Oscillation and Non-oscillation Criteria for Quasi-linear Second Order Differential Equations

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Abstract: Some oscillation and non-oscillation criteria for quasi-linear second order equations are obtained. These results are extensions of earlier results of C.Huang(J.Math Anal. Appl.210(1997), 712-723), A. Elbert(J.Math.Anal.Apll.226(1998), 207-219) and J.Wong(J.Math.Anal.Apll.291(2004), 180-188) which are all about oscillation and non-oscillation criteria of the solution of the second order linear equation. After the proof of the main theorem, two examples are given as the additional remarks of the criteria. At last a special case is discussed. And the uniqueness and periodic criteria of the special case are obtained further by using comparison theorem and Leray-Schauder degree approach.

Key-words: Quasi-linear equation, oscillation, non-oscillations, periodic solution, Leray-Schauder degree

1. Introduction

In this paper, we consider the following second order quasi-liner second order differential equation

$$(r(t)\phi_p(x'))' + q(t)\phi_p(x) = 0, \quad t \ge t_* > 0 \qquad (1)$$

where $r \in C^1(I,(0,\infty))$ and $r'(t) \ge 0$, $t \in I$, $q \in C(I,(0,\infty)$ with $I = [t_*,\infty)$ and $\phi_p(u) = |u|^{p-1}u$ is the $p-$Laplacian for $p > 0$. If a solution $x(t)$ of (1) has arbitrarily large zero, it is called oscillatory, otherwise it is called non-oscillatory. If all solutions of (1) are oscillatory, then (1) is called oscillatory. If all nonzero solutions of (1) are non-oscillatory, then (1) is called non-oscillatory. It is it can be deduced from [3,5,6] that all solutions of (1) exist on I and if one nonzero solution of (1) is oscillatory then (1) is oscillatory and if one solution of (1) is non-oscillatory, then (1) is non-oscillatory. If $p = 1$, $r(t) \equiv 1$, then (1) reduces to the following second order linear equation:

$$x'' + q(t)x = 0, \quad t \ge t_* > 0, \qquad (2)$$

In 1997, Huang[4] proved the following result:

Theorem 1. If there exists $t_0 \ge t_*$ such that for every $n \in N$

$$\int_{2^n t_0}^{2^{n+1} t_0} q(t)dt \le \frac{\alpha_0}{2^{n+1} t_0},$$

where $\alpha_0 = 3 - 2\sqrt{2}$, then (2) is non-oscillatory.

If there exists $t_0 \ge t_*$ and $\alpha > \alpha_0 = 3 - 2\sqrt{2}$ such that for every $n \in N$,

$$\int_{2^n t_0}^{2^{n+1} t_0} q(t)dt \ge \frac{\alpha}{2^n t_0},$$

then (2) is oscillatory.

In 1998, Elbert [1] generalized Huang'results and obtained the following theorem:

Theorem 2. Assume $t_* < t_0 < t_1 < t_2 < \cdots < t_n < \cdots, t_n \to \infty$. Let

$$\beta_n = \frac{t_{n+1} - t_n}{t_1 - t_0}, \qquad n = 0,1,\cdots,$$

then $\beta_0 = 1$, $\beta_n > 0$, $\sum_{n=0}^{\infty} \beta_n = \infty$.

If $q(t)$ satisfies the following inequality

$(t_{n+1}-t_n)\int_{t_n}^{t_{n+1}} q(s)ds \le \alpha_n, \quad 0 \le \alpha_n < 1,$ and $n = 0,1,\cdots,$

for any sequence $\{z_n\}_{n=0}^{\infty}$ satisfying the following relation

$$\begin{cases} z_{n+1} = \dfrac{z_n - \alpha_n}{\theta_n + z_n - \alpha_n}, & n = 0,1,2\cdots, \\ z_0 = 1, \end{cases}$$

we have

$0 < z_n < 1, \quad n = 0,1,2\cdots, \qquad \theta_n = \beta_n / \beta_{n+1}.$

Then (2) is non-oscillatory.

If $q(t)$ satisfies the following inequality

$$(t_{n+1}-t_n)\int_{t_n}^{t_{n+1}} q(s)ds \ge \alpha_n, \quad \alpha_n > 0, \quad n = 0,1,\cdots,$$

and the recurrence relation

$$\begin{cases} u_{n+1} = \alpha_{n+1}\theta_n\left(1 + \frac{u_n}{\alpha_n(1-u_n)}\right), & n = 0,1,\cdots, \\ u_1 = 0, \end{cases}$$

has no solution such that $0 < u_n < 1$ for all $n \in N$, where $\theta = \beta_n / \beta_{n+1}$. Then (2) is oscillatory.

In 2004, Wong [7] generalize Huang's results in another direction and obtained the following results:

Theorem 3. Let $\lambda > 1$. If for some $t_0 \ge t_* > 0$ and every $n \in N$,

$$\int_{\lambda^n t_0}^{\lambda^{n+1} t_0} q(t)dt \le \frac{\alpha}{(\lambda - 1)\lambda^{n+1} t_0},$$

where $\alpha \le k_0(\lambda) = (\sqrt{\lambda} - 1)^2$. Then (2) is non-oscillatory.

Suppose $\lambda > 1$. If for some $t_0 \ge t_*$ and every $n \in N$,

$$\int_{\lambda^n t_0}^{\lambda^{n+1} t_0} q(t)dt \ge \frac{\alpha}{(\lambda - 1)\lambda^{n} t_0},$$

where $\alpha > k_0(\lambda) = (\sqrt{\lambda} - 1)^2$. Then (2) is oscillatory.

