Uniqueness of Positive Solutions For Neumann Problems in Unbounded Domain

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Abstract: In this paper, we consider the existence and uniqueness positive solutions of the following boundary Neumann problem

$$-\Delta u = a(x)u - b(x)f(u)\ x \in T, \frac{\partial u}{\partial n} = 0, \text{ on } \partial T,$$

where $T = \{x = (x_1, x_2, \cdots, x_N) : x_N > 0\}, (N \geq 2)$, $a(x)$ and $b(x)$ are continuous functions with $b(x)$ positive on R^N and n is outward pointing unit normal vector of ∂T, we show that under rather general conditions on $a(x)$ and $b(x)$ for large $|x|$ and $f(u)$ behaves like u^q, where constant $q > 1$, the above problems possesses a minimal positive solution and a maximal positive solution, respectively, Moreover, we establish a relationship between the above problem and the following problem

$$-\Delta u = a(x)u - b(x)f(u)\ x \in R^N,$$

We establish a comparison principal which our proof of the existence results rely essentially on. and make use of a rather intuitive squeezing method to get the existence theorems. Furthermore, by analyzing the behavior of the positive solution for the problem in whole space, we show the boundary Neumann problem in half space has only one positive solution. Our results improve the previous works.

Keyword: Sub-super solution, Neumann problem, Comparison principle, Positive solution, Squeezing method

1 Introduction

In this paper, we are concerned with positive solutions of the following boundary Neumann problem

$$\begin{cases} -\Delta u = a(x)u - b(x)f(u),\ x \in T \subset R^N \\ \dfrac{\partial u}{\partial n} = 0, \text{ on} \partial T \end{cases} \quad (1)$$

where $T = \{x = (x_1, x_2, \cdots, x_N) : x_N > 0\}, (N \geq 2)$, q is a constant greater than 1, $a(x)$ and $b(x)$ are continuous functions with $b(x)$ positive on R^N and n is outward pointing unit normal vector of ∂T, Equations of this kind in bounded or unbounded region with different boundary values have attracted extensive study because of its interest to mathematical biology, Riemannian geometry and generalized reaction-diffusion and in non- Newtonian fluid theory. The existence of exact solution and the asymptotic and numerial solution of problem (1) for different nonlinearities have been attracted considerable interest in the last decades. We refer to [1,5,6,7,8,10,13,20]and the references therein for some of the previous research.

Recently, the Dirichlet problems with different types in the upper half space or rough boundary domains, under two measures on the boundary, have been thoroughly investigated (see [2,3,4,23,24,25,26]). In 2004, Du and Guo in [15] proved that any boundary positive solution of the following Dirichlet problem:

$$\begin{cases} -\Delta u = f(u), x \in T \\ u = 0, x \in \partial T \end{cases}$$

is unique and is a function of x_n only provide that f is locally quasi-monotone on $(0,\infty)$ and satisfies:

(2) for some $a > 0$,

$$f(s) > 0 \text{ in } (0,a), \quad f(s) < 0 \text{ in } (a,\infty),$$

(3) for some small d>0, there exists a constant $\delta > 0$ such that

$$f(s) > \mathrm{d}\, s \quad \text{for all} \quad s \in (0,\delta)$$

We say that $f(s)$ is locally quasi-monotone on $(0,\infty)$ if for any bounded interval $[s_1, s_2] \subset [0,\infty)$, there exists a continuous increasing function $L(s)$ such that $f(s)+L(s)$ is non-decreasing in s for $s \in [s_1, s_2]$.

Clearly, this condition is less restrictive than requiring $f(s)$ to be locally Lipschitz continuous on $[0,\infty)$.

In 2005, for α is a positive constant (or ∞), Dong in [12] showed that the folowing problem

$$\begin{cases} -\Delta u = f(u), x \in T \\ u = \alpha, x \in \partial T \end{cases}$$

has a unique positive solution if $f(s)$ is locally quasi-monotone on $(0,\infty)$ and satisfies (2).

In the present paper, we will consider the boundary Neumann problem in the upper half space for more general nonlinearity. We only consider the existence of positive solutions. By a positive solution to (1), we mean a function $u \in W^{1,2} \cap C(T)$ satisfying $u > 0$ in T such that

$$\int_T Du \cdot D\psi dx = \int_T g(x,u)\psi dx, \forall \psi \in C_0^\infty(T \cup \partial T)$$

and

$$\frac{\partial u}{\partial n} = 0, \text{ on } \partial T$$

Where $g(x,u) = a(x)u - b(x)f(u)$.

Through out this paper, we always assume that for some γ and ξ such that $\gamma > 0$, there exist positive numbers α_1, α_2 and β_1, β_2 such that

$$\begin{aligned} \alpha_1 &= \underline{\lim}_{|x|\to\infty} \frac{a(x)}{|x|^\gamma}, \alpha_2 = \underline{\lim}_{|x|\to\infty} \frac{a(x)}{|x|^\gamma} \\ \beta_1 &= \underline{\lim}_{|x|\to\infty} \frac{b(x)}{|x|^\xi}, \beta_2 = \underline{\lim}_{|x|\to\infty} \frac{b(x)}{|x|^\xi} \end{aligned} \quad (4)$$

and $f(u)$ satisfies the conditions(5)and $(A2)$ listed below.

