The Asymptotical Behavior of Probability Measures for the Fluctuations of Stochastic Models

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Abstract: - We consider the fluctuations of shapes of two phases boundaries of the one-dimensional statistical mechanics models. By applying the theory of one-dimensional random walk, the models of the two phases boundaries are constructed by assuming that there is a specified value of the large area in the intermediate region of the two phases boundaries. Then we investigate the asymptotical behavior of the corresponding sequence of probability measures describing the statistical properties of the two phases boundaries. We show that the limiting probability measures coincide with some conditional probability distribution of certain Gaussian distribution. Further we discuss the properties of fluctuations of phase separation lines for the Ising model, and we obtain the asymptotic properties of the two interfaces S.O.S. model.

Key-Words: - Stochastic models; random phase boundaries; central limit theory; random walk; Gibbs measure; Hamiltonian

1 Introduction

The problem of description of shapes of phase boundaries is a well-known problem in statistical mechanics systems. In recent years, some research work has been done to investigate the statistical properties of the random phase boundaries for some statistical physics models, for example see Refs. [1-8]. In this paper, we consider the statistical limiting properties of the two random phases boundaries model. This work originates in an attempt to describe the fluctuations of the phase boundaries in two random phases boundaries models (e.g. one-dimensional two random phases boundaries S.O.S. model). In Ref. [1], the statistical properties of random walks and the interface of Widom-Rowlinson model (conditioned by fixing a large area under their paths and conditioned by fixing the terminating point) are considered, and the central limit theorem for these conditional distributions is proved. In [7], the interfaces of supercritical Ising model (see [9-11]) on the lattice fractal---the Sierpinski carpet is studied. The similar problems arise in describing the fluctuations of two random phases boundaries models. In the first part of the present paper, with the conditions “fixed area” of the intermediate layer and “fixed end points” in a two random paths model, we study the limiting properties of the two random phases boundaries, see [12]. In the second part of this paper, the research results of the first part will be extended and improved, the statistical properties of the interfaces of S.O.S. model and the two-dimensional stochastic Ising model are studied. We show that the heights of the fluctuations of phase separation lines of the Ising model occur on a scale $l^{1/2}(\ln l)^{1/2}$ for a large parameter β and a large l (where the Ising model is considered on a rectangle of horizontal side length $2l$). Then we discuss the asymptotic properties of the two interfaces S.O.S. model, and obtain the corresponding limiting results for the two interfaces S.O.S. model.

In this paper, we consider the phase boundaries (or interfaces) models consisting of the interfaces without overhangs, and therefore its configurations of the horizontal length L are represented by set of heights $h_x \in Z, \quad x \in L_X = \{x_0, x_0+1, ..., x_0+L\} \subset Z$. At each site x of the one dimensional lattice Z, we attach the variable of “heights” $h_x \in Z$, therefore the configurations of the random interfaces model on a horizontal set L_X (with the length of L) are represented by sets of heights $\Omega_L = \{h\} = \{h_x\}_{x \in L_X}$, for the simplicity, we assume $x_0 = 0$. The energy of the configuration $\{h\}=\{h_x\}_{x\in L_X}$ is determined by the Hamiltonian

$$H_L(h)=\sum_{i=0}^{L-1}U\left(\left|h_{i+1}-h_i\right|\right)$$

where $U(\cdot)$ is a real-valued function. There are many possible choices for the function $U(\cdot)$, this means that the results of the present paper can be extended to some other interfaces models. For the sake of simplicity we restrict ourselves to the case of integer-valued heights $h_x\in Z$. Let a positive parameter β be an inverse temperature, and the finite partition function of this system be

$$Z_{L,\beta}=\sum_{h_1\in Z}\cdots\sum_{h_L\in Z}\exp\left[-\beta H_L(h)\right].$$

Then the corresponding Gibbs probability distribution on Ω_L is given by

$$P_{L,\beta}(h)=\left(Z_{L,\beta}\right)^{-1}\exp\left[-\beta H_L(h)\right] .$$

Next we consider the two phases boundaries statistical mechanics model. At each site x of the one dimensional lattice Z, we attach two variables of "heights" $h_x^1,h_x^2\in Z$, therefore the configurations of the random paths model on a horizontal set L_X are represented by sets of heights $\{h^1,h^2\}=\{h_x^1,h_x^2\}_{x\in L_X}$, for the simplicity, we also assume $x_0=0$. Now we define the interfaces of the two random interfaces model as followings, for $t\in[0,1]$,

$$X_L^{h^1}\left(\frac{j}{L}\right)=h_j^1,\qquad j\in L_X$$

$$X_L^{h^1}(t)=(j+1-Lt)X_L^{h^1}\left(\frac{j}{L}\right)+(Lt-j)X_L^{h^1}\left(\frac{j+1}{L}\right),$$
$$j\le Lt\le j+1$$

and $X_L^{h^2}\left(\frac{j}{L}\right)$, $X_L^{h^2}(t)$ are defined similarly as above definitions.

For $x\in L_X$ and $x\ge 1$, let $\xi_x=h_x^1-h_{x-1}^1$, $\eta_x=h_x^2-h_{x-1}^2$, so we have $h_x^1=\sum_{i=0}^x\xi_x$ and $h_x^2=\sum_{i=0}^x\eta_x$, where let $\xi_0=0$, $\eta_0=0$. Let $\xi=\{\xi_x,x\in L_X\}$, $\eta=\{\eta_x,x\in L_X\}$, then the Hamiltonian of the model on the horizontal set of L_X is given by

$$H_L(\xi,\eta)=\sum_{i=1}^{L}\left(U_1(\xi_i)+U_2(\eta_i)\right)$$

where $U_1(\cdot)$, $U_2(\cdot)$ are real-valued functions. The partition function of the dynamic system is defined as following

$$Z_{L,\beta}=\sum_{\xi,\eta}\exp\left[-\beta H_L(\xi,\eta)\right],$$

where β is a positive parameter called an inverse temperature. The corresponding Gibbs probability distribution on is given by

$$P_{L,\beta}(\xi,\eta)=\left(Z_{L,\beta}\right)^{-1}\exp\left[-\beta H_L(\xi,\eta)\right] .$$

Thus we have the corresponding interfaces $X_L^{\xi}\left(\frac{j}{L}\right),X_L^{\xi}(t),X_L^{\eta}\left(\frac{j}{L}\right),X_L^{\eta}(t)$.