In this paper, by using a similar method in [1], we generalize Elbert's results to equation (1) and obtained the following theorem:

Theorem 4. Assume $t_* < t_0 < t_1 < t_2 < \cdots < t_n < \cdots, t_n \to \infty$. Let β_n be given by Theorem 12 and

$$\theta = (\beta_n / \beta_{n+1})^p, \quad n = 0,1\cdots$$

Assume $0 < p \le 1$ and $q(t)$ satisfies the following inequalities:

$$(t_{n+1}-t_n)^p \int_{t_n}^{t_{n+1}} q(t)dt \le \alpha_n r(t_n), \qquad 0 \le \alpha_n < 1, \quad n = 0,1,\cdots, \tag{3}$$

and there exists a sequence of numbers $\{z_n\}_{n=0}^{\infty}$ satisfying the following relations:

$z_0 = 1,$

$$z_{n+1} = \frac{r(t_n)z_n - \alpha_n}{r(t_{n+1})\theta_n + r(t_n)(z_n - \alpha_n)}, \quad n = 0,1,2,\cdots,$$

with $0 < z_n < 1, \quad n = 1,2,\cdots$. Then (1) is non-oscillatory.

Assume $p \ge 1$ and $q(t)$ satisfies the following inequalities:

$$(t_{n+1}-t_n)^p \int_{t_n}^{t_{n+1}} q(t)dt \ge \alpha_n r(t_n), \qquad \alpha_n > 0, \quad n = 0,1,2,\cdots, \tag{4}$$

and the recurrence relation

$u_0 = 0,$

$$u_{n+1} = \alpha_{n+1}\theta_n\left(1 + \frac{r(t_{n+1})u_n}{r(t_n)(1-u_n)\alpha_n}\right) \quad n = 0,1,\cdots, \tag{5}$$

has no solution such that $0 < u_n < 1$ for all $n \in N$, Then (1) is oscillatory.

Corollary 5. let $r(t) \equiv 1$ and $0 < p \le 1$, $\alpha_n = \alpha \in (0,1), \quad \theta_n = \theta \in (0,1)$ such that

$$\sqrt{\theta} + \sqrt{\alpha} \le 1.$$

Then (1) is non-oscillatory.

Let $r(t) \equiv 1$ and $p \ge 1$, $\alpha_n = \alpha > 0$, $\theta_n = \theta \in (0,1)$ such that

$$\sqrt{\theta} + \sqrt{\alpha\theta} \ge 1.$$

Then (1) is oscillatory.

Remark 6. Let $r(t) \equiv 1$, $p = 1$, then Theorem 4 reduces to Theorem 2. Let $\lambda > 1, \quad t_n = \lambda t_0$, then $\theta_n = 1/\lambda \in (0,1)$, then it is not difficult to verify that Corollary 5 implies Theorem 3. Therefore, Theorem 4 generalizes the results of Huang, Elbert and Wong.

2. Proofs

Lemma 7. Assume $r'(t) \ge 0, \quad q(t) \ge 0, \quad t \in [t_0, \infty)$. If $x(t) \ge 0$, $x'(t) > 0, \quad t \in [t_a, t_b] \subset [t_0, \infty)$, then

$\phi_p(x'(t))$ is non-increasing on $[t_a, t_b]$. Especially, if $x'(t) \ge 0,\ t \in [t_a, t_b]$, then $x'(t)$ is non-increasing on $[t_a, t_b]$.

Proof. For any $t_1 < t_2$ such that $[t_1, t_2] \subset [t_a, t_b]$, assume $x(t) > 0,\ r'(t) \ge 0,\ q(t) \ge 0$ on $[t_1, t_2]$, integrating (1) on $[t_1, t_2]$, we get

$$r(t_1)\phi_p(x'(t_1)) - r(t_2)\phi_p(x'(t_2))$$
$$= \int_{t_1}^{t_2} q(t)(x(t))^p\, dt \ge 0$$

which yields

$$\phi_p(x'(t_2)) \le \frac{r(t_1)}{r(t_2)}\phi_p(x'(t_1)) \le \phi_p(x'(t_1)).$$

Since $t_2 > t_1$ is arbitrary on $[t_a, t_b]$, we see that $\phi_p(x'(t))$ is non-increasing on $[t_a, t_b]$.

Lemma 8. Let $x(t)$ be a nonzero solution of (1). If $x(t)$ satisfies $x(a) = 0,\ x'(\tau) = 0$, where $t_0 \le a < \tau$, then

$$r(a) < (\tau - a)^p \int_a^\tau q(s)ds;$$

if $x(t)$ satisfies $x'(a) = 0,\ x(\tau) = 0$, then

$$r(\tau) < (\tau - a)^p \int_a^\tau q(t)dt.$$

Proof. We prove the first inequality only, the second inequality can be treated similarly. Since if x is a solution of (1), by uniqueness result [2], we have $x'(a) \ne 0$. Since if $x(t)$ is a solution of (1), $-x(t)$ is also a solution of (1). We can assume without loss of generality that $x'(a) > 0$. Let $\tau_0 = \inf\{t > a, x'(t) = 0\} \le \tau$, then $x'(t) > 0$, $t \in [a, \tau_0)$, by Lemma 7, $x'(t)$ is non-increasing on $[a, \tau_0]$. Integrating (1) from a to τ_0 to obtain

$$0 < r(a)(x'(a))^p = \int_a^{\tau_0} q(t)(x(t))^p dt < \int_a^{\tau_0} q(t)(x(\tau_0))^p dt$$
$$\le (x'(a))^p(\tau_0 - a)^p \int_a^\tau q(t)dt,$$

which implies that

$$r(a) < (\tau_0 - a)^p \int_a^{\tau_0} q(t)dt \le (\tau - a)^p \int_a^\tau q(t)dt$$

Lemma 9. if $A \ge 0,\ B \ge 0,\ 0 < p \le 1$, then $A^p + B^p \ge (A+B)^p$; if $p \ge 1$, then $A^p + B^p \le (A+B)^p$.

Proof. Simple calculation yields above results.

Proof of the first part of Theorem 4.