(5): $f(t) \ge 0$, $f(t)/t$ is increasing on $(0,\infty)$ and $\lim_{t\to 0} f(u)/u = 0$;

(6): $\int_1^\infty F(t)^{-\frac{1}{2}} dt < \infty$, where $F(t) = \int_0^t f(s)ds$

It is easily shown that under these conditions, problem (1) has at least one (weak) positive solution. By standard regularity theory of elliptic equations, any $W_{loc}^{1,2}(R^N)$ solution of (1) belongs to $C^1(R^N)$.

Let us now describe our results in more details. In section 2, we establish a comparison principal which our proofs of the existence results rely essentially on. We make use of a rather intuitive squeezing method as follows to obtain the existence theorem as follows.

Let B_r be a ball on R^N with centered at origin with radius r, $\Omega_r = B_r \cap T$, $\Gamma_1 = \partial B_r \cap T$ and $\Gamma_2 = \partial T \cap \Omega_r$. Then for large $r > 0$, the following problem:

$$\begin{cases} -\Delta u = a(x)u + b(x)f(u), x \in \Omega_r \\ u = 0, x \in \Gamma_1 \\ \dfrac{\partial u}{\partial n} = 0, x \in \Gamma_2 \end{cases}$$

has a unique positive solution u_r. On the other hand, the mixed boundary problem

$$\begin{cases} -\Delta u = a(x)v + b(x)f(v), x \in B_n \\ u = \infty, x \in \Gamma_1 \\ \dfrac{\partial v}{\partial n} = 0, x \in \Gamma_2 \end{cases}$$

has a positive solution v_r. When r increases to infinity, u_r and v_r converges to a minimal positive solution and a maximal positive solution for (1), respectively, namely:

Theorem 1 Problem (1) possesses a minimal positive solution $\underline{u}$ and a maximal positive solution $\overline{u}$, respectively.

In order to obtain a complete understanding of problem (1), in section 3, we need to study the following problem:

$$-\Delta u = a(x)u - b(x)f(u), x \in R^N \quad (7)$$

Under the assumptions on $a(x), b(x)$ and $f(t)$, furthermore, for some positive constants d_1, d_2 and $q > 1$, $f(t)$ satisfies

$$\underline{\lim}_{t\to 0}\frac{f(t)}{t^q} \geq d_1 > 0, \overline{\lim}_{t\to\infty}\frac{f(t)}{t^q} \leq d_2 < \infty \quad (8)$$

We obtain the following asymptotic behavior of positive solutions for (7) as $|x| \to \infty$ first.

Theorem 2 Suppose $u \in C^1(R^N)$ is a positive solution of (7). If (4) and (8) are satisfied, then for some positive constants c_1 and c_2 such that $0 < c_1 \leq c_2 < \infty$, we have

$$\underline{\lim}_{|x|\to\infty}\frac{u^{q-1}(x)}{|x|^{\gamma-\tau}} \geq c_1 \quad (9)$$

and

$$\overline{\lim}_{|x|\to\infty}\frac{u^{q-1}(x)}{|x|^{\gamma-\tau}} \leq c_2 \quad (10)$$

Next we combine the squeezing method in [18] with the iteration argument motivated by one attributed to Safonov (see also [14,17]) to obtain the uniqueness result in whole space.

Theorem 3 Suppose $f(t)$ satisfies (8). Furthermore, if $f(u)$ satisfies:

$$\begin{cases} \text{when } \gamma > \tau, \lim\limits_{u\to\infty}\dfrac{f(u)}{u^q} = k_1 > 0 \\ \text{when } \gamma < \tau, \lim\limits_{u\to 0}\dfrac{f(u)}{u^q} = k_2 > 0 \\ \text{when } \gamma = \tau, f(u) = Cu^q, C > 0 \end{cases} \quad (11)$$

Then problem (7) has a unique positive solution.

In section 4, we establish a relationship between the positive solutions of (1) and ones of (7), and utilizing the uniqueness result for problem (7), we obtain our main uniqueness result:

Theorem 4 Assume that $f(t)$ satisfies (9) and (11), then problem (1) has a unique positive solution.

2 Existence of Positive Solutions of Problem (1)

In this section, we adapt the comparison principle in [16] and modify it, we obtain the following new comparison principle.

Lemma 5 (Comparison principle) Suppose that Ω is a bounded domain in R^N which $\partial\Omega$ splits into Γ_1 and Γ_2. $\alpha(x)$ and $\beta(x)$ are continuous with $\beta(x) \geq 0$, $\beta(x) \neq 0$ on Ω and $\|a\|_{L^\infty(\Omega)} < \infty$. Let $u_1, u_2 \in C^1(\Omega)$ be positive in Ω and satisfy (in the weak sense)

$$\Delta u_1 + \alpha(x)u_1 - \beta(x)f(u_1) \leq 0 \leq \Delta u_2 + \alpha(x)u_2 - \beta(x)f(u_2)$$

in Ω and

$$\overline{\lim}_{dist(x,\Gamma_1 \to 0)}(u_2 - u_1) \leq 0$$

$$\frac{\partial u_1}{\partial n} \geq \frac{\partial u_2}{\partial n}, x \in \Gamma_2.$$

where $f(u)$ is a continuous function which for every $x \in \Omega$, $f(u)/u$ is strictly increasing for u in the range $\inf_\Omega\{u_1, u_2\} < u < \sup_\Omega\{u_1, u_2\}$. Then $u_2 \leq u_1$ in Ω.

This Lemma can be easily derived from Lemma 2.1 in [16].