From above definitions, $\xi=\{\xi_x,x\in L_X\}$ and $\eta=\{\eta_x,x\in L_X\}$ can be seen as the sequences of i.i.d. random variables respectively. So, the two random interfaces model has two independent random *SOS* paths, that is, the model corresponds to the ensemble of two independent self-avoiding paths in $[0,L]\times Z$ starting from $(0,0)$ and ending at sites z in the line $\{x=L\}$ (where $z=(x,y)$), which do not go back in the horizontal direction. Next we introduce the generating function of the height of the endpoints for one "step", that is, for a fixed $x\in L_X$, let

$$Q(\mu,\nu)=$$
$$\sum_{\xi_x,\eta_x}e^{\beta\mu\xi_x+\beta\nu\eta_x-\beta\left(|\xi_x|+|\eta_x|\right)}\Big/\sum_{\xi_x,\eta_x}\exp\left[-\beta H_L\left(\xi_x,\eta_x\right)\right]$$

where $Q(\mu,\nu)$ is independent of x and $-\infty<\xi_x,\eta_x<+\infty$. Due to the independence of the random variables $\{\xi_x,x\in L_X\}$ and $\{\eta_x,x\in L_X\}$, thus

$$Q(\mu,\nu)^L=$$
$$\sum_{\xi,\eta}\exp\left[\beta\mu\overline{\xi}+\beta\nu\overline{\eta}\right]\exp\left[-\beta H_L(\xi,\eta)\right]\Big/Z_{L,\beta}$$

where

$$\overline{\xi}=\sum_{x=1}^{L}\xi_x\quad\text{and}\quad\overline{\eta}=\sum_{x=1}^{L}\eta_x .$$

For $(\mu,\nu)\in R\times R$, we define

$$\varphi(\mu,\nu)=\lim_{L\to 0}\frac{1}{L}\times$$
$$\ln\left(\sum_{\xi,\eta}\exp[\beta\mu\overline{\xi}+\beta\nu\overline{\eta}]\exp[-\beta H_L(\xi,\eta)]\Big/Z_{L,\beta}\right)$$

by the Refs. [1][3][6], it is known that this limit exists if (μ,ν) is in some neighborhood of the origin.

The aim of this paper is to study the asymptotes of fluctuations of the two random interfaces conditioned by fixing a large area between the two random interfaces. Denote by a_L^ξ, a_L^η representing the areas under the paths $X_L^\xi\left(\frac{j}{L}\right)$, $X_L^\eta\left(\frac{j}{L}\right)$ respectively, and denote by $a_L^{\eta-\xi}=a_L^\eta-a_L^\xi$ representing the area of the intermediate layer between the two random interfaces. For a real $\bar\zeta_0$ and $0\le s\le 1$, assume that

$$F\left(\bar\zeta_0,\beta,s\right)=\frac{d}{d\zeta_0}\varphi\left(-(1-s)\zeta_0,(1-s)\zeta_0\right)\bigg|_{\zeta_0=\bar\zeta_0},$$

$$\frac{1}{\beta}\int_0^1 F\left(\bar\zeta_0,\beta,s\right)ds=a \tag{1}$$

where $a>0$ is some constant. Above (1) is an important condition for this paper, we will use this condition to fulfill our proof in the followings. Then we state the main results of this paper.

Theorem 1 Assume that for some $\delta(\beta)>0$ and $a>0$, there exists a real $\bar\zeta_0$ satisfying above condition (1) and $\left|\bar\zeta_0\right|<\delta(\beta)$, then the process

$$Y_L(t)=\frac{1}{\sqrt{L}}\left\{X_L^\eta(t)-X_L^\xi(t)-\frac{L}{\beta}\int_0^t F\left(\bar\zeta_0,\beta,s\right)ds\right\}$$

under $P_{L,\beta}\left(\cdot\middle|a_L^{\eta-\xi}=\lfloor aL\rfloor\right)$, converges weakly to the process

$$Y(t)=\frac{1}{\beta}\int_0^t\sqrt{\varphi''\left(-(1-s)\bar\zeta_0,(1-s)\bar\zeta_0\right)}dB(s)$$

conditioned that $\int_0^1 Y(t)dt=0$, where $\{B(s)\}_{s\ge0}$ is the one dimensional standard Brownian motion, and $\lfloor aL\rfloor$ is the integer part of aL.

Remark 1 In Theorem 1, the model is only conditioned by fixing a large area between the two random interfaces and having the same starting endpoints. The results can also be proved similarly for the two random interfaces with fixed value of area and the two same endpoints.

Theorem 2 Let $\varphi_\mu{}'(\mu,\nu)=\frac{\partial}{\partial\mu}\varphi(\mu,\nu)$, and $F_\mu\left(\bar\zeta_0,\beta,s\right)=-\varphi'\left(-(1-s)\bar\zeta_0,(1-s)\bar\zeta_0\right)$. With the same conditions of Theorem 1, the probability distribution of the random process $-X_L^\xi(t)/L$, under $P_{L,\beta}\left(\cdot\middle|a_L^{\eta-\xi}=\lfloor aL\rfloor\right)$, converges weakly to the corresponding probability distribution concentrated on the function

$$Y_1(t)=\frac{1}{\beta}\int_0^t\sqrt{F_\mu\left(\bar\zeta_0,\beta,s\right)}ds.$$

2 Convergence of Probability Measures for the Two Random Interfaces Model

In this section, we begin discussing the area between the two random paths. Then we show the some results about the weak convergence (see [12]) of random vector of the two random interfaces for the model. Now we define the areas of a_L^ξ, a_L^η, $a_L^{\eta-\xi}$ as followings,

$$a_L^\xi=\sum_{x=1}^L h_x^1/L=\sum_{x=1}^L(1-x/L)\xi_x,$$

$$a_L^\eta=\sum_{x=1}^L h_x^2/L=\sum_{x=1}^L(1-x/L)\eta_x,$$

$$a_L^{\eta-\xi}=a_L^\eta-a_L^\xi=\sum_{x=1}^L(1-x/L)\left(\eta_x-\xi_x\right).$$

By the independence of $\{\xi_x,x\in L_X\}$ and $\{\eta_x,x\in L_X\}$, the generation function of the area $a_L^{\eta-\xi}$ is defined by

$$\begin{aligned}
&Q_{a_L^{\eta-\xi}}(\zeta)\\
&=\sum_{\eta,\xi}\exp\left\{\beta\zeta a_L^{\eta-\xi}\right\}\exp\left\{-\beta H_L(\xi,\eta)\right\}\Big/Z_{L,\beta}\\
&=\prod_{x=1}^L\sum_{\xi_x,\eta_x}\exp\left\{\beta\zeta(1-x/L)\left(\eta_x-\xi_x\right)-\beta\left(|\xi_x|+|\eta_x|\right)\right\}\\
&\quad\times\frac{1}{Z_{L,\beta}}\\
&=\prod_{x=1}^L Q\left(-\zeta(1-x/L),\zeta(1-x/L)\right).
\end{aligned}$$

Let q be a natural number, and let $\{t_i,1\le i\le q\}$ be any set of real numbers, such that $0<t_1<...<t_q\le1$. Set a random vector as

$$\hat X_L^{(q)}\left(t_1,...,t_q\right)=\left(a_L^{\eta-\xi},h_{\lfloor t_1L\rfloor}^2-h_{\lfloor t_1L\rfloor}^1,...,h_{\lfloor t_qL\rfloor}^2-h_{\lfloor t_qL\rfloor}^1\right).$$

