence $\mu_A(x) = \mu_B(y)$. Likewisely, we have $f(\lambda_A(x)) = f(\lambda_A(y)) \Longrightarrow \lambda_A(x) = \lambda_B(y)$ for all $x \in L$. Finally, f is a homomorphism because for $x, y \in G$,

$$f(\mu_A(x \odot y)) = \mu_B(\phi(x \odot y)) = \mu_B(\phi(x) \odot \phi(y)),$$

$$f(\lambda_A(x \odot y)) = \lambda_B(\phi(x \odot y)) = \lambda_B(\phi(x) \odot \phi(y)).$$

Hence $A = (\mu_A, \lambda_A)$ is isomorphic to $B = (\mu_B, \lambda_B)$. This completes the proof. □

5 Cartesian product of bifuzzy ideals

Definition 31 Let G be a nonempty set. Then we call a mapping $A = (\mu_A, \lambda_A) : G \times G \to [0, 1] \times [0, 1]$ a *bifuzzy relation* on G if $\mu_A(x, y) + \lambda_A(x, y) \leq 1$ for all $(x, y) \in G \times G$.

Definition 32 Let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be bifuzzy sets on a set G. If $A = (\mu_A, \lambda_A)$ is a bifuzzy relation on a set G, then $A = (\mu_A, \lambda_A)$ is called a *bifuzzy relation* on $B = (\mu_B, \lambda_B)$ if $\mu_A(x, y) \leq \min(\mu_B(x), \mu_B(y))$ and $\lambda_A(x, y) \geq \max(\lambda_B(x), \lambda_B(y))$ for all $x, y \in G$.

Definition 33 Let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be two bifuzzy sets on a set G. Then the *Cartesian product* $A \times B$ is defined as follow:

$$\begin{aligned} A \times B &= (\mu_A, \lambda_A) \times (\mu_B, \lambda_B) \\ &= (\mu_A \times \mu_B, \lambda_A \times \lambda_B), \end{aligned}$$

where $(\mu_A \times \mu_B)(x, y) = \min(\mu_A(x), \mu_B(y))$ and $(\lambda_A \times \lambda_B)(x, y) = \max(\lambda_A(x), \lambda_B(y))$.

We note that the Cartesian product $A \times B$ is always a bifuzzy set in $G \times G$ if

$$0 \leq \min(\mu_A(x), \mu_B(y)) + \max(\lambda_A(x), \lambda_B(y)) \leq 1.$$

The proof of the following proposition is trivial.

Proposition 34 Let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be bifuzzy sets on a set G. Then

(i) $A \times B$ is a bifuzzy relation on G,

(ii) $U(\mu_A \times \mu_B; t) = U(\mu_A; t) \times U(\mu_B; t)$ and $L(\lambda_A \times \lambda_B; t) = L(\lambda_A; t) \times L(\lambda_B; t)$ for all $t \in [0, 1]$.

Theorem 35 If $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ are two bifuzzy ideals of a K-algebra $\mathcal{K}$. Then $A \times B$ is a bifuzzy ideal of $\mathcal{K} \times \mathcal{K}$.

Proof: For any $x = (x_1, x_2) \in G \times G$, we have

$$\begin{aligned}(\mu_A \times \mu_B)(e) &= (\mu_A \times \mu_B)((e_1, e_2)) \\ &= \min(\mu_A(e_1), \mu_B(e_2)) \\ &\geq \min(\mu_A(x_1), \mu_B(x_2)) \\ &= (\mu_A \times \mu_B)(x_1, x_2) \\ &= (\mu_A \times \mu_B)(x),\end{aligned}$$

$$\begin{aligned}(\lambda_A \times \lambda_B)(e) &= (\lambda_A \times \lambda_B)((e_1, e_2)) \\ &= \max(\lambda_A(e_1), \lambda_B(e_2)) \\ &\leq \max(\lambda_A(x_1), \lambda_B(x_2)) \\ &= (\lambda_A \times \lambda_B)(x_1, x_2) \\ &= (\lambda_A \times \lambda_B)(x).\end{aligned}$$