Lemma 6. Suppose that Ω is a bounded domain in R^N and $\beta(x)$ are continuous with $\beta(x) > 0$. If $\lambda_1(\Omega, \alpha) < 0$ and f satisfies $(A_1) - (A_2)$, then, the following problems

$$\begin{cases} -\Delta u = \alpha(x)u - \beta(x)f(u), x \in \Omega \\ u = 0, x \in \partial\Omega \end{cases} \quad (12)$$

has a unique positive solution.

Proof. Let ϕ be a positive eigenfunction corresponding to $\lambda_1(\Omega, \alpha)$. Since $\lim_{t\to 0} f(t)/t = 0$, then for all small positive constant ε, it easily checked that $\varepsilon\phi$ is a subsolution of problem (12). Since f satisfies (8), we can easily obtain

$$\lim_{t\to\infty} f(t)/t = \infty.$$

Hence there exists a large number $M_0 > 0$ such that for all

$$M > M_0, \ \alpha(x)M - \beta(x)f(M) \leq 0.$$

Thus, M is a supersolution of (12). A standard sub-and super solution argument (see [10,19]) implies problem (12) has at least one positive solution.

Let $u_1(x)$ and $v_1(x)$ be two arbitrary positive solutions of (6), by Lemma 5 with $\Gamma_1 = \partial\Omega$, we have

$$u_1(x) \leq v_1(x) \text{ and } u_1(x) \geq v_1(x).$$

Then it has a unique positive solution.

Next we will show the existence result Theorem 2. Let B_r be a ball on R^N with centered at origin with radius r, $\Omega_r = B_r \cap T$, $\Gamma_1 = \partial B_r \cap T$ and $\Gamma_2 = \partial T \cap \Omega_r$. Now we consider the following problem:

$$\begin{cases} -\Delta u = a(x)u + b(x)f(u), x \in \Omega_r \\ u = 0, x \in \Gamma_1 \\ \dfrac{\partial u}{\partial n} = 0, x \in \Gamma_2 \end{cases} \quad (13)$$

Since condition (4) holds, by the properties of the first eigenvalue (see [11,21]), there exists a large $r_0 > 0$ such that for all $r \geq r_0$, $\lambda(\Omega_r, a) < 0$. By Lemma 6, the problem (13) with $\Omega = \Omega_r$ has a unique positive solution u_1. It is clear that u_1 satisfies

$$u_1 = 0,\ x \in \Gamma_1 \text{ and } \frac{\partial u_1}{\partial n} \leq 0,\ x \in \Gamma_2$$

Then by the comparison principle Lemma 5, we obtain u_1 is a subsolution of equation (13). Since f satisfies $(A2)$, we can easily obtain

$$\lim_{t \to \infty} \frac{f(t)}{t} = \infty$$

Hence there exists a large $M_0 > u_1$ such that for all $M > M_0$, $a(x)M - b(x)f(M) \leq 0$. Thus, M is a supersolution of (13).

By standard sub-supersolution method for elliptic equation, the problem (13) has at least one positive solution u_r in the order interval $[u_1, M]$. It follows from Lemma 5 that it has a unique positive solution.

Let us choose an increasing sequence of positive real numbers $r_n > r_0$ and $r_n \to \infty$ as $n \to \infty$. By the discussion above, problem (13) with $\Omega_r = \Omega_{r_n}$ has a unique positive solution u_n. It follows from Lemma 5 that $u_n \leq u_{n+1}$. If we can find an upper bound for $u_n(x)$ on any fixed Ω_R, then by a standard regularity argument, $u(x) = \lim_{n \to \infty} u_n(x)$ is well-defined in T and it would be a positive solution of problem (1). To find such an upper bound, we consider the problem

$$-\Delta v = \alpha(x)v - \beta(x)f(v),\ x \in \Omega_R,\ v|_{\partial \Omega_R} = \infty$$

By Theorem 1.1 in [9], the above problem has a positive solution $v(x)$. Then clearly by the comparison principle Lemma 5, we obtain

$$u_n(x) \leq v(x), \quad \forall x \in \Omega_R$$

for all large n such that $r_n > R$. This is the bound we are looking for, and hence the existence of a solution for (1) is proved.

From $u_n \leq u_{n+1}$ we find

$$\underline{u}(x) \geq u_n(x) > 0$$

for each n, and hence $\underline{u}$ is a positive solution of (1). For an arbitrary positive solution u of (1), we can see that u satisfies

$$-\Delta u = \alpha(x)u - \beta(x)f(u),\ u|_{\Gamma_1} > 0$$

By Lemma 5 $u \geq u_n$ on Ω_{r_r} for each n, and hence

$$u \geq \underline{u} = \lim_{n \to \infty} u_n$$

So $\underline{u}$ is the minimal positive solution of (1).

Next we will show the existence of a maximal positive solution of (1). To this end, we choose an increasing sequence of real number r_n such that $r_n \to \infty$ as $n \to \infty$ and denote $B_n = \Omega_{r_n}$. We consider the following mixed boundary problem

$$\begin{cases} -\Delta u = a(x)u - b(x)f(u), x \in B_n \\ u = \infty, x \in \Gamma_1 \\ \dfrac{\partial u}{\partial n} = 0, x \in \Gamma_2 \end{cases} \quad (14)$$

Obviously $u = 0$ is a subsolution of problem (14). By Theorem 1.1 in [9], the following equation

$$\begin{cases} -\Delta v = \alpha(x)v - \beta(x)f(v), x \in B_n \\ v = 0, x \in \partial B_n \end{cases}$$

has a positive solution and we denote it as v_n. It is easy to show $\frac{\partial v_n}{\partial n} \geq 0$ and v_n is a supersolution of (14). Thus problem (14) has at least one positive solution u_n.