Then for $\underline{\zeta}=\left(\zeta_0,\zeta_1,...,\zeta_q\right)\in R^{q+1}$, we have

$$\sum_{\eta,\xi}e^{\beta\underline{\zeta}\hat X_L^{(q)}\left(t_1,...,t_q\right)}e^{-\beta H_L(\xi,\eta)}\Big/Z_{L,\beta}=$$

$$\prod_{x=1}^L Q\left(-\zeta_L\left(x;\underline{\zeta}\right),\zeta_L\left(x;\underline{\zeta}\right)\right)$$

where

$$\zeta_L\left(x;\underline{\zeta}\right)=\zeta_0\left(1-x/L\right)+\sum_{i=1}^{q}\zeta_i 1_{[0,Lt_i]}\left(x\right).$$

For the real $\overline{\zeta}_0$ defined in (1) and some small constant $\alpha>0$, let $\underline{\zeta}\in R^{q+1}$ satisfy the following conditions

$$D_{\alpha,\overline{\zeta}_0}=\left\{\zeta:-\alpha<\zeta_0<\overline{\zeta}_0+\alpha,\left|\zeta_i\right|<\alpha,i=1,...,q\right\}.$$

Next we introduce the corresponding quadratic form, a $(q+1)\times(q+1)$ matrix denote by

$$V_L\left(\underline{\zeta}\right)=\frac{1}{\beta^2 L}Hess\ln\left(\sum_{\eta,\xi}e^{\beta\underline{\zeta}\hat{X}_L^{(q)}\left(t_1,...,t_q\right)}e^{-\beta H_L(\xi,\eta)}\Big/Z_{L,\beta}\right)$$

where $V_L\left(\underline{\zeta}\right)$ is analytic in $D_{\alpha,\overline{\zeta}_0}$. Assume that $\underline{\zeta}\in D_{\alpha,\overline{\zeta}_0}$, and according to the definition of $V_L\left(\underline{\zeta}\right)$, then uniformly in $\underline{\zeta}$ and $\underline{y}=\left(y_0,...,y_q\right)\in R^{q+1}$ such that $\left|\underline{y}\right|=1$, we have

$$\underline{y}.V_L\left(\underline{\zeta}\right)\underline{y}\to\underline{y}.V\left(\underline{\zeta}\right)\underline{y},\qquad \text{as } L\to\infty$$

where

$$V\left(\underline{\zeta}\right)=\frac{1}{\beta^2}Hess\int_0^1\ln Q\left(-\zeta\left(s\right),\zeta\left(s\right)\right)ds$$

and

$$\zeta\left(s\right)=\zeta_0\left(1-s\right)\sum_{i=1}^{q}\zeta_i 1_{[0,t_i]}\left(s\right),\quad \text{for } 0\le s\le 1.$$

Let $\hat{P}_L^{(q)}$ be the probability distribution of $\hat{X}_L^{(q)}\left(t_1,...,t_q\right)$ under $P_{L,\beta}$, and $\hat{P}_{L,\underline{\zeta}}^{(q)}$ be given by

$$\hat{P}_{L,\underline{\zeta}}^{(q)}(\underline{z})=e^{\beta\underline{\zeta}\underline{z}}\hat{P}_L^{(q)}(\underline{z})\Big/E_{L,\beta}\left(\exp\left\{\beta\underline{\zeta}\hat{X}_L^{(q)}\left(t_1,...,t_q\right)\right\}\right)$$

for all $\underline{\zeta}\in D_{\alpha,\overline{\zeta}_0}$ and $\underline{z}\in Z_L^{(q)}=\left(L^{-1}Z\right)\times Z^q$. Denote by $\hat{E}_{L,\underline{\zeta}}^{(q)}(\cdot)$ the corresponding expectation function for $\hat{E}_{L,\underline{\zeta}}^{(q)}(\cdot)$. By the uniform boundedness of the family of analytical functions $V_L\left(\underline{\zeta}\right)$ for all L and all $\underline{\zeta}$ in $D_{\alpha,\overline{\zeta}_0}$, according to Lemma 2.6 and Proposition 2.7 in Ref. [1], we have the following Lemma 1 and Lemma 2.

Lemma 1 Let $\underline{\zeta}_L,\underline{\zeta}\in D_{\alpha,\overline{\zeta}_0}$, and $\underline{\zeta}_L\to\underline{\zeta}$ as $L\to\infty$. Then the random vector

$$\hat{Y}_L^{(q)}\left(t_1,...,t_q\right)=\frac{1}{\sqrt{L}}\left(\hat{X}_L^{(q)}\left(t_1,...,t_q\right)-\hat{E}_{L,\underline{\zeta}_L}^{(q)}\hat{X}_L^{(q)}\left(t_1,...,t_q\right)\right)$$

converges weakly to a Gaussian random vector $\hat{Y}^{(q)}\left(t_1,...,t_q\right)$ of which covariance matrix is given by $V\left(\underline{\zeta}\right)$.

Let $g_{\underline{\zeta}}$ be the density function of the Gaussian vector $\hat{Y}^{(q)}\left(t_1,...,t_q\right)$ given in Lemma 1, then we have the following Lemma 2.

Lemma 2 Let $Z_L^{(q)}=\left(L^{-1}Z\right)\times Z^q$, then for each $\underline{z}_L\in Z_L^{(q)}$ and $\underline{\zeta}_L\in D_{\alpha,\overline{\zeta}_0}$, define

$$\underline{z}_L\in Z_L^{(q)}\underline{y}_L=\frac{1}{\sqrt{L}}\left(\underline{z}_L-\hat{E}_{L,\underline{\zeta}_L}^{(q)}\hat{X}_L^{(q)}\left(t_1,...,t_q\right)\right).$$

Then we have

$$L^{(q+3)/2}\hat{P}_{L,\underline{\zeta}_L}^{(q)}(\underline{z}_L)-g_{\underline{\zeta}_L}\left(\underline{y}_L\right)\to 0,\quad \text{as}\quad L\to\infty$$

uniformly in $\underline{z}_L\in Z_L^{(q)}$ and $\underline{\zeta}_L\in D_{\alpha,\overline{\zeta}_0}$.

3 Convergence of Finite Dimensional Distributions

In this section, we discuss the limiting properties of the random vector $\hat{X}_L^{(q)}\left(t_1,...,t_q\right)$ defined in Section 2, and show the proofs of Theorem 1. Then we give the proof of Theorem 2, in fact, by using the proofing method of Theorem 1, we can prove Theorem 2.