Let $x = (x_1, x_2)$ and $y = (y_1, y_2) \in G \times G$. Then

$$\begin{aligned}&(\mu_A \times \mu_B)(x) = (\mu_A \times \mu_B)((x_1, x_2)) \\ &= \min(\mu_A(x_1), \mu_B(x_2)) \\ &\geq \min(\min(\mu_A(x_1 \odot y_1), \mu_A(y_1 \odot (y_1 \odot x_1))), \\ &\min(\mu_B(x_2 \odot y_2), \mu_B(y_2 \odot (y_2 \odot x_2)))) \\ &= \min(\min(\mu_A(x_1 \odot y_1), \mu_B(x_2 \odot y_2)), \\ &\min(\mu_A(y_1 \odot (y_1 \odot x_1)), \mu_B(y_2 \odot (y_2 \odot x_2)))) \\ &= \min((\mu_A \times \mu_B)((x_1, x_2) \odot (y_1, y_2)), \\ &(\mu_A \times \mu_B)((y_1, y_2) \odot ((y_1, y_2) \odot (x_1, x_2))) \\ &= \min((\mu_A \times \mu_B)(x \odot y), (\mu_A \times \mu_B)(y \odot (y \odot x))),\end{aligned}$$

$$\begin{aligned}&(\lambda_A \times \lambda_B)(x) = (\lambda_A \times \lambda_B)((x_1, x_2)) \\ &= \max(\lambda_A(x_1), \lambda_B(x_2)) \\ &\leq \max(\max(\lambda_A(x_1 \odot y_1), \lambda_A(y_1 \odot (y_1 \odot x_1))), \\ &\max(\lambda_B(x_2 \odot y_2), \lambda_B(y_2 \odot (y_2 \odot x_2)))) \\ &= \max(\max(\lambda_A(x_1 \odot y_1), \lambda_B(x_2 \odot y_2)), \\ &\max(\lambda_A(y_1 \odot (y_1 \odot x_1)), \lambda_B(y_2 \odot (y_2 \odot x_2)))) \\ &= \max((\lambda_A \times \lambda_B)((x_1, x_2) \odot (y_1, y_2)), \\ &(\lambda_A \times \lambda_B)((y_1, y_2) \odot ((y_1, y_2) \odot (x_1, x_2))) \\ &= \max((\lambda_A \times \lambda_B)(x \odot y), (\lambda_A \times \lambda_B)(y \odot (y \odot x))).\end{aligned}$$

Hence $A \times B$ is a bifuzzy ideal of $\mathcal{K} \times \mathcal{K}$. □

Definition 36 Let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be bifuzzy sets on a set G, then *the strongest bifuzzy relation* on G that is bifuzzy relation on B is A_B, defined by

$$A_B = (\mu_{A_{\mu_B}}, \lambda_{A_{\lambda_B}})$$

where $\mu_{A_{\mu_B}}(x, y) = \min(\mu_B(x), \mu_B(y))$ and $\lambda_{A_{\lambda_B}}(x, y) = \max(\lambda_B(x), \lambda_B(y))$ for all $x, y \in G$.

The proofs of the following propositions are obvious.

Proposition 37 For given bifuzzy sets $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ on a set G, let A_B be the strongest bifuzzy relation on G. Then $U(\mu_{A_{\mu_B}}; t) = U(\mu_B; t) \times_{\min} U(\mu_B; t)$ and $L(\lambda_{A_{\lambda_B}}; t) = L(\lambda_B; t) \times_{\max} L(\lambda_B; t)$ for $t \in [0, 1]$.

Proposition 38 For given bifuzzy sets $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ on a set G, let A_B be the strongest bifuzzy relation on G. If A_B is a bifuzzy ideal of $\mathcal{K} \times \mathcal{K}$, then $\mu_A(x) \leq \mu_A(e)$ and $\lambda_A(x) \geq \lambda_A(e)$ for all $x \in G$.

Theorem 39 Let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be bifuzzy ideals in a K-algebra $\mathcal{K}$ and A_B the strongest bifuzzy relation on $\mathcal{K}$. Then $B = (\mu_B, \lambda_B)$ is a bifuzzy ideal of $\mathcal{K}$ if and only if A_B is a bifuzzy ideal of $\mathcal{K} \times \mathcal{K}$.