Applying Lemma 5, we see

$$u_n \geq u_{n+1} > \underline{u}, x \in B_n \text{ for all } n.$$

So $\overline{u} = \lim_{n \to \infty} u_n$ is well-defined on T. Furthermore, by standard regularity considerations, we know $\overline{u}$ satisfies (1) on T and $\overline{u} \geq \underline{u}$, so $\overline{u}$ is a positive solution of (1).

Clearly any positive solution u of (1) satisfies, for each n,

$$-\Delta u = a(x)u - b(x)f(u),$$

$$u|_{\Gamma 1} < \infty, \frac{\partial u}{\partial n} = 0.$$

It follows from Lemma 5 that we see

$$u_n \geq u \text{ on } B_n \text{ for all } n,$$

and hence

$$\overline{u} = \lim_{n \to \infty} \omega_n \geq u$$

The proof is now finished.

3 The Whole Space Problem

In this section, we will prove the asymptotic behavior of the positive solution of problem (7), and then make use of this result to prove the uniqueness result in Theorem 3.

Before we start to prove our uniqueness result

Theorem 3, we need the following existence lemma.

Lemma 7. If condition (4) is satisfied, then problem (7) possesses a minimal positive solution $\underline{u}$ and a maximal positive solution $\overline{u}$.

Proof. By condition (4), there exists a large $r > 0$, such that $\lambda_1(B_r, a) < 0$, and it follows from Lemma 6 that the following problem

$$\begin{cases} -\Delta u = \alpha(x)u - \beta(x)f(u), x \in B_r \\ u = 0, x \in \partial B_r \end{cases} \tag{15}$$

has a unique positive solution u_r.

Let us choose an increasing sequence of positive real numbers r_n with $r_1 > r$ and $r_n \to \infty$ as $n \to \infty$ .By the properties of the first eigenvalue in [11,21], and by Lemma 6, problem (915 with $r = r_n$ has a unique positive solution u_n. By the comparison principle Lemma 5, we deduce

$$u_n \le u_{n+1}.$$

If we can find an upper bound for u_n on any mixed B_R, then by a standard regularity argument, $\underline{u} = \lim_{n\to\infty} u_n$ is well-defined in R^N and it would be a positive solution of (7).

To find such an upper bound, we consider the problem

$$\begin{cases} -\Delta u = \alpha(x)u - \beta(x)f(u), x \in B_R \\ u = 0, x \in \partial B_R \end{cases} \tag{16}$$

Theorem 1.1 in [9] implies that (16) has a positive solution v. Then by Lemma 5,

$$u_n(x) \le v(x), \forall x \in B_R$$

for all large n such that $r_n > R$. This is the bound we are looking for, and hence the existence of a solution for (7) is proved.

From $u_n \le u_{n+1}$ we find $\underline{u} \ge u_n(x) > 0$ for each n ,and hence $\underline{u}$ is a positive solution of (7). For an arbitrary positive solution u of (7), we can see that u satisfies

$$-\Delta u = \alpha(x)u - \beta(x)f(u), \quad u|_{\partial B_{r_n}} > 0$$

So $\underline{u}$ is the minimal positive solution of (7).

Next we will show the existence of a maximal positive solution of (7). To this end, we choose an increasing sequence of real number r_n such that $r_n \to \infty$ as $n \to \infty$ and denote $B_n = B_{r_n}$ .We consider the boundary blow-up problem:

$$-\Delta\omega = \alpha(x)\omega - \beta(x)f(\omega) \text{ in } B_n \quad u|_{\partial B_n} = \infty \tag{17}$$

It follows from Theorem1.1 in [9] that (17) has a positive solution and we denote it as ω_n. Applying Lemma 5, we see

$$\omega_n \ge \omega_{n+1} \ge \underline{u}, \quad x \in B_n \text{ for all } n.$$

Thus

$$\overline{u} = \lim_{n\to\infty} \omega_n$$

is well-defined on R^N. Furthermore, by standard regularity considerations, we know $\overline{u}$ satisfies (7) on R^N and $\overline{u} \ge \underline{u}$, so $\overline{u}$ is a positive solution of (7).

Clearly any positive solution u of (7) satisfies, for each n,

$$-\Delta u = \alpha(x)u - \beta(x)f(u), \quad u|_{\partial B_{r_n}} < \infty$$

It follows from Lemma 5 that we see

$$\omega_n \ge u \text{ on } B_n \text{ for all } n.$$

And hence

$$\overline{u} = \lim_{n\to\infty} \omega_n > u$$

This finishes the proof.

Next we will show the asymptotic behavior of positive solutions for (7) as $|x| \to \infty$ and use the result to prove Theorem 3.

Proof of Theorem 3: Because $f(u)$ satisfies (8), then there exist two positive constant $0 < h_1 \le h_2$ such that

$$h_1 t^q \le f(t) \le h_2 t^q \tag{18}$$

By Proposition 3.2 in [10], the following problem:

$$-\Delta v = a(x)v - h_2 b(x)v^q, x \in R^N \tag{19}$$

possesses a minimal positive v.