Proof of Theorem 1. In Section 2, the random vector $\hat{X}_L^{(q)}\left(t_1,...,t_q\right)$ is given. First we consider the convergence of the finite-dimensional distribution of the random vector $\hat{Y}_L^{(q)}\left(t_1,...,t_q\right)$ defined in Lemma 1. Let $\underline{\zeta}_L^0,\underline{\zeta}^0$ be a special sequence in $D_{\alpha,\underline{\zeta}_L^0}$, such that

$$\underline{\zeta}_L^0=\left(\underline{\zeta}_{L,0},0,...0\right),\quad \underline{\zeta}^0=\left(\overline{\zeta}_0,0,...0\right)$$

where $\overline{\zeta}_0$ is defined in (1), and $\underline{\zeta}_{L,0}$ satisfies the following condition

$$\frac{d}{d\zeta_0}\ln Q_{\alpha_L^{\eta-\varepsilon}}\left(\zeta_0\right)\Big|_{\zeta_0=\zeta_{L,0}}=\left\lfloor aL\right\rfloor. \tag{2}$$

by (1)(2), it can be proved that $\underline{\zeta}_L^0\to\underline{\zeta}^0$ as $L\to\infty$, here we omit the proof. Let

$$\varphi_L\left(\underline{\zeta};t_1,...,t_q\right)=\frac{1}{L}\ln\left(\sum_{\xi,\eta}e^{\beta\underline{\zeta}\cdot\hat{X}_L^{(q)}\left(t_1,...,t_q\right)_e-\beta H_L(\xi,\eta)}\Big/Z_{L,\beta}\right)$$

and denote by

$$\varphi^{(q)}\left(\zeta;t_1,...,t_q\right)=\lim_{L\to\infty}\varphi_L\left(\zeta;t_1,...,t_q\right)$$

for $\underline{\zeta}\in D_{\alpha,\underline{\zeta}_L^0}$. By the uniform boundedness of $Hess_{\underline{\zeta}}\varphi_L$, we have

$$\begin{aligned}&\hat{E}^{(q)}_{L,\underline{\zeta}_L^0}\hat{X}_L^{(q)}\left(t_1,...,t_q\right)=\\&\left(\lfloor aL\rfloor,\hat{E}^{(q)}_{L,\underline{\zeta}_L^0}\left(h^2_{\lfloor t_1L\rfloor}-h^1_{\lfloor t_1L\rfloor}\right),...,\hat{E}^{(q)}_{L,\underline{\zeta}_L^0}\left(h^q_{\lfloor t_1L\rfloor}-h^q_{\lfloor t_1L\rfloor}\right)\right)\\&=\frac{L}{\beta}\left(\nabla_{\underline{\zeta}}\varphi_L\right)\left(\underline{\zeta}_L^0;t_1,...,t_q\right)\\&=\frac{L}{\beta}\left(\nabla_{\underline{\zeta}}\varphi^{(q)}\right)\left(\underline{\zeta}^0;t_1,...,t_q\right)+o(1).\end{aligned}$$

By Lemma 2, we have for $-\infty<a_j<b_j<\infty,\ 1\le j\le q$,

$$\begin{aligned}&\lim_{L\to\infty}\hat{P}_L^{(q)}\left(y_j\in\left[a_j,b_j\right],\ 1\le j\le q\,\middle|\,z_0=\lfloor aL\rfloor\right)\\&=\lim_{L\to\infty}\hat{P}^{(q)}_{L,\underline{\zeta}_L^0}\left(y_j\in\left[a_j,b_j\right],\ 1\le j\le q\,\middle|\,z_0=\lfloor aL\rfloor\right)\\&=\frac{\int_{[a_1,b_1]\times...\times[a_q,b_q]}g_{\underline{\zeta}^0}\left(0,y_1,...,y_q\right)dy_1,...,y_q}{\int_{R^q}g_{\underline{\zeta}^0}\left(0,y_1,...,y_q\right)dy_1,...,y_q}.\end{aligned}$$

According to Lemma 1, let

$$\hat{Y}^{(q)}\left(t_1,...,t_q\right)=\left(Y_0,Y(t_1),...,Y(t_q)\right)$$

be a Gaussian random vector with distribution density $g_{\underline{\zeta}}\left(y_0,...,y_q\right)$. Then its covariance matrix is given by

$$\begin{aligned}&E\left[Y\left(t_j\right)Y\left(t_K\right)\right]\\&=\frac{1}{\beta^2}\int_0^{t_j\wedge t_k}\varphi''\left(-(1-s)\bar{\zeta}_L^0,(1-s)\bar{\zeta}^0\right)ds\end{aligned}$$

$$E\left[Y_0Y\left(t_j\right)\right]=\frac{1}{\beta^2}\int_0^{t_j}\varphi''\left(-(1-s)\bar{\zeta}_L^0,(1-s)\bar{\zeta}^0\right)ds$$

$$E\left[Y_0^2\right]=\frac{1}{\beta^2}\int_0^{1}\varphi''\left(-(1-s)\bar{\zeta}_L^0,(1-s)\bar{\zeta}^0\right)ds$$

for $j,k=1,...,q$, where $a\wedge b=\min\{a,b\}$. This means that $\left\{Y_0,\{Y(t)\}_{t\in[0,1]}\right\}$ is a Gaussian random process with covariance matrix given above for every $q\ge1$. In above proof, we suppose that

$$h^2_{\lfloor t_iL\rfloor}-h^1_{\lfloor t_iL\rfloor}=X_L^{\eta}\left(\frac{\lfloor Lt_i\rfloor}{L}\right)-X_L^{\xi}\left(\frac{\lfloor Lt_i\rfloor}{L}\right),$$

for $i=1,...,q$. Similarly to Ref. [1], the above argument is also true if we replace $X_L^{\eta}\left(\frac{\lfloor Lt_i\rfloor}{L}\right)-X_L^{\xi}\left(\frac{\lfloor Lt_i\rfloor}{L}\right)$ with $X_L^{\eta}\left(t_i\right)-X_L^{\xi}\left(t_i\right)$ for every $1\le i\le q$. Then the distribution of $\hat{X}_L^{(q)}\left(t_1,...,t_q\right)$, under $P_{L,\beta}\left(\cdot\middle|\alpha_L^{\eta-\varepsilon}=\lfloor aL\rfloor\right)$, converges weakly to the corresponding distribution of Gaussian random vector $\hat{Y}_L^{(q)}\left(t_1,...,t_q\right)$.

Secondly, the tightness of above conditional distribution of the random process $Y_L(t)$ should be discussed, see [1]. Following the similar argument of Section 3 in Ref. [1], we can prove a sufficient condition for the tightness of the considered process $Y_L(t)$. So by the theory of weak convergence (see [12]), together with the first part of this proof, this completes the proof of Theorem 1.

Remark 2 According to the arguments of [1], and with the results of Theorem 1, the probability distribution of the random process

$$\left(X_L^{\eta}(t)-X_L^{\xi}(t)\right)/L$$

under $P_{L,\beta}\left(\cdot\middle|\alpha_L^{\eta-\xi}=\lfloor aL\rfloor\right)$, converges weakly to the corresponding distribution concentrated on the function $\frac{1}{\beta}\int_0^1F\left(\bar{\zeta}_0,\beta,s\right)ds$.