Proof: Let $B = (\mu_B, \lambda_B)$ be a bifuzzy ideal of $\mathcal{K}$. For $x = (x_1, x_2) \in G \times G$, we have

$$\begin{aligned}(\mu_{A_{\mu_B}})(e) &= \mu_{A_{\mu_B}}(e_1, e_2) \\ &= \min(\mu_B(e_1), \mu_B(e_2)) \\ &\geq \min(\mu_B(x_1), \mu_B(x_2)) \\ &= \min(\mu_{A_{\mu_B}}(x_1, x_2)) \\ &= \min(\mu_{A_{\mu_B}}(x)),\end{aligned}$$

$$\begin{aligned}(\lambda_{A_{\lambda_B}})(e) &= \lambda_{A_{\lambda_B}}(e_1, e_2) \\ &= \max(\lambda_B(e_1), \lambda_B(e_2)) \\ &\leq \max(\lambda_B(x_1), \lambda_B(x_2)) \\ &= \max(\lambda_{A_{\lambda_B}}(x_1, x_2)) \\ &= \max(\lambda_{A_{\lambda_B}}(x)).\end{aligned}$$

Take $x = (x_1, x_2)$ and $y = (y_1, y_2) \in G \times G$. Then

$$\begin{aligned}&(\mu_{A_{\mu_B}})(x) = \mu_{A_{\mu_B}}((x_1, x_2)) \\ &= \min(\mu_B(x_1), \mu_B(x_2)) \\ &\geq \min(\min(\mu_B(x_1 \odot y_1), \mu_B(y_1 \odot (y_1 \odot x_1))), \\ &\min(\mu_B(x_2 \odot y_2), \mu_B(y_2 \odot (y_2 \odot x_2)))) \\ &= \min(\min(\mu_B(x_1 \odot y_1), \mu_B(x_2 \odot y_2)), \\ &\min(\mu_B(y_1 \odot (y_1 \odot x_1)), \mu_B(y_2 \odot (y_2 \odot x_2)))) \\ &= \min(\mu_{A_{\mu_B}}(x_1, x_2) \odot (x_2, y_2)), \\ &\mu_{A_{\mu_B}}(y_1, y_2) \odot ((y_1, y_2) \odot (x_1, x_2)) \\ &= \min(\mu_{A_{\mu_B}}(x \odot y), \mu_{A_{\mu_B}}(y \odot (y \odot x))),\end{aligned}$$

$$(\lambda_{A_{\lambda_B}})(x) = \lambda_{A_{\lambda_B}}((x_1, x_2))$$
$$= \max(\lambda_B(x_1), \lambda_B(x_2))$$
$$\leq \max(\max(\lambda_B(x_1 \odot y_1), \lambda_B(y_1 \odot (y_1 \odot x_1))),$$
$$\max(\lambda_B(x_2 \odot y_2), \lambda_B(y_2 \odot (y_2 \odot x_2))))$$
$$= \max(\max(\lambda_B(x_1 \odot y_1), \lambda_B(x_2 \odot y_2)),$$
$$\max(\lambda_B(y_1 \odot (y_1 \odot x_1)), \lambda_B(y_2 \odot (y_2 \odot x_2))))$$
$$= \max(\lambda_{A_{\lambda_B}}(x_1, x_2) \odot (x_2, y_2)),$$
$$\lambda_{A_{\lambda_B}}(y_1, y_2) \odot ((y_1, y_2) \odot (x_1, x_2))$$
$$= \max(\lambda_{A_{\lambda_B}}(x \odot y), \lambda_{A_{\lambda_B}}(y \odot (y \odot x))).$$

This shows that A_B is a bifuzzy ideal of $\mathcal{K} \times \mathcal{K}$.
Conversely, suppose that $A_B = (\mu_{A_{\mu_B}}, \lambda_{A_{\lambda_B}})$ is a bifuzzy ideal of $\mathcal{K} \times \mathcal{K}$. Then

$$\begin{aligned} \min\{\mu_B(e), \mu_B(e)\} &= \mu_{A_{\mu_B}}(e, e) \\ &\geq \mu_{A_{\mu_B}}(x, y) \\ &= \min\{\mu_B(x), \mu_B(y)\}, \end{aligned}$$