By the constructions of the minimal positive solutions v, on any fixed B_R, we have

$$-\Delta v > a(x)v - b(x)f(v)$$

By Lemma 5, we can easily obtain $v \le \underline{u}$, where $\underline{u}$ is the minimal positive solution of (7). By Lemma 3.1 in [11], we have

$$\underline{\lim}_{|x|\to\infty} \frac{v^{q-1}(x)}{|x|^{\gamma-\tau}} \ge \frac{\alpha_1}{h_2\beta_2}$$

Thus there exists $c_1 > 0$ such that

$$\underline{\lim}_{|x|\to\infty} \frac{\underline{u}^{q-1}(x)}{|x|^{\gamma-\tau}} \ge c_1 \tag{20}$$

By the same method as above, the following problem:

$$-\Delta v = a(x)\omega - h_1 b(x)\omega^q, x \in R^N \tag{21}$$

has a maximal positive solution such that $\overline{u} \le \omega$, where $\overline{u}$ is the maximal positive solution of (7). By the Lemma 3.1 in [14], we have

$$\overline{\lim}_{|x|\to\infty} \frac{\omega^{q-1}(x)}{|x|^{\gamma-\tau}} \le \frac{\alpha_2}{h_1\beta_1} \tag{22}$$

Thus

$$\overline{\lim}_{|x|\to\infty} \frac{\overset{-q-1}{u}(x)}{|x|^{\gamma-\tau}} \le \frac{\alpha_2}{h_1\beta_1} \quad (23)$$

It follows from (20) and (23) that the Theorem 3 is complete.

What remains is to show the uniqueness result for problem (7). The following technical lemma is the core of our iteration argument to be used in the uniqueness proof.

Lemma 8 Suppose that (4), (5), (6), (8) and (11) hold, u_1, u_2 are positive solutions of (7). Then there exists $R > 1$ large so that, if $x_0 \in R^N$ satisfies, for some $k_* \ge k > 1$,

$$|x_0| > R,\ u_2(x_0) > k_* u_1(x_0),$$

thus we can find $y_0 \in R^N$, and positive constants $c_0 = c_0(R,k)$ and $r_0 = r_0(R,k)$ independent of x_0 and k_* such that

$$|y_0 - x_0| = r_0 |x_0|^{-\gamma/2}, u_2(y_0) > (1+c_0)k_* u_1(y_0) \quad (24)$$

Proof. By (4), (9) and (10), for all large $R_1 > 1$ and $|x| > R_1$, we have

$$(1/2)\alpha_1 |x|^{\gamma} < a(x) < 2\alpha_2 |x|^{\gamma}$$
and
$$(1/2)\beta_1 |x|^{\gamma} < a(x) < 2\beta_2 |x|^{\gamma} \quad (25)$$

And, for $i = 1,2$,

$$\mu_1 |x|^{(\gamma-\tau)/(q-1)} < u_i(x) < \mu_2 |x|^{(\gamma-\tau)/(q-1)} \quad (26)$$

where

$$\mu_1 = (1/2)\left(\frac{\alpha_1}{h_2\beta_2}\right)^{1/(q-1)}, \mu_2 = 2\left(\frac{\alpha_2}{h_1\beta_1}\right)^{1/(q-1)}$$

We now fixed $R_1 > 1$ large enough so that $R^{-1-(\gamma/2)} < 1/2$ and (25), (26) hold for all x satisfying $|x| > R_1/2$. Then we define

$$\Omega_0 := \{x \in R^N : u_2(x) > k_* u_1(x)\} \cap B_r(x_0),$$

where

$$r = r_0 |x_0|^{-\gamma/2}, B_r(x_0) = \{x \in R^N : |x - x_0| < r\},$$

and $r_0 \in (0,1)$ is to be determined below.

Clearly $x \in \Omega_0$ implies

$$|x_0| - r \le |x| \le |x_0| + r,$$

which in turn implies, due to $|x_0| > R_1$, |x| >R and our choice of R_1,

$$(1/2)|x_0| < |x| < (3/2)|x_0| \quad (27)$$

Using (25)-(27) and the assumption that $u_2 - k_* u_1 > 0$ in Ω_0, we now consider $\Delta(u_2 - k_* u_1)$ in Ω_0 in three cases.

Case 1: $\gamma > \tau$ **.**

By Theorem 2, if $\gamma > \tau$, then $u(x) \to \infty$ as $|x| \to \infty$. Then, it follows from (11) that

$$\lim_{|x|\to\infty} \frac{f(u(x))}{u(x)^q} = k_1$$

So for some $\varepsilon > 0$ small enough, there exists a large $R_2 > R_1$ such that if $|x| > R_2$,

we have

$$(k_1 - \varepsilon)u^q \le f(u) \le (k_1 + \varepsilon)u^q$$

and

$$(k_1 - \varepsilon){k_*}^{q-1} - (k_1 + \varepsilon) > 0$$

Then we deduce, for $x \in \Omega_0$

$$\begin{aligned}
&\Delta(u_2(x) - k_* u_1(x)) \\
&= -a(x)(u_2(x) - k_* u_1(x)) + b(x)(f(u_2) - k_* f(u_1)) \\
&\ge -a(x)(u_2(x) - k_* u_1(x)) + b(x)((k_1 - \varepsilon)u_2{}_2^q \\
&\quad - k_*(k_1 + \varepsilon)u_1^q) \\
&\ge -a(x)(u_2(x) - k_* u_1(x)) + b(x)((k_1 - \varepsilon)u_2{}_1^q \\
&\quad - k_*(k_1 + \varepsilon)u_1^q) \\
&\ge -2\alpha_2 |x|^{\gamma} (u_2(x) - k_* u_1(x)) + b(x)k_* u_1^q ((k_1 - \varepsilon)k_*^{q-1} \\
&\quad - (k_1 + \varepsilon)) \\
&\ge -M |x_*|^{\gamma} (u_2(x) - k_* u_1(x)) + \frac{1}{2} k_* |x|^{\tau} A_1^q \beta_1 \\
&\quad |x|^{\frac{(\gamma-\tau)^q}{q-1}} ((k_1 - \varepsilon)k_*^{q-1} - (k_1 + \varepsilon)) \\
&\ge -M |x_*|^{\gamma} (u_2(x) - k_* u_1(x)) + m_1 k_* |x_*|^{\sigma}
\end{aligned}$$