Since it is proved in[4,5] that (1) can not has non-oscillatory and nonzero oscillatory solutions at the same time, we need only to prove that (1) has a non-oscillatory solution. Therefore we need only to prove that the solution $x(t)$ of (1) satisfying initial condition $x(t_0) = 0, x'(t_0) > 0$ satisfies $x(t) > 0,\ \forall t > t_0$. By the existence and uniqueness result proved in [2], we see that the solution $x(t)$ is unique and exists on $I = [t_0, \infty)$.

In fact, it follows from (3) that $(t_1 - t_0)^p \int_{t_0}^{t_1} q(s)ds < \alpha_0 r(t_0) < r(t_0)$ and Lemma 8 implies that $x'(t) > 0,\ t \in [t_0, t_1]$. We get therefore $x(t) > 0,\ t \in [t_0, t_1]$ and by Lemma 7, $x'(t) > 0$ is non-increasing on $[t_0, t_1]$. Moreover, by integrating (1) from t_0 to t_1 to get

$$r(t_0)(x'(t_0))^p - r(t_1)(x'(t_1))^p = \int_{t_0}^{t_1} q(t)x^p(t)dt$$
$$\le ((x'(t_0)(t_1 - t_0))^p \int_{t_0}^{t_1} q(t)dt$$
$$\le \alpha_0 r(t_0)(x'(t_0))^p,$$

which yields

$$r(t_1)(x'(t_1))^p \ge (x'(t_0))^p r(t_0) - \alpha_0 r(t_0)(x'(t_0))^p \\ \ge r(t_0)(1 - \alpha_0)(x'(t_0))^p > 0 \tag{6}$$

Claim 10. $x(t) > 0,\ \forall t \in [t_1, t_2]$. Otherwise, let $\tau = \inf\{t \in [t_1, t_2] | x(t) = 0\}$, then by Rolle's Theorem, there exists $t = a \in [t_1, \tau)$ such that $x'(a) = 0$. As $a \in [t_1, \tau) \subset [t_1, t_2]$, it follows from Lemma 8 that

$$r(t_1)\alpha_1 \ge (t_2 - t_1)^p \int_{t_1}^{t_2} q(t)dt$$
$$\ge (\tau - a)^p \int_a^\tau q(t)dt > r(\tau) \ge r(t_1),$$

which contradicts the assumption $\alpha_1 < 1$. Claim 10 is thus proved and we have

$$x(t) > 0,\ t \in [t_1, t_2]. \tag{7}$$

Next we show the following inequalities by using mathematical induction.

$$(x'(t_n))^p \ge \frac{z_n}{\beta_n^p}\left(\sum_{i=0}^n \beta_i x'(t_i)\right)^p \tag{8}$$

$$r(t_{n+1})(x'(t_{n+1}))^p \\ \ge r_n\left[(x'(t_n))^p - \frac{\alpha_n}{\beta_n^p}\left(\sum_{i=0}^n \beta_i x'(t_i)\right)^p\right] \tag{9}$$

$$x(t) > 0, \quad t \in [t_{n+1}, t_{n+2}], \tag{10}$$

where z_n is defined in Theorem 4.

The case $n = 0$ follows from (6) and (7). Assume (8)-(10) hold for $0,1,\cdots,n$. We show that (8)-(10) holds also for $n+1$. As $z_{n+1} > 0$, it follows from (8) and (9) that

$$r(t_{n+1})(x'(t_{n+1}))^p \geq \frac{r(t_n)(z_n - \alpha_n)}{\beta_n^p}\left(\sum_{i=0}^{n}\beta_i x'(t_i)\right)^p > 0$$

From Lemma 9, for $0 < p \leq 1$, let $A = \sum_{i=0}^{\infty}\beta_i x'(t_i)$ $B = \beta_{n+1}x'(t_{n+1})$, we get

$$\begin{aligned}&(\beta_{n+1}^p + \tfrac{r(t_{n+1})\beta_n^p}{r(t_n)(z_n - \alpha_n)})(x'(t_{n+1}))^{p+1}\\ &\geq (\textstyle\sum_{i=0}^{\infty}\beta_i x'(t_i))^p + \beta_{n+1}^p(x'(t_{n+1}))^p\\ &\geq (\textstyle\sum_{i=1}^{n+1}\beta_i x'(t_i))^p\end{aligned}$$

or equivalently,

$$(x'(t_{n+1}))^p \geq \frac{z_{n+1}}{\beta_{n+1}^p}\left(\sum_{i=0}^{n+1}\beta_i x'(t_i)\right)^p > 0 \tag{11}$$

By (1), (10) and (11), Lemma 7 implies that $\phi_p(x'(t))$ is non-increasing on $t \in [t_{n+1}, t_{n+2}]$, hence $x'(t) \leq |x'(t)| \leq x'(t_{n+1})$ for $t \in [t_{n+1}, t_{n+2}]$, which yields for $t \in [t_{n+1}, t_{n+2}]$,

$$\begin{aligned}x(t) &= x(t_{n+1}) + x'(\tau_t)(t - t_{n+1})\\ &\leq x(t_{n+1}) + x'(t_{n+1})(t - t_{n+1})\\ &\leq x(t_{n+1}) + x'(t_{n+1})(t_{n+2} - t_{n+1}),\end{aligned}$$

where $\tau_t \in (t_{n+1}, t)$.