Proof of Theorem 2 Let q be a natural number, and let $\{t_i,1\le i\le q\}$ be any set of real numbers, such that $0<t_1<...<t_q\le1$. Set a random vector as

$$\hat{X}_L^{\xi}\left(t_1,...,t_q\right)=\left(a_L^{\eta-\xi},-h^1_{\lfloor t_1L\rfloor},...,-h^1_{\lfloor t_qL\rfloor}\right).$$

Let $\underline{\zeta}_L^0,\underline{\zeta}^0$ be a special sequence in $D_{\alpha,\underline{\zeta}_L^0}$, such that

$$\underline{\zeta}_L^0=\left(\underline{\zeta}_{L,0},0,...0\right)\ ,\quad \underline{\zeta}^0=\left(\bar{\zeta}_0,0,...0\right)$$

where $\bar{\zeta}_0$ is defined in (1), and $\underline{\zeta}_{L,0}$ is defined in (2). Then we have the corresponding function as following

$$\begin{aligned}&\varphi_L^1\left(\underline{\zeta};t_1,...,t_q\right)\\&=\frac{1}{L}\ln\left(\sum_{\xi,\eta}e^{\beta\underline{\zeta}\cdot\hat{X}_L^{\xi}\left(t_1,...,t_q\right)}e^{-\beta H_L(\xi,\eta)}\Big/Z_{L,\beta}\right)\\&=\frac{1}{L}\ln\prod_{x=1}^{L}Q\left(-\zeta_L^{\xi}\left(x;\underline{\zeta}\right),\zeta_L^{\eta}\left(x;\zeta_0\right)\right)\end{aligned}$$

where

$$\zeta_L^{\xi}\left(x;\underline{\zeta}\right)=\zeta_0\left(1-x/L\right)+\sum\nolimits_{i=1}^{q}\zeta_i1_{[0,Lt_i]}(x),$$

$$\zeta_L^{\eta}\left(x;\underline{\zeta}\right)=\zeta_0\left(1-x/L\right).$$

For any $\underline{\zeta}=\left(\zeta_0,\zeta_1,...,\zeta_q\right)\in R^{q+1}$ satisfy the following conditions

$$D_{\alpha,\bar{\zeta}_0}=\left\{\zeta:-\alpha<\zeta_0<\bar{\zeta}_0+\alpha,\left|\zeta_i\right|<\alpha,i=1,...,q\right\}.$$

Let

$$\varphi^1\left(\underline{\zeta};t_1,...,t_q\right)=\lim_{L\to\infty}\varphi_L^1\left(\underline{\zeta};t_1,...,t_q\right)$$

for $\underline{\zeta} \in D_{\alpha,\underline{\zeta}_L^0}$, and $\hat{E}^1_{L,\underline{\zeta}}(\cdot)$ is the corresponding expectation function for $\hat{X}_L^{\xi}\left(t_1,...,t_q\right)$. By the uniform boundedness of $Hess_{\underline{\zeta}}\varphi_L$, we have

$$\hat{E}^1_{L,\underline{\zeta}_L^0}\hat{X}_L^{\xi}\left(t_1,...,t_q\right)=$$
$$\left(\lfloor aL\rfloor,\hat{E}^1_{L,\underline{\zeta}_L^0}\left(-h^1_{\lfloor t_1L\rfloor}\right),...,\hat{E}^1_{L,\underline{\zeta}_L^0}\left(-h^1_{\lfloor t_qL\rfloor}\right)\right)$$
$$=\frac{L}{\beta}\left(\nabla_{\underline{\zeta}}\varphi_L^1\right)\left(\underline{\zeta}_L^0;t_1,...,t_q\right)$$
$$=\frac{L}{\beta}\left(\nabla_{\underline{\zeta}}\varphi^1\right)\left(\underline{\zeta}^0;t_1,...,t_q\right)+o(1)$$

where $1\le j\le q$, and

$$\hat{E}^1_{L,\underline{\zeta}_L^0}\left(-h^1_{\lfloor t_1L\rfloor}\right)=$$
$$\sum_{x=1}^{L}\frac{\partial}{\partial\xi_i}\ln Q\left(-\zeta_L^{\xi}\left(x;\underline{\zeta}\right),\zeta_L^{\eta}\left(x;\zeta_0\right)\right)\Bigg|_{\underline{\zeta}=\underline{\zeta}_L^0}.$$

For the random vector $\hat{X}_L^{\xi}\left(t_1,...,t_q\right)$, by using the methods of Lemma 2.6 and Proposition 2.7 in [1], we can have the similar results as that of Lemma 1 and Lemma 2. Then following the steps in the proof of Theorem 1, we can prove that the probability distribution of the random process

$$\frac{1}{\sqrt{L}}\left\{-X_L^{\xi}(t)-\frac{L}{\beta}\int_0^t F_\mu\left(\overline{\zeta}_0,\beta,s\right)ds\right\}$$

under $P_{L,\beta}\left(\cdot\middle|\alpha_L^{\eta-\xi}=\lfloor aL\rfloor\right)$, converges weakly to some Gaussian distribution. Thus by Remark 2, the probability distribution of the random process $-X_L^{\xi}(t)/L$, under $P_{L,\beta}\left(.\middle|a_L^{\eta-\xi}=\lfloor aL\rfloor\right)$, converges weakly to the corresponding probability distribution concentrated on the function

$$Y_1(t)=\frac{1}{\beta}\int_0^t\sqrt{F_\mu\left(\overline{\zeta}_0,\beta,s\right)}ds.$$

This completes the proof of Theorem 2.

According to the results of Theorem 1 and Theorem 2, we have the following Corollary 1.

Corollary 1 Suppose that the definitions and conditions of Theorem 2 hold, then the probability distribution of the random process $X_L^{\eta}(t)/L$, under $P_{L,\beta}\left(\cdot\middle|\alpha_L^{\eta-\xi}=\lfloor aL\rfloor\right)$, converges weakly to the corresponding probability distribution concentrated on the function

$$\frac{1}{\beta}\int_0^t F\left(\overline{\zeta}_0,\beta,x\right)dx-\frac{1}{\beta}\int_0^t F_\mu\left(\overline{\zeta}_0,\beta,x\right)dx.$$

Proof. The random process $\left(X_L^{\eta}(t)\middle|\alpha_L^{\eta-\xi}=\lfloor aL\rfloor\right)$ can be written as

$$\left(X_L^{\eta}(t)\middle|\alpha_L^{\eta-\xi}=\lfloor aL\rfloor\right)$$
$$=\left(X_L^{\eta}(t)-X_L^{\xi}(t)\middle|\alpha_L^{\eta-\xi}=\lfloor aL\rfloor\right)$$
$$+\left(X_L^{\xi}(t)\middle|\alpha_L^{\eta-\xi}=\lfloor aL\rfloor\right).$$

For the first term of above equation, under $P_{L,\beta}\left(\cdot\middle|\alpha_L^{\eta-\xi}=\lfloor aL\rfloor\right)$ and by Theorem 1 and Remark 2, we have that the probability distribution of the random process $\left(X_L^{\eta}(t)-X_L^{\xi}(t)\middle|\alpha_L^{\eta-\xi}=\lfloor aL\rfloor\right)$ converges weakly to the corresponding probability distribution of the function

$$\frac{1}{\beta}\int_0^t F\left(\overline{\zeta}_0,\beta,x\right)dx.$$

For the second term of above equation, under $P_{L,\beta}\left(\cdot\middle|\alpha_L^{\eta-\xi}=\lfloor aL\rfloor\right)$ and according to Theorem 2 and Remark 2, the probability distribution of the random process $\left(-X_L^{\xi}(t)\middle|\alpha_L^{\eta-\xi}=\lfloor aL\rfloor\right)$ converges weakly to the corresponding probability distribution of the function

$$\frac{1}{\beta}\int_0^t F_\mu\left(\overline{\zeta}_0,\beta,x\right)dx.$$

This completes the proof of Corollary 1.