where

$$M = 2\alpha_2 \max((\tfrac{1}{2})^{\gamma}, (\tfrac{3}{2})^{\gamma}), \sigma = \tau + \frac{(\gamma-\tau)q}{q-1}$$

$$m_1 = \frac{1}{2} A_1^q \beta_1 ((k_1 - \varepsilon)k_*^{q-1} - (k_1 + \varepsilon)) \min((\tfrac{1}{2})^{\sigma}, (\tfrac{3}{2})^{\sigma})$$

Case 2: $\gamma < \tau$.

By Theorem 2, if $\gamma < \tau$, then $u(x) \to \infty$ as $|x| \to \infty$. Then, it follows from (11) that

$$\lim_{|x|\to\infty} \frac{f(u(x))}{u(x)^q} = k_2 .$$

So for some $\varepsilon > 0$ small enough, there exists a large $R_3 > R_1$ such that if $|x| > R_3$,

We have

$$(k_2 - \varepsilon)u^q \le f(u) \le (k_2 + \varepsilon)u^q$$

and

$$(k_2 - \varepsilon){k_*}^{q-1} - (k_2 + \varepsilon) > 0$$

Then we deduce, for $x \in \Omega_0$

$$\begin{aligned}
&\Delta(u_2(x) - k_* u_1(x)) \\
&= -a(x)(u_2(x) - k_* u_1(x)) + b(x)(f(u_2) - k_* f(u_1))
\end{aligned}$$

$$\geq -a(x)(u_2(x)-k_*u_1(x))+b(x)((k_2-\varepsilon)u_2{}_2^q -k_*(k_2+\varepsilon)u_1^q)$$
$$\geq -a(x)(u_2(x)-k_*u_1(x))+b(x)((k_2-\varepsilon)u_2{}_1^q -k_*(k_2+\varepsilon)u_1^q)$$
$$\geq -2\alpha_2\,|x|^\gamma\,(u_2(x)-k_*u_1(x))+b(x)k_*u_1^q ((k_2-\varepsilon)k_*^{q-1}-(k_2+\varepsilon))$$
$$\geq -M\,|x_*|^\gamma\,(u_2(x)-k_*u_1(x))+\frac{1}{2}k_*\,|x|^\tau A_1^q\beta_1|x|^{\frac{(\gamma-\tau)q}{q-1}}((k_2-\varepsilon)k_*^{q-1}-(k_2+\varepsilon))$$
$$\geq -M\,|x_*|^\gamma\,(u_2(x)-k_*u_1(x))+m_2k_*\,|x_*|^\sigma$$

where

$$M=2\alpha_2\max((\frac{1}{2})^\gamma,(\frac{3}{2})^\gamma),\sigma=\tau+\frac{(\gamma-\tau)q}{q-1}$$
$$m_2=\frac{1}{2}A_1^q\beta_1((k_2-\varepsilon)k_*^{q-1}-(k_2+\varepsilon))\min((\frac{1}{2})^\sigma,(\frac{3}{2})^\sigma)$$

Case 3: $\gamma=\tau$.

It follows from (11) that there exists a large $R_4>R_1$ such that $f(u)=Cu^q$. Then we deduce, for $x\in\Omega_0$

$$\Delta(u_2(x)-k_*u_1(x))$$
$$=-a(x)(u_2(x)-k_*u_1(x))+b(x)(f(u_2)-k_*f(u_1))$$
$$\geq -a(x)(u_2(x)-k_*u_1(x))+b(x)(Cu_2{}_2^q-Cu_1^q)$$
$$\geq -a(x)(u_2(x)-k_*u_1(x))+b(x)C(k_*^qu_2{}_1^q-k_*u_1^q)$$
$$\geq -2\alpha_2|x|^\gamma(u_2(x)-k_*u_1(x))+b(x)C(k_*^qu_1^q-k_*u_1^q)$$
$$\geq -M|x_*|^\gamma(u_2(x)-k_*u_1(x))+\tfrac{1}{2}|x|^\tau A_1^q\beta_1|x|^{\frac{(\gamma-\tau)q}{(q-1)}}C(k_*^q-k_*)$$
$$\geq -M|x_*|^\gamma(u_2(x)-k_*u_1(x))+m_2k_*|x_*|^\sigma$$

where

$$M=2\alpha_2\max((\frac{1}{2})^\gamma,(\frac{3}{2})^\gamma),\sigma=\tau+\frac{(\gamma-\tau)q}{(q-1)},$$
$$m_3=\frac{1}{2}\beta_1A_1^qC(k_*^{q-1}-1)\min((\frac{1}{2})^\sigma,(\frac{3}{2})^\sigma)$$

Overall, for

$$R>\max\{R_1,R_2,R_3,R_4\},m=\min\{m_1,m_2,m_3\}>0$$

we have

$$\Delta(u_2(x)-k_*u_1(x))\geq -M\,|x_*|^\gamma\,(u_2(x)-k_*u_1(x))+mk_*\,|x_*|^\sigma$$