Integrating (1) over $[t_{n+1}, t_{n+2}]$, we obtain

$$\begin{aligned}&r(t_{n+1})(x'(t_{n+1}))^p - r(t_{n+2})|x'(t_{n+2})|^{p-1}x'(t_{n+2})\\ &= \int_{t_{n+1}}^{t_{n+2}} q(t)(x(t))^p dt\\ &\leq |x(t_{n+1}) + x'(t_{n+1})(t_{n+2} - t_{n+1})|^p \int_{t_{n+1}}^{t_{n+2}} q(t)dt.\end{aligned}$$

By the Lagrange mean theorem and the definition of β_n, θ_n, we have

$$\begin{aligned}x(t_{n+1}) &= \sum_{i=0}^{n}[x(t_{i+1}) - x(t_i)] = \sum_{i=0}^{n}(t_{i+1} - t_i)x'(t_i^*)\\ &\leq \sum_{i=0}^{n}(t_{i+1} - t_i)x'(t_i) = (t_1 - t_0)\sum_{i=0}^{n}\beta_i x'(t_i),\end{aligned}$$

where $t_i < t_i^* < t_{i+1}$, $i = 0,1,\cdots n$.

Hence

$$\begin{aligned}&r(t_{n+1})(x'(t_{n+1}))^p - r(t_{n+2})|x'(t_{n+2})|^{p-1}x'(t_{n+2})\\ &\leq (t_1 - t_0)^p\left(\textstyle\sum_{i=0}^{n+1}\beta_i x'(t_i)\right)^p\int_{t_{n+1}}^{t_{n+2}} q(t)dt\\ &= \left(\frac{t_{n+2} - t_{n+1}}{\beta_{n+1}}\right)^p\int_{t_{n+1}}^{t_{n+2}} q(s)ds\left(\textstyle\sum_{i=0}^{n+1}\beta_i x'(t_i)\right)^p\\ &\leq \frac{\alpha_{n+1}r(t_{n+1})}{\beta_{n+1}^p}\left(\textstyle\sum_{i=0}^{n+1}\beta_i x'(t_i)\right)^p\end{aligned}$$

That is,

$$\begin{aligned}&r(t_{n+2})|x'(t_{n+2})|^{p-1}x'(t_{n+2})\\ &\geq r(t_{n+1})(x'(t_{n+1}))^p\\ &\qquad - \tfrac{\alpha_{n+1}r(t_{n+1})}{\beta_{n+1}^p}\left(\textstyle\sum_{i=0}^{n+1}\beta_i x'(t_i)\right)^p\\ &= r(t_{n+1})\left[\begin{matrix}(x'(t_{n+1}))^p\\ - \tfrac{\alpha_{n+1}r(t_{n+1})}{\beta_{n+1}^p}\left(\sum_{i=0}^{n+1}\beta_i x'(t_i)\right)^p\end{matrix}\right]\end{aligned} \tag{12}$$

Since $z_{n+2} > 0$ implies $z_{n+1} > \alpha_{n+1}$ and by (11), we get

$$\begin{aligned}&|x'(t_{n+2})|^{p-1}x'(t_{n+2})\\ &\qquad\geq \frac{z_{n+1} - \alpha_{n+1}}{\beta_{n+1}^p}\left(\sum_{i=0}^{n+1}\beta_i x'(t_i)\right)^p,\end{aligned} \tag{13}$$

which implies $x'(t_{n+2}) > 0$. Lemma 7 and the inequality

$$(t_{n+3} - t_{n+2})\int_{t_{n+2}}^{t_{n+3}} q(s)ds \leq \alpha_{n+2}r(t_{n+2}) < r(t_{n+2})$$

implies that $x(t) > 0, \ t \in [t_{n+2}, t_{n+3}]$, which, together with (11)-(13), completes the induction step. We have showed that $x(t) > 0$ and $x'(t) > 0$ holds for all $t > t_0$, hence $x(t)$ is a positive and hence a non-oscillatory solution of (1). This completes the proof of the first part of Theorem 4.

Proof of the second part of Theorem 4.

If the results of the second part of Theorem 4 is false, then we can assume without loss of generality that there exists a solution $x(t)$ of (1) such that for all $t > t_0$, $x(t) > 0$ and it is not difficult to verify that $x'(t) \geq 0, \forall t > t_0$. By Lemma 7, $x'(t)$ is non-increasing for all $t > t_0$.

By using the Lagrange mean value theorem again and by the definition of β_n and θ_n, we obtain, by noticing $x(t_0) > 0$, that

$$\begin{aligned} x(t_n) &= x(t_0) + \sum_{i=1}^{n}[x(t_i) - x(t_{i-1})] \\ &> \sum_{i=1}^{n}(t_i - t_{i-1})x'(t_i^*) \\ &\geq (t_1 - t_0)\sum_{i=1}^{n}\beta_{i-1}x'(t_i), \end{aligned}$$

where $t_{i-1} < t_i^* < t_{i+1}$, $i = 1,2,\cdots,n$ and by (4), we get

$$\begin{aligned} & r(t_n)(x'(t_n))^p - r(t_{n+1})(x'(t_{n+1}))^p \\ &= \int_{t_n}^{t_{n+1}} q(t)x^p(t)dt > \int_{t_n}^{t_{n+1}} q(t)x^p(t_n)dt \\ &> \left(\sum_{i=1}^{n}\beta_{i-1}x'(t_i)\right)^p (t_1 - t_0)^p \\ &\quad \bullet \int_{t_n}^{t_{n+1}} q(t)dt \\ &= \frac{(t_{n+1}-t_n)^p \int_{t_n}^{t_{n+1}} q(t)dt}{\beta_n^p}\left(\sum_{i=1}^{n}\beta_{i-1}x'(t_i)\right)^p \\ &\geq \frac{\alpha_n r(t_n)}{\beta_n^p}\left(\sum_{i=1}^{n}\beta_{i-1}x'(t_i)\right)^p \end{aligned} \tag{14}$$

which implies the following two inequalities:

$$(x'(t_n))^p > \frac{\alpha_n}{\beta_n^p}\left(\sum_{i=1}^{n}\beta_{i-1}x'(t_i)\right)^p \tag{15}$$

and

$$\begin{aligned} & r(t_{n+1})(x'(t_{n+1}))^p \\ &< r(t_n)[(x'(t_n))^p - \frac{\alpha_n}{\beta_n^p}(\sum_{i=1}^{n}\beta_{i-1}x'(t_i))^p]. \end{aligned} \tag{16}$$