4 The Fluctuations of S.O.S. Model and Ising Model

In this section, we discuss the relations between the two random interfaces model and the two interfaces S.O.S. model and Ising model. The statistical properties of the interfaces of S.O.S. model and Ising model are studied in this section. The Hamiltonian $H_L^S\left(h^1,h^2\right)$ of two interfaces S.O.S. model has the same definition of $H_L\left(h^1,h^2\right)$ in Section 1. But the partition function of two interfaces S.O.S. model is given by

$$Z_{L,\beta}^S=\sum_{h_x^1\le h_x^2,x\in L_X}\exp\left[-\beta H_L^S\left(h^1,h^2\right)\right]$$

and according to the definitions in Section 1, we have the corresponding partition function

$$Z_{L,\beta}^S=\sum_{\xi\le\eta}\exp\left[-\beta H_L^S\left(\xi,\eta\right)\right]$$

where $\xi\le\eta$ denote that $\xi_x\le\eta_x$ for all $x\in L_X$.

From above definitions, for the two interfaces S.O.S. model, the two interfaces of the model don't intersect, so that, the two interfaces are not independent. The corresponding probability measure is defined as following

$$P_{L,\beta}\left(\xi,\eta\right)=\left(Z^{S}_{L,\beta}\right)^{-1}\exp\left[-\beta H^{S}_{L}(\xi,\eta)\right] \qquad (3)$$

on the set $\left\{(\xi,\eta):\ \xi_x \le \eta_x,\ \forall x \in L_X\right\}$.

In this paper, we discuss the most popular ferromagnetic model that is the stochastic Ising model. First we discuss the fluctuations of two-dimensional Ising model. Since the S.O.S. model is the simple case of Ising model, so our results of Ising model in this paper can be done similar to the S.O.S. model. Let Z^2 be the usual two-dimensional square lattice with sites $u=\left(u_1,u_2\right)$, equipped with the l_1-norm:

$$\|u\| = |u_1| + |u_2|.$$

Given $\Lambda \subset Z^2$, $\Omega_\Lambda = \{-1,+1\}^\Lambda$ is the configuration space. An element of $\Omega_\Lambda = \{-1,+1\}^\Lambda$ will usually denote by $\xi_\Lambda = \{\xi(u):\ u \in \Lambda\}$. Whenever confusion does not arise, we will also omit the subscript Λ in the notation ξ_Λ. Given a boundary condition τ, we consider the Hamiltonian

$$H^{\tau}_{\Lambda}(\xi) = -\frac{1}{2}\sum_{\substack{(u,u')\in\Lambda\times\Lambda \\ \|u-u'\|=1}} (\xi(u)\xi(u') - 1) - \sum_{\substack{(u,u')\in\Lambda\times\Lambda^c \\ \|u-u'\|=1}} (\xi(u)\tau(u') - 1).$$

The Gibbs measure associated with the Hamiltonian is defined as

$$\mu^{\beta,\tau}_{\Lambda}(\xi) = \exp\left[-\beta H^{\tau}_{\Lambda}(\xi)\right] \Big/ \sum_{\xi\in\Omega_\Lambda} \exp\left[-\beta H^{\tau}_{\Lambda}(\xi)\right]$$

where β is a parameter. Note that we use $\mu^{\beta,\tau}_{\Lambda}(\xi)$ (not P) to denote the probability measure for the Ising model. The stochastic dynamics which is studied in the present paper is defined by the Markov generator

$$\left(A^{\beta,\tau}_{\Lambda} f\right)(\xi) = \sum_{u\in\Lambda} c(u,\xi)\left[f(\xi^u) - f(\xi)\right]$$

acting on $L^2(\Omega_\Lambda, d\mu^{\beta,\tau}_\Lambda)$, where $\xi^u = +\xi(v)$, if $v \ne u$, and $\xi^u = -\xi(v)$, if $v = u$. $c(u,\xi)$ is the transition rates for the process ([9-11]), satisfying nearest neighbour interactions, attractivity, boundedness and detailed balance condition

$$c(u,\xi)\ \mu^{\beta,\tau}_{\Lambda}(\xi) = c(u,\xi^u)\ \mu^{\beta,\tau}_{\Lambda}\left(\xi^u\right).$$

Let $Q_{l,m} \subset Z^2$ be a rectangle of side length $2l$ (horizontal size) and $2m$. For the two-dimensional Ising model (see [9-11]), by using the techniques of correlation functions for estimating the fluctuation of phase separation (or interface) line, when $\beta > \beta_c^{I\sin g}$ ($\beta_c^{I\sin g}$ is the critical point of Ising model), we can prove that, with probability larger than $1-\exp\left[-c_1(\beta)\ln l\right]$, the interface has a height less than $c(\beta)\left(l\ln l\right)^{1/2}$, where l large enough and $c_1(\beta), c(\beta)$ are positive constants. Let $\Omega_{Q_{l,m}}$ be the configuration space of the Ising model, and $\mu^{\tau}_{Q_{l,m}}$ be the corresponding Gibbs measure with the boundary condition τ, where τ is defined by

$$\tau = \begin{cases} -1 & \text{if} \quad u_2 \ge m+1 \\ +1 & \text{if} \quad u_2 \le m \end{cases}$$

for $u = \left(u_1,u_2\right) \in Z^2$.

Let Z^2_* be the dual lattice of Z^2, i.e., $Z^2_* = Z^2 + (1/2,1/2)$. For $u,v \in R^2$, let $[u,v]$ be the closed segment with u,v as its endpoints. The edges of Z^2_* (Z^2) are those $e=[u,v]$ with u,v nearest neighbours in Z^2_* (Z^2). Given an edge e of Z^2, e^* is the unique edge in Z^2_* that intersects e. We denote by B_Λ the set of edges such that both endpoints are in Λ, and by $\overline{B}_\Lambda$ the set of all edges with at least one endpoint in Λ. Given $\Lambda \subset Z^2$, we let $\Lambda^c = Z^2 \setminus \Lambda$ and define Λ^* as the set of all $u \in Z^2_*$ such that $d(u,\Lambda) = 1/\sqrt{2}$, where

$$d(u,\Lambda) = \inf\{|u-v|:\ v\in\Lambda\}.$$

The set of the dual edges is defined as $B^*_\Lambda = \{e^*:\ e \in \overline{B}_\Lambda\}$. The interior and exterior boundaries of Λ are defined by

$$\partial_{\text{int}}\Lambda = \{u\in\Lambda:\ \exists v\notin\Lambda,\ |u-v|=1\}$$

$$\partial_{ext}\Lambda = \{u\notin\Lambda:\ \exists v\in\Lambda,\ |u-v|=1\}$$

and $\partial_{\text{int}}\Lambda^*$, $\partial_{ext}\Lambda^*$ are defined in the similar way.