$$\begin{aligned} \max\{\lambda_B(e), \lambda_B(e)\} &= \lambda_{A_{\lambda_B}}(e, e) \\ &\leq \lambda_{A_{\lambda_B}}(x, y) \\ &= \max\{\lambda_B(x), \lambda_B(y)\}. \end{aligned}$$

for all $(x, y) \in G \times G$. It follows that $\mu_B(x) \leq \mu_B(e)$ and $\lambda_B(x) \geq \lambda_B(e)$ for all $x \in G$.
For any $x = (x_1, x_2)$, $y = (y_1, y_2) \in G \times G$,

$$\min\{\mu_B(x_1), \mu_B(x_2)\} = \mu_{A_{\mu_B}}(x_1, x_2)$$
$$\geq \min\{\mu_{A_{\mu_B}}((x_1, x_2) \odot (y_1, y_2)),$$
$$\mu_{A_{\mu_B}}((y_1, y_2) \odot ((y_1, y_2) \odot (x_1, x_2))\}$$
$$= \min\{\mu_{A_{\mu_B}}(x_1 \odot y_1, x_2 \odot y_2),$$
$$\mu_{A_{\mu_B}}(y_1 \odot (y_1 \odot x_1), y_2 \odot (y_2 \odot x_2))\}$$
$$= \min\{\min\{\mu_B(x_1 \odot y_1), \mu_B(x_2 \odot y_2)\},$$
$$\min\{\mu_B(y_1 \odot (y_1 \odot x_1)), \mu_B(y_2 \odot (y_2 \odot x_2))\}\}$$
$$= \min\{\min\{\mu_B(x_1 \odot y_1), \mu_B(y_1 \odot (y_1 \odot x_1))\},$$
$$\min\{\mu_B(x_2 \odot y_2), \mu_B(y_2 \odot (y_2 \odot x_2))\}\},$$

$$\max\{\lambda_B(x_1), \lambda_B(x_2)\} = \lambda_{A_{\lambda_B}}(x_1, x_2)$$
$$\leq \max\{\lambda_{A_{\lambda_B}}((x_1, x_2) \odot (y_1, y_2)),$$
$$\lambda_{A_{\lambda_B}}((y_1, y_2) \odot ((y_1, y_2) \odot (x_1, x_2))\}$$
$$= \max\{\lambda_{A_{\lambda_B}}(x_1 \odot y_1, x_2 \odot y_2),$$
$$\lambda_{A_{\lambda_B}}(y_1 \odot (y_1 \odot x_1), y_2 \odot (y_2 \odot x_2))\}$$
$$= \max\{\max\{\lambda_B(x_1 \odot y_1), \lambda_B(x_2 \odot y_2)\},$$
$$\max\{\lambda_B(y_1 \odot (y_1 \odot x_1)), \lambda_B(y_2 \odot (y_2 \odot x_2))\}\}$$
$$= \max\{\max\{\lambda_B(x_1 \odot y_1), \lambda_B(y_1 \odot (y_1 \odot x_1))\},$$
$$\max\{\lambda_B(x_2 \odot y_2), \lambda_B(y_2 \odot (y_2 \odot x_2))\}\}.$$

Putting $x_2 = y_2 = e$ gives

$$\mu_B(x_1) \geq \min\{\mu_B(x_1 \odot y_1), \mu_B(y_1 \odot (y_1 \odot x_1))\},$$
$$\lambda_B(x_1) \leq \max\{\lambda_B(x_1 \odot y_1), \lambda_B(y_1 \odot (y_1 \odot x_1))\}.$$

Hence $B = (\mu_B, \lambda_B)$ is a bifuzzy ideal of $\mathcal{K}$. □

6 Conclusions

It is known that logic is an essential tool for giving applications in mathematics and computer science. Non-classical logic takes the advantage of the classical logic to handle information with various facts of uncertainty such as the fuzziness and randomness. The non-classical logic has become a formal and useful tool for computer science to deal with fuzzy information and uncertain information. In the present paper we have introduced notion of bifuzzy ideals in logical algebras:K-algebras and investigate some interesting properties. Thus our obtained results can be applied in various fields such as artificial intelligence, signal processing, multiagent systems, pattern recognition, robotics, computer networks, expert systems, decision making, automata theory and medical diagnosis.

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