For $n \in N$, we have therefore

$$\begin{aligned} \frac{\alpha_n}{\beta_n^p}\left(\sum_{i=1}^{n}\beta_{i-1}\right)^p (x'(t_n))^p &= \frac{\alpha_n}{\beta_n^p}\left(\sum_{i=1}^{n}\beta_{i-1}x'(t_n)\right)^p \\ &\leq \frac{\alpha_n}{\beta_n^p}\left(\sum_{i=1}^{n}\beta_{i-1}x'(t_i)\right)^p < (x'(t_n))^p, \end{aligned} \tag{17}$$

which yields

$$\frac{\alpha_n}{\beta_n^p}\left(\sum_{i=1}^{n}\beta_{i-1}\right)^p < 1. \tag{18}$$

Let $u_0, u_1, \cdots$ be given by Theorem 4, we are going to prove the following claim.

Claim 11. For $p \geq 1$ and $n = 1,2,\cdots,$ we have

$$\begin{aligned} & u_n(x'(t_n))^p \\ &\leq \frac{\alpha_n}{\beta_n^p}\left(\sum_{i=1}^{n}\beta_{i-1}x'(t_i)\right)^p < (x'(t_n))^p, \end{aligned} \tag{19}$$

and

$$0 < u_n < 1. \tag{20}$$

We prove Claim 11 by induction method. By using (15) and (18), it is easy to see that (19)-(20) hold for $n = 1$. Assume (19)-(20) hold for n, we show next that they hold also for $n+1$.

We get from (19)

$$\begin{aligned} & (x'(t_n))^p - \frac{\alpha_n}{\beta_n^p}\left(\sum_{i=1}^{n}\beta_{i-1}x'(t_i)\right)^p \\ &\leq (1-u_n)(x'(t_n))^p \end{aligned} \tag{21}$$

It follows from (16), (19) and (21) that

$$\begin{aligned} r(t_{n+1})(x'(t_{n+1}))^p &< r(t_n)(1-u_n)(x'(t_n))^p \\ &= \frac{r(t_n)(1-u_n)u_n}{u_n}(x'(t_n))^p \\ &\leq \frac{r(t_n)(1-u_n)\alpha_n}{u_n\beta_n^p}\left(\sum_{i=1}^{n}\beta_{i-1}x'(t_i)\right)^p \end{aligned}$$

that is,

$$\frac{u_n\beta_n^p r(t_{n+1})}{r(t_n)(1-u_n)\alpha_n}(x'(t_{n+1}))^p < \left(\sum_{i=1}^{n}\beta_{i-1}x'(t_i)\right)^p \tag{22}$$

Adding $\beta_n^p(x'(t_{n+1}))^p$ to both sides of (22) and by using Lemma 8 for $p \geq 1$, we get

$$\begin{aligned} & \beta_n^p\left(1 + \frac{u_n r(t_{n+1})}{r(t_n)(1-u_n)\alpha_n}\right)(x'(t_{n+1}))^p \\ &< \left(\sum_{i=1}^{n+1}\beta_{i-1}x'(t_i)\right)^p \end{aligned} \tag{23}$$

Multiplying both sides of (23) by $\alpha_{n+1}/\beta_{n+1}^p$ and by using the definition of u_{n+1} in Theorem 4, we obtain by using (17) for $n+1$,

$$\begin{aligned} & u_{n+1}(x'(t_{n+1}))^p \\ &= \alpha_{n+1}\theta_n\left(1 + \frac{u_n r(t_{n+1})}{(1-u_n)r(t_n)\alpha_n}\right)(x'(t_{n+1}))^p \\ &< \frac{\alpha_{n+1}}{\beta_{n+1}^p}\left(\sum_{i=1}^{n+1}\beta_{i-1}x'(t_i)\right)^p < (x'(t_{n+1}))^p. \end{aligned} \tag{24}$$

Hence Claim 11 is proved.

Now it follows from (24) and definition of u_{n+1} that $0 < u_{n+1} < 1$. This completes the induction step shows that (19)-(20) hold for any $n \in N$. But this contradicts the assumption of Theorem 4. The second part of Theorem 4 is thus proved.

Proof of Corollary 5.

Assume $\alpha, \theta \in (0,1)$ and $\sqrt{\alpha} + \sqrt{\theta} \leq 1$, we can define a function $f : [\alpha, 1] \to [0,1)$ by

$$f(x) = \frac{x - \alpha}{\theta + x - \alpha}, \qquad \alpha \leq x \leq 1.$$

It is easy to see that

$$f'(x) = \frac{\theta}{(\theta + x - \alpha)^2} > 0, \qquad \alpha \le x \le 1.$$

Let $x_1 \le x_2$ be the two fixed points of f, then

$$2x_1 = 1 - \theta + \alpha - \sqrt{\Delta} \le 2x_2 = 1 - \theta + \alpha + \sqrt{\Delta}$$

where

$$\begin{aligned}\Delta &= (1 - \theta + \alpha)^2 - 4\alpha \\ &= ((1 + \sqrt{\theta})^2 - \alpha)(1 + \sqrt{\alpha} - \sqrt{\theta}) \\ &\quad (1 - \sqrt{\alpha} - \sqrt{\theta}) \ge 0\end{aligned}$$

Then by $\theta \in (0,1)$, it is easy to verify that $\alpha < x_1 \le x_2 < 1$ and $x_1 < f(1) < 1$. Since $f'(x) \in [0,1], \forall x \in (\alpha,1]$, we obtain

$$f[x_{1,} 1] \subset [x_{1,}, f(1)] \subset (x_{1,}, 1),$$

which implies that

$$\alpha < x_{1,} < z_n = f(z_{n-1}) \le f(1) < 1, \quad n \in N.$$

Hence (1) is non-oscillatory by the first part of Theorem 4.