For simplicity, we call an edge in Z^2_* by a bond, so that we can distinguish it from edges in Z^2. We say that a neighbouring pair u and v in Z^2 are separated by a bond e^* if the edge $e=[u,v]$ intersects e^*. Let $\Lambda \subset Z^2$ and $\tau \in \{-1,0,+1\}^{Z^2}$ be fixed, for every configuration $\xi \in \Omega_\Lambda$, we denote by $\Gamma(\xi)$ the collection of all bonds separating neighbouring sites u and v such that:

(i) $u,v\in\Lambda$, and $\xi(u)\xi(v)=-1$ or

(ii) $u\in\Lambda$, $v\in\partial_{ext}\Lambda$ and $\xi(u)\tau(v)=-1$.

We divide $\Gamma(\xi)$ into connected components. Further we use the convention that any pair of orthogonal bonds that intersect in a given site u^* of the dual lattice Z_*^2 are a linked pair of bonds iff they are both on the same side of the forty-five degrees line across u^*, then we regard that two linked pairs at u^* are not connected at u^*. By this convention, each connected component of $\Gamma(\xi)$, say Γ, has the following properties:

(i) if $u^* \in \Lambda^* \setminus \partial_{\text{int}} \Lambda^*$, then the number of bonds in Γ that intersect u^* is always even;

(ii) bonds in Γ can be ordered as $e_0^*, e_1^*, \cdots, e_n^*$, so that e_i^* and e_{i+1}^* have a common vertex for every i, and if Γ has a point u^* at which 4 bonds in Γ which intersect u^*, then there are $i \neq j$ such that these 4 bonds are divided into two linked pairs $\{e_i^*, e_{i+1}^*\}$ and $\{e_j^*, e_{j+1}^*\}$.

We call these components of $\Gamma(\xi)$ by contours in ξ (with boundary condition τ). If for any $u^* \in Z_*^2$, the number of bonds in the contour Γ which intersect u^* is even, then we call Γ a closed contour. A contour which is not closed is called by an open contour. The length $|\Gamma|$ of a contour is simply the number of bonds in Γ.

Now we give the following Lemma 3 and Lemma 4. They are important for us to estimate the heights of the interfaces.

Lemma 3 For the two-dimensional stochastic Ising model, let $Q_{l,m}$ be defined as above and let $\beta > \beta_c^{I\sin g}$. For some $\bar{k}(\beta) > 0$, set $m = [\bar{k}(\beta) l^{1/2} (\ln l)^{1/2}]$, $k = [\bar{k}(\beta) l^{1/2} (\ln l)^{1/2} / 10]$. Suppose that

$$Q_k = \{(u_1, u_2) \in Q_{l,m}: \ u_2 \le m - 3k\},$$

then there are $c_1(\beta) > 0$ and $l_0 = l_0(\beta) > 0$ independent of $Q_{l,m}$, such that for all $l > l_0$ and $u \in Q_k$, we have

$$\mu_{Q_{l,m}}^{\tau}\left(\left(F_{Q_{l,m}}^{\tau}\right)^c\right) \le \exp\left[-c_1(\beta) \ln l\right]$$

where $F_{Q_{l,m}}^{\tau}$ is the event

$$F_{Q_{l,m}}^{\tau} = \left\{\xi \in \Omega_{Q_{l,m}} : \Gamma_{open}^{\tau}(\xi) \subset \{u \in Q_{l,m} : u_2 \ge \frac{13m}{16}\}\right\}$$

and $\Gamma_{open}^{\tau}(\xi)$ denote those open contours produced by the configuration $\xi \in \Omega_{Q_{l,m}}$ with boundary condition τ on $Q_{l,m}$.

Proof. The proof of Lemma 3 depends on the estimates of the heights of the interfaces for the Ising model. By the Lemma 6.10 of [8], for $\beta > \beta_c^{I\sin g}$ and some large constant $M > 0$, when l is large enough we have

$$\mu_{Q_{l,m}}^{\tau}\left(\Gamma_{open}^{\tau}(\xi) \not\subset S(A, B: \ M \ln l)\right) \le \exp\left[-\kappa(\beta, M) \ln l\right] \quad (4)$$

where $A = (-l, m)$, $B = (l, m)$, and $\kappa(\beta, M) > 0$ is a positive parameter, let

$$S(A, B: \ M \ln l) = \left\{u \in Q_{l,m} : |u - A| + |u - B| \le |A - B| + M \ln l\right\}.$$

According to the definition of $Q_{l,m}$ and Q_k, by the computation of $S(A, B: \ M \ln l)$ and above (4), the fluctuations of phase separation line occur on a scale $l^{1/2} (\ln l)^{1/2}$, that is, there are $\bar{k}(\beta) > 0$, $c_1(\beta) > 0$ (dependent on M) such that

$$\mu_{Q_{l,m}}^{\tau}\left(\Gamma_{open}^{\tau}(\xi) \not\subset \{u \in Q_{l,m}: \ u_2 \ge 13m/16\}\right) \le \exp\left[-c_1(\beta) \ln l\right].$$

This inequality proves the inequality of Lemma 3.

Lemma 4 For the two-dimensional stochastic Ising model, let $Q_{l,m}$ be defined as above and let $\beta > \beta_c^{I\sin g}$. For some $\bar{k}(\beta) > 0$, set $m = [\bar{k}(\beta) l^{1/2} (\ln l)^{1/2}]$, $k = [\bar{k}(\beta) l^{1/2} (\ln l)^{1/2} / 10]$. Suppose that

$$Q_k = \{(u_1, u_2) \in Q_{l,m}: \ u_2 \le m - 3k\},$$

then there are $c_1(\beta) > 0$ and $l_0 = l_0(\beta) > 0$ independent of $Q_{l,m}$, such that for all $l > l_0$ and $u \in Q_k$, we have

$$\mu_{Q_{l,m}}^{+}(\xi(u) = 1) - \mu_{Q_{l,m}}^{\tau}(\xi(u) = 1) \le \mu_{Q_{l,m}}^{\tau}\left(\left(F_{Q_{l,m}}^{\tau}\right)^c\right)$$

where $F_{Q_{l,m}}^{\tau}$ is the event

$$F_{Q_{l,m}}^{\tau} = \left\{\xi \in \Omega_{Q_{l,m}} : \Gamma_{open}^{\tau}(\xi) \subset \{u \in Q_{l,m} : u_2 \ge \frac{13m}{16}\}\right\}$$

and $\Gamma_{open}^{\tau}(\xi)$ denote those open contours produced by the configuration $\xi \in \Omega_{Q_{l,m}}$ with boundary condition τ on $Q_{l,m}$. Further we have

$$\mu_{Q_{l,m}}^{+}(\xi(u) = 1) - \mu_{Q_{l,m}}^{\tau}(\xi(u) = 1) \le \mu_{Q_{l,m}}^{\tau}\left(\left(F_{Q_{l,m}}^{\tau}\right)^c\right) \le \exp\left[-c_1(\beta) \ln l\right]$$

where $\mu_{Q_{l,m}}^{+}$ is the Gibbs measure with the plus boundary condition on $Q_{l,m}$.