With these preparations, we now define

$$\omega(x)=(2N)^{-1}mk_*\,|x_0|^\delta\,(r^2-|x-x_0|^2).$$

Clearly $\omega(x)>0$ in $B_r(x_0)$ and

$$\Delta\omega=-mk_*\,|x_0|^\delta.$$

It follows that, for $x\in\Omega_0$

$$\Delta(u_2-k_*u_1+\omega)\geq -M|x_0|^\gamma(u_2-k_*u_1)\geq -M|x_0|^\gamma(u_2-k_*u_1+\omega) \tag{28}$$

If we denote by $\lambda_1(\Omega)$ the first eigenvalue of $-\Delta$ over Ω under homogeneous Dirichlet boundary conditions, we have

$$\lambda_1(\Omega_0)\leq(B_r(x_0))=r^{-2}\lambda_1(B_1(x_0)).$$

Therefore, by the definition of r_0, we obtain

$$\lambda_1(\Omega_0)\geq r_0^{-2}\,|X_0|^\gamma\,\lambda_1,$$

where $\lambda_1=\lambda_1(B_1(x_0))$ is independent of x_0. we now choose $r_0\in(0,1)$ small enough so that

$$r_0^{-2}\lambda_1>M$$

And hence

$$\lambda_1(\Omega_0)\geq M\,|x_0|^\gamma.$$

Then by the maximum principle (see [5]), due to (28),

$$u_2(x_0)-k_*u_1(x_0)+\omega(x_0)\leq\max_{\partial\Omega_0}(u_2-k_*u_1+\omega).$$

We observe that the maximum of $(u_2-k_*u_1+\omega)$. over $\partial\Omega_0$ has to be achieved by some $y_0\in\partial B_r(x_0)$ since any $y_0\in\partial\Omega_0\setminus\partial B_r(x_0)$ satisfies, by the definition of Ω_0, $u_2(y)=k_*u_1(y)$ and hence

$$u_2(y)-k_*u_1(y)+\omega(y)=\omega(|y)\leq\omega(x_0)<u_2(x_0)-k_*u_1(x_0)+\omega(x_0).$$

Thus we can find $y_0\in\partial\Omega_0$ satisfying $|y_0-x_0|=r$ (hence $\omega(y_0)=0$) such that

$$u_2(y_0)-k_*u_1(y_0)=u_2(y_0)-k_*u_1(y_0)+\omega(y_0).$$
$$\geq u_2(x_0)-k_*u_1(x_0)+\omega(x_0)$$
$$>\omega(x_0)=(2N)^{-1}mk_*|x_0|^\delta r^2$$
$$=(2N)^{-1}mk_*r_0^2|x_0|^{(\gamma-\tau)/(q-1)}$$
$$\geq dk_*|y_0|^{(\gamma-\tau)/(q-1)},$$

where

$$d=(2N)^{-1}mr_0^2\min\{(1/2)^{-(\gamma-\tau)/(q-1)},(2/3)^{-(\gamma-\tau)/(q-1)}\}>0,$$

and we have used (27). Making use of (26), we finally deduce

$$u_2(y_0)-k_*u_1(y_0)>dk_*\,|y_0|^{(\gamma-\tau)/(q-1)}\geq c_1\mu_2^{-1}k_*u_1(y_0).$$

Therefore we can take $c_0=d\mu_2^{-1}$ and the proof is complete.

Proof of Theorem 3: By Lemma 7 above and Theorem 2, under conditions (4) and (8), problem (1) possesses a minimal positive solution u_1 and a maximal positive solution u_2 and any positive solution of (1) satisfies (9) and (10).

Let $k_1=\overline{\lim}_{|x|\to\infty}\dfrac{u_2}{u_1}$.

By (4) and (5) we know that $k_1\geq 1$ is finite. If $k_1=1$, then for any $\varepsilon>0$ there exists $R_\varepsilon>0$ such that for all x satisfying $|x|>R_\varepsilon$

$$u_2(x)\leq(1+\varepsilon)u_1(x)$$

Since $(1+\varepsilon)u_1$ is a supersolution of (7), we apply

Lemma 5 over $\Omega = B_R(0), R > R_\varepsilon$, and deduce

$$u_2(x) \le (1+\varepsilon)u_1(x), \quad x \in R^N.$$

Letting $\varepsilon \to 0$ we obtain $u_1 \equiv u_2$. This complete the proof.

Next we will prove the result is true when $k_1 > 1$. Therefore there exists a constant $k \in (1, k_1)$ and a sequence $\{x_n\}$ such that

$$|x_n| \to \infty, \quad u_2(x_n)/u_1(x_n) > k, \quad n = 1, 2, \ldots$$

We are now ready to apply Lemma 8. Let R, r_0 and $c_0 = c_0(R,k)$ be determined by Lemma 8. We recall that R satisfies $R^{-1-(\gamma/2)} < 1/2$. We first find an integer $j > 1$ such that

$$(1+c_0)^j k > \sup_{|x|>R} \frac{u_2}{u_1}.$$

Since $|x_n| \to \infty$, we can then find n_0 large enough such that

$$|x_{n_0}|(1/2)^j > R.$$

Taking $x_0 = x_{n_0}$ and $k_* = k$ in Lemma 8, we can find $y_0 = y_1$ such that

$$|y_1 - x_0| = r_0 |x_0|^{-\gamma/2}, \quad u_2(y_1) > (1+c_0)ku_1(y_1)$$

Clearly

$$|y_1| \ge |x_0| - r_0 |x_0|^{-\gamma/2} \ge |x_{n_0}|(1 - R^{-1-(-\gamma/2)})$$
$$> |x_{n_0}|(1/2) > R$$