Assume now $\alpha > 0,\ \theta \in (0,1)$ and $\sqrt{\theta} + \sqrt{\alpha\theta} > 1$. Define a function $g : [0,1) \to (\theta\alpha, \infty)$ by

$$g(x) = \theta\left(\alpha + \frac{x}{1-x}\right), \qquad 0 \le x < 1.$$

Then

$$g'(x) = \frac{\theta}{(1-x)^2} > 0, \qquad 0 \le x < 1.$$

We shall prove the value $\{u_n\}$ defined by

$$u_0 = 0, \ \ u_{n+1} = g(u_n), \tag{25}$$

can not stay in $[0,1)$ for all $n \in N.$ Since $u_1 = g(u_0) = g(0) = \theta\alpha.$ If $\theta\alpha \ge 1,$ we are done. We consider only the case $\theta\alpha < 1$. Since the function g is strictly increasing, we see that

$$0 = u_0 < u_1 < u_2 < \cdots < u_n < \cdots.$$

If $u_n \in [0,1)$ for all $n \ge 0$, then the limit $u^* = \lim_{n\to\infty} u_n$ exists and u^* is a fixed point of g in $[0,1)$, which means that u^* is a positive solution of the quadratic equation

$$x^2 - (1 - \theta + \alpha\theta)x + \alpha\theta = 0.$$

But this equation has no real root since its discriminant δ is negative:

$$\begin{aligned}\Delta &= (1 - \theta + \alpha\theta)^2 - 4\alpha\theta \\ &= (1 + \sqrt{\theta} + \sqrt{\alpha\theta})(1 + \sqrt{\theta} - \sqrt{\alpha\theta}) \\ &\quad \bullet (1 + \sqrt{\alpha\theta} - \sqrt{\theta})(1 - \sqrt{\theta} - \sqrt{\alpha\theta}) \\ &< 0.\end{aligned}$$

Because it is easy to see that the first three factors of Δ are positive and the last one is negative if $\sqrt{\theta} + \sqrt{\alpha\theta} > 1$ and $\theta \in (0,1)$.

Let $r(t) \equiv 1$ in (1), then the following example shows that there exists $p > 1,\ q(t) > 0$ satisfying the conditions of the first part of Theorem 4, but Theorem 4 does not hold.

Example 1. Let $r(t) \equiv 1,\ \ q(t) = cp/t^{p+1}$. Then it is well known that (1) is oscillatory if $c > \frac{p^p}{(p+1)^{p+1}}$ and (1) is non-oscillatory if $c \le \frac{p^p}{(p+1)^{p+1}}$.(see, for example, [2])

Let $t_n = 2^n, \quad n = 0,1,\cdots,$ then $t_{n+1} - t_n = 2^n$, $\beta_n = 2^n$, $\theta_n = \theta = 2^{-p}$ and

$$(t_{n+1} - t_n)^p \int_{t_n}^{t_{n+1}} q(s)ds = k\left(1 - \frac{1}{2^p}\right).$$

Hence

$$\alpha_n = \alpha = k\left(1 - \frac{1}{2^p}\right) < 1, \quad n = 1,2,\cdots.$$

In this case, we shall show that there exists $p^* > 1$, and $ak > 0$ such that if $p > p^*$, then (1) is non-oscillatory by the first part of Theorem 4. That is, we look for $p^* > 1$ such that if $p > p^*$, then for k such that

$$k > \frac{p^p}{(p+1)^{p+1}}$$

and

$$\sqrt{\theta} + \sqrt{\alpha} = 2^{-\frac{p}{2}} + \sqrt{k(1 - 2^{-p})} \le 1.$$

The above inequalities implies that k satisfies

$$\frac{p^p}{(p+1)^{p+1}} < k < \frac{2^{\frac{p}{2}}}{2^{\frac{p}{2}} + 1}. \tag{26}$$

Define a function $f : [1, \infty) \to R$ as

$$f(x) = \frac{(\sqrt{2})^x - 1}{(\sqrt{2})^x + 1} - \frac{x^x}{(x+1)^{x+1}}.$$

Then it is to see that $f(1) < 0,\ f(2) > 0$ and $f'(x) > 0,\ \forall x \ge 1$. This means that there exists a $x^* \in (1,2)$ such that $f(x^*) = 0$ and $f(x) > 0$ if

$x > x^*$. Let $p^* = x^*$, then for $p > p^*$ and k satisfies (26), we see that (1) is oscillatory. But by the first part of Theorem 4, (1) is non-oscillatory. Hence Theorem 4 may be incorrect if $p > 1$.

Example 2. Assume $r(t) \equiv 1,\ 0 < P < 1$, $q(t) = cp/t^{p+1}$, where c satisfies

$$\frac{((\sqrt{2})^p - 1)2^p}{(\sqrt{2})^p + 1} < c < \min\left\{\frac{2^p}{2^p - 1}, \frac{p^p}{(p+1)^{p+1}}\right\} \quad (27)$$

Since the left side of (27) approaches zero when $p \to 0^+$, and the right side of (27) approaches 1 when $p \to 0^+$, we can choose $0 < p << 1$ such that c satisfies (27). In this case, (1) is non-oscillatory by the choose of c. But if we choose $t_n = 2^n, \theta_n = \theta = 2^{-p}$, $\alpha_n = \alpha = c\left(\frac{2^p - 1}{2^p}\right) < 1$. Then it is easy to verify that $\sqrt{\theta} + \sqrt{\theta\alpha} > 1$. Hence the condition of Corollary 5 holds, but at the same time (4) and $0 < z_n < 1,\ n = 1,2,\cdots$, hold. Which shows that the second part of Theorem 4 may be incorrect for some $p \in (0,1)$.

In addition, by using the methods in paper [8-10], we can also obtain the numerical solutions of (1).