Proof. Let $F_{Q_{l,m}}^{\tau}$ be the event defined as above. We can write

$$\mu_{Q_{l,m}}^{\tau}\left(\xi(u)=1\right)=\mu_{Q_{l,m}}^{\tau}\left(\xi(u)=1 \mid F_{Q_{l,m}}^{\tau}\right)\mu_{Q_{l,m}}^{\tau}\left(F_{Q_{l,m}}^{\tau}\right) + \mu_{Q_{l,m}}^{\tau}\left((\xi(u)=1)\bigcap\left(F_{Q_{l,m}}^{\tau}\right)^{c}\right)$$

where $\left(F_{Q_{l,m}}^{\tau}\right)^{c}$ is just the complement event. By the FKG inequality

$$\mu_{Q_{l,m}}^{\tau}\left(\xi(u)=1 \mid F_{Q_{l,m}}^{\tau}\right)\mu_{Q_{l,m}}^{\tau}\left(F_{Q_{l,m}}^{\tau}\right) \geq \mu_{Q_{l,m}}^{+}\left(\xi(u)=1\right).$$

Then we have the difference

$$\mu_{Q_{l,m}}^{+}\left(\xi(u)=1\right)-\mu_{Q_{l,m}}^{\tau}\left(\xi(u)=1\right) \leq \mu_{Q_{l,m}}^{\tau}\left(\left(F_{Q_{l,m}}^{\tau}\right)^{c}\right).$$

Combing the result of Lemma 3, this inequality proves the inequality of Lemma 4.

Remark 3 Lemma 3 and Lemma 4 are proved for the two-dimensional stochastic Ising model, they describe the statistical properties of the interfaces of the Ising model. The simple case of this problem arises in the one-dimensional S.O.S. model. Through the similar arguments in the proof of Lemma 3 and Lemma 4, we can have the similar result as that of Lemma 3 and Lemma 4 for one-dimensional S.O.S. model, that is, the interfaces of S.O.S. model have a height less than $c(\beta)(l\ln l)^{1/2}$ with large probability.

In above Remark 3, we discuss the interface height for one-interface S.O.S. model. The aim of this paper is to study two random paths model and two interfaces S.O.S. model. From Section 1 to Section 3, we have studied the interface of the two random paths model conditioned on a fixed area in the intermediate layer and fixed end points. In this Section, by using Lemma 3, Lemma 4, and Remark 3, we study the relations between the two random paths model and the two interfaces S.O.S. model.

In the definitions of Section 1, with the starting points $\xi_0=0$ and $\eta_0=0$, we discussed the two random paths model with the partition function of $Z_{L,\beta}=\sum_{\xi,\eta}\exp\left[-\beta H_L(\xi,\eta)\right]$. While in this Section, we modify the end points of the model. Let $\Psi_{L_x}^{\xi,\eta}$ denote the event

$$\xi_0=\xi_L=0, \quad \eta_0=\eta_L=M(\beta)(L\ln L)^{\frac{1}{2}},$$

where $M(\beta)(>4c(\beta))$ is a large positive constant. The random paths $X_L^{\xi}\left(\frac{j}{L}\right)$, $X_L^{\eta}\left(\frac{j}{L}\right)$ are defined in Section 1, and let $\Upsilon_{L_x}^{\xi,\eta}$ denote the event that the random paths $X_L^{\xi}\left(\frac{j}{L}\right)$ and $X_L^{\eta}\left(\frac{j}{L}\right)$ don't intersect each other on L_x, then we have the following Lemma 5.

Lemma 5 For the two random paths model defined in Section 1, there are $c_2(\beta)>0$, $L_2=L_2(\beta)>0$ and $\beta_2>0$ such that for all $L>L_2$ and for all $\beta>\beta_2$,

$$P_{L,\beta}\left(\left(\Upsilon_{L_x}^{\xi,\eta}\right)^{c}\middle|\Psi_{L_x}^{\xi,\eta}\right)\leq 2\exp\left[-c_2(\beta)\ln L\right]$$

where $P_{L,\beta}$ is the corresponding probability measure for two interfaces S.O.S. model, which is defined in (3).

Proof. The proof of Lemma 5 follows directly from Lemma 3, Lemma 4, Remark 3 and the condition $M(\beta)>4c(\beta)$. This lemma shows that, with large probability, the two random paths don't intersect each other.

Let

$$\left(X_L^{\xi}(t),X_L^{\eta}(t)\right)^{*}=\left(\left(X_L^{\xi}(t),X_L^{\eta}(t)\right)\middle|\Upsilon_{L_x}^{\xi,\eta},\Psi_{L_x}^{\xi,\eta}\right)$$

and $P_{L,\beta}^{*}=P_{L,\beta}\left(\cdot\middle|\Upsilon_{L_x}^{\xi,\eta},\Psi_{L_x}^{\xi,\eta}\right)$ be the corresponding conditional probability distribution of the random process $\left(X_L^{\xi}(t),X_L^{\eta}(t)\right)^{*}$. According to above preparation and Lemma 5, we have the following Corollary 2.

Corollary 2 With the same conditions of Lemma 5, we have the following

$$\lim_{L\to\infty}\left\{P_{L,\beta}\left(\left(X_L^{\xi}(t),X_L^{\eta}(t)\right)\in G\middle|\Psi_{L_x}^{\xi,\eta}\right) - P_{L,\beta}^{*}\left(\left(X_L^{\xi}(t),X_L^{\eta}(t)\right)\in G\right)\right\}=0$$

where $G=[a_1,a_2]\times[b_1,b_2]$, $-\infty<a_i<b_i<\infty$, for $i=1,2$.

From the definition of the process $\left(X_L^{\xi}(t),X_L^{\eta}(t)\right)^{*}$, it is known that the process $\left(X_L^{\xi}(t),X_L^{\eta}(t)\right)^{*}$ is a conditional two interfaces S.O.S. model (with the special fixed end points). Corollary 2 shows a limiting relation between the two random paths model and the conditional two interfaces S.O.S. model. This result is useful to study the asymptotic properties of the two interfaces S.O.S. model by using the results of two random paths model, for example, we consider the two interfaces S.O.S. model with a large fixed area between the two interfaces, etc.

5 Conclusion

In this paper, we studied the statistical properties of the two random interfaces model. Under some conditions, that there is a specified value of the large area in the intermediate region of the two random interfaces, Theorem 1 shows the weak convergence of the fluctuations for the two random interfaces. In Section 4, the research results in Section 1-3 are extended and improved for the two interfaces S.O.S. model. The results of the present paper can also be applied to other fields, for example, see [13-15].

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