We now take $x_0 = y_1$ and $k_* = (1+c_0)k$ in Lemma 8, and we can find y_2 such that

$$|y_2 - y_1| = r_0 |y_1|^{-\gamma/2}, \quad u_2(y_2) > (1+c_0)^2 ku_1(y_2)$$

Let us note that

$$|y_2| \ge |y_1|(1/2) \ge |x_{n_0}|(1/2)^2 > R.$$

We can repeat the above process until we obtain y_j which satisfies

$$u_2(y_j) > (1+c_0)^j ku_1(y_j), \quad |y_j| \ge |x_{n_0}|(1/2)^j > R.$$

Therefore

$$\frac{u_2(y_j)}{u_1(y_j)} \ge (1+c_0)^j k > \sup_{|x|>R} \frac{u_2}{u_1}$$

This contradiction completes our proof.

Remark If $a(x) = a(|x|)$ and $b(x) = b(|x|)$, then the unique positive solution of (7) must be radially symmetric solution, we can use the methods in [7,8,20] to obtain the analytic solution.

4 Proof of the Main Theorem

In this section, we will span the positive solution of problem (1) to whole space, and use the results in section 3 to prove the uniqueness Theorem 4.

To start, we should prove the following lemma.

Lemma 9. Assume u_1 to be an arbitrary positive solution of problem (1), letting

$$u = \begin{cases} u, & x \in T \\ u_2, & x \in R^N \setminus T \end{cases}$$

Where $u_2 = u_1(x_1, x_2, \cdots, -x_N), \quad x \in R^N \setminus T$, then u is the positive solution of the following problem

$$-\Delta u = a(x)u - b(x)u^q, \quad x \in R^N.$$

Proof For any R > 0, we denote

$$\Gamma = B_R \cap \partial T, \quad \Omega_1 = B_R \cap T, \quad \Omega_2 = B_R \setminus \Omega_1$$

By a simple computation, we can obtain that u_2 is a positive solution of

$$-\Delta u = a(x)u - b(x)u^q, x \in \Omega_2, \frac{\partial u}{\partial n} = 0.$$

Next we will show that

$$u = u_1|_{\Omega_1}; \quad x \in \Omega_1 \quad u = u_2|_{\Omega_2}; \quad x \in \Omega_2$$

is a positive solution of the following equation

$$-\Delta u = a(x)u - b(x)u^q, \quad x \in B_R. \quad (29)$$

For $\forall \varphi \in C_c^\infty(B_R)$, since

$$\begin{aligned}
&(-\Delta u, \varphi)_{L^2(B_R)} \\
&= \int_{B_R} -\Delta u \cdot \varphi dx \\
&= \int_{B_R} DuD\varphi dx - \int_{\partial B_R} \frac{\partial u}{\partial \nu} \varphi dx \\
&= \int_{\Omega_1} DuD\varphi + \int_{\Omega_2} DuD\varphi dx \\
&= \int_{\Omega_1} -\Delta u_1 \varphi dx + \int_{\partial \Omega_1} \frac{\partial u_1}{\partial \nu} \varphi dx + \int_{\Omega_2} -\Delta u_2 \varphi dx \\
&\quad + \int_{\partial \Omega_2} \frac{\partial u_2}{\partial \nu} \varphi dx \\
&= \int_{\Omega_1} \left[a(x)u_1 - b(x)u_1^q \right] \varphi dx \\
&\quad + \int_{\Omega_2} \left[a(x)u_2 - b(x)u_2^q \right] \varphi dx \\
&= \int_{B_R} \left[a(x)u - b(x)u^q \right] \varphi dx \\
&= (a(x)u - b(x)u^q, \varphi)_{L^2(B_R)},.
\end{aligned}$$

Hence u is a positive solution of problem (29). It follows from the arbitrary of R that

$$u = u_1, \quad x \in T, \quad u = u_2, \quad x \in R^N \setminus T$$

is a positive solution of

$$-\Delta u = a(x)u - b(x)u^q \; x \in R^N.$$

The proof is complete.

Now we are ready to complete the proof of Theorem 4.

Proof of Theorem 4 Let $u_1(x)$ and $v_1(x)$ be two arbitrary positive solutions of (1). By Lemma 9, letting

$$u=\begin{cases}u_1, x\in T\\ u_2, x\in R^N\setminus T\end{cases}$$

and

$$v=\begin{cases}v_1, x\in T\\ v_2, x\in R^N\setminus T\end{cases}$$

We know that $u(x)$ and $v(x)$ corresponding to u_1 and v_1, respectively, is the positive solution of

$$-\Delta u = a(x)u - b(x)u^q \ x\in R^N.$$

By Theorem 3 above, the problem in whole space has only one positive solution. It follows that

$$u(x)=v(x),\ x\in R^N$$

Thus $\quad u_1 = v_1, x\in T$

This completes our proof.

5 Conclusion

In this paper, under less restricted conditions on coefficients $a(x)$ and $b(x)$, we obtain existence and uniqueness theorem for a class of semilinear logistic equations with Neumann boundary value in unbounded domain in R^N. It improves the previous result. We can use the same method to handle with more complicated cases. For example, assume that the coefficient b(x)≥0 and b(x)≠0 on R^N, named degenerate logistic type semilinear equations, if the volume of the set D={$x: x\in R^N$, b(x)=0} is small enough we can show the unique result in our paper remains the same.

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