3. Special case

In this section, we will consider the equation

$$(p(t)x')' + f(t,x) = 0 \quad (28)$$

where $p \in C^1((0,2\pi),(0,+\infty))$ is 2π-periodic, $f \in C(R^2, R)$ is 2π-periodic in t.

Let $p = 1$ in equation (1), it is easy to see that the first part of (28) is same as (1) when $p = 1$, but only the nonlinear part $f(t,x)$ of (28) is more complex than $q(t)x$ of equation (1) when $p = 1$. So we can consider the equation (28) as the special case of the equation (1).

Landsman-Lazer type conditions play an important role in the study of the existence of and uniqueness of periodic solutions for second order nonlinear differential equations. There has been many results on the existence and uniqueness of periodic solution for the following scalar equation (28) or for the following vector Newton equation

$$x'' + gradG(x) = h(t) \quad (29)$$

See, for example, [11,13-17] and the references therein. Recently, Cong[12] considers equation (28), by using bilinear form theorem combined with the well-known Leary-Schauder degree principle, he proved the following theorem.

Theorem 12. Assume ($A1$) $f \in C^1(R^2, R)$ is 2π-periodic with respect to the first variable, and $p \in PC(R)$-the space of all 2π-periodic functions in $C^1(R)$, satisfies $0 < M_1 \le p(t) \le M_2$ on R for some constants M_1 and M_2.

($A2$) the exist two positive constants a and b such that

$$a \le f_x \le b \ \text{ on } R \times R$$

and there exists a nonnegative integer m satisfying the condition

$$m^2 M_2 < a \le b < (m+1)^2 M_1.$$

Then equation (28) has a unique 2π-periodic solution.

However, the results of [12] is not sharp. In this paper, by using comparison principle combined with Leray-Schauder degree method, we obtain the fowling results.

Theorem 13. Assume $f(t,x) \in C([0,2\pi] \times R, R)$ can be written as

$$f(t,x) = g(t,x)x + h(t,x), \quad (30)$$

where g,h are conditions functions and are 2π-periodic in t with h satisfying

$$\lim_{|x|\to\infty} \frac{h(t,x)}{x} = 0,\ \forall t \in [0.2\pi], \quad (31)$$

and there exists a nonnegative integer n such that

$$n^2 M_2 \le \liminf_{|x|\to\infty} g(t,x) \le \limsup_{|x|\to\infty} g(t,x) \le (n+1)^2 M_1 \quad (32)$$

where

$$0 < M_1 = \min_{t\in[0,2\pi]} p(t) \le \max_{t\in[0,2\pi]} p(t) = M_2 < \infty.$$

If at least one of the following conditions holds: (i) the function $p(t)$ is not constant, that is, $M_1 < M_2$; (ii) the first and last two inequalities in (32) are strict in some subsets of $[0,2\pi]$ of positive measure, then (28) has at least one 2π-periodic solution. Moreover, if $f \in C^1(R^2, R)$ and satisfies

$$n^2 M_2 \le \liminf_{x\in R} \frac{\partial f}{\partial x}(t,x) \le \limsup_{x\in R} \frac{\partial f}{\partial x}(t,x) \le (n+1)^2 M_1 \tag{33}$$

and the first and last two strict inequalities hold on subsets of $[0,2\pi]\times R$ of positive measure. Then (28) has a unique 2π -periodic solution.

Proof of Theorem

Lemma 14. Consider the homogeneous equation

$$(p(t)x')' + q(t)x = 0 \tag{34}$$

Assume $p(t)\in PC(R)$ satisfies the assumptions in Theorem 13, $q(t)\in L^1[0,2\pi]$ is 2π -periodic and satisfies for all $t\in R$,

$$n^2 M_2 \le q(t) \le (n+1)^2 M_1 \tag{35}$$

where n is a nonnegative integer. If either the function $p(t)$ is not constant or the two inequalities in (35) are strict in some subsets of $[0,2\pi]$ of positive measure, then (34) has only the trivial 2π -periodic solution $x(t)\equiv 0$.

Proof. We only prove the case $n \ge 1$, the case $n=0$ can be proved similarly. Define a generalized polar coordinates transformation $T_1:(r,\theta)$ with $r>0, \theta\in R$ as

$$T_1: x = r\sin\theta, \quad p(t)x' = c_1 r\cos\theta \tag{36}$$

where $c_1 = (n+1)M_1$, then it is easy to see that the periodic condition

$$x(0) = x(2\pi), \qquad x'(0) = x'(2\pi)$$

for a nontrivial solution $x(t)$ of (34) is equivalent to

$$\begin{array}{l} r(0) = r(2\pi) > 0, \\ \theta(2\pi) = \theta(0) + 2k\pi \end{array} \quad \text{for some } k\in Z.$$

Taking derivatives of (36) and substituting them into (34), we obtain the following generalized polar coordinates system:

$$\begin{aligned} r' &= r(n+1)\left[\frac{M_1}{p(t)} - \frac{q(t)}{M_1(n+1)^2}\right]\sin\theta\cos\theta, \\ \theta' &= (n+1)\left[\frac{M_1}{p(t)}\cos^2\theta + \frac{q(t)\sin^2\theta}{M_1(n+1)^2}\right] \end{aligned} \tag{37}$$

The first equation of (37) implies that $r(t) = r(0)\exp\int_0^t \eta(s)ds$, where

$$\eta(t) = \left[\frac{M_1}{P(t)} - \frac{q(t)}{M_1(n+1)^2}\right]\sin\theta(t)\cos\theta(t),$$

which implies that $r(t)>0$ whenever $r(0)>0$, $t\in R$. Now the inequality $M_1 \le p(t)$ and inequalities (35) implies that the second equation of (37) satisfies $\theta' \le n+1$ and the strict inequality holds on a sunset of $[0,2\pi]$ of positive measure if the assumptions of Lemma 14 are satisfied. We get therefore

$$\theta(2\pi) - \theta(0) < 2(n+1)\pi. \tag{38}$$

Similarly, we define transformation T_2 as

$$T_2: x = \rho\sin\theta, \quad p(t)x' = d_1\rho\cos\theta,$$

where $d_1 = nM_2$ and $\rho = k_n r$ with $k_n = \frac{(n+1)M_1}{nM_2}$. Comparing T_2 with T_1, we see that only the variable r is replaced by $\rho = k_n r$, and the variable θ is unchanged in T_2. Under the transformation T_2, we obtain

$$\begin{aligned} \rho' &= \rho n\left[\frac{M_2}{r(t)} - \frac{q(t)}{M_2 n^2}\right]\sin\theta\cos\theta, \\ \theta' &= n\left[\frac{M_2}{p(t)}\cos^2\theta + \frac{q(t)\sin^2\theta}{M_2 n^2}\right] \end{aligned} \tag{39}$$

The first equation of (39) implies that $r(t) = r(0)\exp\int_0^t \xi(s)ds$, where

$$\xi(t) = n\left[\frac{M_2}{P(t)} - \frac{q(t)}{M_2 n^2}\right]\sin\theta(t)\cos\theta(t),$$

the second equation of (38) and (35) implies that $\theta' \ge n$ and the strict inequality holds on a subset of $[0,2\pi]$ of positive measure, which implies that

$$2n\pi < \theta(2\pi) - \theta(0). \tag{40}$$

Combining (38) and (40), we see that $x(t)$ can not be a 2π -periodic solution of (36). A contraction. Hence (34) has only the trivial 2π -periodic solution $x \equiv 0$.

Proof of Theorem 13

Let $\|x\|$ be the usual C^1 norm for $x\in PC(R)$, i.e., $\|x\| = \max_{t\in[0,2\pi]}|x(t)| + \max_{t\in[0,2\pi]}|x'(t)|$. For any constant $\mu \in \left(n^2M_2, (n+1)^2M_1\right)$, Lemma 14. implies that the equation

$$(p(t)x')' + \mu x = 0$$

has only the trivial solution $x \equiv 0$.

First, we show the existence part. Introduce an auxiliary equation

$$(p(t)x')' + (1-\lambda)\mu x + \lambda f(t,x) = 0, \quad \lambda \in [0,1] \tag{41}$$

We claim that there exists a constant $M_0 > 0$ such that if $x_\lambda(t)$ is a 2π -periodic solution of (14), then

$$\|x_\lambda\| \le M_0 \tag{42}$$

Suppose (42) is false, then there would exist sequences $\{x_{\lambda_i}\}$ of $2\pi_p$ periodic solutions of (41) and $\{\lambda_i\} \subset [0,1]$ such that $\|x_{\lambda_i}\| \to \infty$ as $i \to \infty$. Denote $w_i(t) = \frac{x_{\lambda_i}(t)}{\|x_{\lambda_i}\|}$, then $w_i(t)$ satisfies equation

$$(p(t)w_i'(t))' + \begin{bmatrix}(1-\lambda_i)\mu + \\ \lambda_i g(t, x_{\lambda_i}(t))\end{bmatrix} w_i(t) + \frac{\lambda_i h(t, x_{\lambda_i}(t))}{\|x_{\lambda_i}\|} = 0 \tag{43}$$

Since $g(t,x)$ satisfies (33), by passing to subsequence, Arzela-Ascoli theorem implies that $\lambda_i \to \lambda_0 \in [0,1]$, $w_i \to w_0 \in PC(R)$, $g(t, x_{\lambda_i}(t)) \to g_0(t) \in L^1[0,2\pi]$ as $i \to \infty$. By assumption, $\lim_{|x|\to\infty} \frac{h(t,x)}{x} = 0$, we see that from (43) that $w_0(t)$ is a 2π -periodic solution of the equation

$$(p(t)w')' + q(t)w = 0 \tag{44}$$

with $\|w_0\| = 1$, where $q(t) = (1-\lambda_0)\mu + \lambda_0 g_0(t)$. On the hand, as $g(t,x)$ satisfies (32) and by the choice of μ, we see that $q(t)$ satisfies (35). Lemma 14 implies that (44) has only trivial solution $w \equiv 0$. This is a contraction. Hence (42) holds. The Leacy-Schauder principle implies that (28) has at least one 2π -periodic solution.

Next we prove the uniqueness part. Let $f(t,x) = xg(t,x) + f(t,0)$, where.

$$g(t,x) = \int_0^1 \frac{\partial f}{\partial x}(t, \theta x)d\theta$$

Since f satisfies (33), g must satisfies (32). Theorem 13 implies that (28) has at least one 2π -periodic solution. Assume there exist two 2π -periodic solutions $x_1(t)$, $x_2(t)$ of (28), then $x(t) = x_1(t) - x_2(t)$ is a 2π -periodic solution of the following equation:

$$(r(t)x')' + x\bar{g}(t,x) = 0, \tag{45}$$

where

$$\bar{g}(t,x) = \int_0^1 \frac{\partial f}{\partial x}(t, x_2(t) + \theta x(t))d\theta.$$

By assumption (33), $\bar{g}(t,x)$ satisfies (32) with g replaced by $\bar{g}$. Now Lemma 14 implies that (45) has only the trivial solution $x \equiv 0$, hence $x_1(t) \equiv x_2(t)$.

Let $p(t) = 1$ in (28), the equation is linear. Because it is the special case of (28), so we can get easily the corresponding theorem with Theorem 13 for $p(t) = 1$.

4. Conclusion

In this paper, we consider the solutions of equation (1) and (28). The oscillation and non-oscillation criteria are obtained by studying the quasi-linear equation (1) and while considering the special case , that is, equation (28) which is linear when $p(t) = 1$, we can get the periodic and uniqueness criteria. This shows that we maybe obtain the wilder conclusion when the simpler problem is studied.

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