Weights, inequalities and a local Hölder norm for solutions to (∂/∂t-L)(u)=divf on bounded domains

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Abstract: The rate of change of u, a solution to Lu=divf in a bounded, rough domain Ω_T, u=g on $\partial_p\Omega_T$, is investigated using a local Hölder norm of u and different measures on Ω_T and on $\partial_p\Omega_T$. Results are discussed for both L a strictly elliptic operator and for L=∂/∂t-L_0, with L_0 a strictly parabolic divergence form operator; the coefficients are bounded and measurable, and in the case of L_0, time dependent.

Key words: elliptic, parabolic equations, Lipschitz domains, Borel measures, kernels, Hölder norms.

1 Introduction

The question of the rate of change of a temperature function or of a potential function in a limited environment is of fundamental importance in many applications of mathematics. The focus of this paper is to answer one part of the following general question: For which measures, μ, η, and νdω, where μ and η are Borel measures on a bounded domain Ω in Euclidean space R^{d}, d>2, or R^{d+1},d≥ 2, and νdω is a measure on ∂Ω, so that, for a solution u to the second order boundary value problem,

$$(1) \qquad Lu(x) = \mathrm{div}(\vec{g})(x) \quad \text{for } x \in \Omega$$
$$u(z) = f(z) \quad \text{for } z \in \partial\Omega$$

the following inequality is valid:

$$(2) \qquad \left(\int_\Omega \|u(x)\|^q d\mu(x)\right)^{1/q} \le$$
$$C\left(\int_\Omega \left(|\vec{g}(x)|^r + |\mathrm{div}\,\vec{g}(x)|^r\right) d\eta(x)\right)^{1/r}$$
$$+\, C\left(\int_{\partial\Omega} |f(z)|^p \nu(z)\, d\omega(z)\right)^{1/p}.$$

for as large a range of indices q, p, and r as possible. $\|u\|_{H^\alpha}(x)$ denotes either the gradient of u, $|\nabla u(x)|$, or a local Hölder norm for u at x (see the definition below). The functions f and g are assumed to be in some test class: say, f belongs to the space $L^\infty(\partial\Omega)$ and g is in the usual Sobolev space $H^1(\Omega)$. The boundary measure νdω is composed of a non-negative, locally integrable weight function ν(x′) multiplied by dω, the "harmonic measure" generated by the operator L on the domain.

The operators I will be considering will be either strictly elliptic divergence form operators, i.e.

$$L = \sum_{i,j-1}^{d} \frac{\partial}{\partial x_i}\left(a_{i,j}(x)\frac{\partial}{\partial x_j}\right)$$

or their parabolic counterpart, (∂/∂t)-L, with

$$L = \sum_{i,j-1}^{d} \frac{\partial}{\partial x_i}\left(a_{i,j}(x,t)\frac{\partial}{\partial x_j}\right)$$

where $(1/\lambda)|\xi|^2 \leq \sum_{i,j=1}^{d} \xi_i a_{i,j} \xi_j \leq \lambda|\xi|^2$ for a constant $\lambda \geq 1$. The coefficients $a_{(i,j)}(x)$, $a_{(i,j)}(x,t)$ are symmetric, bounded and measurable.

A standard method for finding solutions to this equation is to break it up into two different problems and then to use superposition to obtain the solution to the original equation. The two problems are: 1. the Dirichlet problem, namely to find a solution u to (DP)

$$Lu(x) = 0 \quad \text{for } x \in \Omega$$
$$u(w) = g(w) \quad \text{for } w \in \partial\Omega$$

and 2. Poisson's equation, namely to find a solution v(x) to (PE)

$$Lu(x) = \operatorname{div}(\vec{f})(x) \quad \text{for } x \in \Omega$$
$$u(w) = 0 \quad \text{for } w \in \partial\Omega$$

For the Dirichlet problem one has only to consider the two measures, μ on Ω, and $\nu d\omega$ the measure on the boundary of Ω. For the second problem, solving the inhomogeneous equation with zero boundary data, one has only the two Borel measures on Ω, μ and η, to deal with.

In the case of Ω being a classical domain, i.e. for the unit disk or the upper half space, the question for the Dirichlet problem has been thoroughly investigated. In 1995 for solutions to the heat equation and for harmonic functions in the upper half space, Wheeden and Wilson [37] proved necessary and sufficient conditions on a Borel measures, μ, defined in $R^{(d+1)}$, and a non-negative weight $\nu(x')$, so that $\nu(x')\,dx'$ defines a measure on R^d, and

$$\left(\int_{\mathbb{R}_+^{d+1}} |\nabla u(x)|^q d\mu(x)\right)^{1/q} \leq$$

$$C\left(\int_{\mathbb{R}^d} |f(x')|^p \nu(x') dx'\right)^{1/p}$$

for any solution to the Dirichlet problem Lu=0 in $R^{(d+1)}$, u=f on R^d. Here $L=\Delta$ or $\partial/\partial t-\Delta$, and $1<p\leq q<\infty$ with $q\geq 2$. In other words they solved the problem in the case of harmonic functions, considering each partial derivative separately, on the domain R_+^{d+1}, for $1<p\leq q<\infty$ and $q\geq 2$. Wheeden and Wilson used the dual operator approach, which was again employed by Sweezy and Wilson when they considered extending the situation to harmonic functions on Lipschitz domains [31]. Prior to the time Wheeden and Wilson proved their theorems for the upper half space, work of Luecking, Shirokov, Verbitsky and Videnskii had completely characterized the measures μ for which one could obtain the weighted norm inequality for harmonic functions with $\nu d\omega=ds$, ds being the surface measure on $\partial\Omega$ when Ω is the upper half space [15], [16], [21], [22], [34], [35]. Their results cover all indices $0<p,q<\infty$; f's L^p norm must be replaced by the H^p Hardy space norm if $0<p\leq 1$. Since that time, J. M. Wilson and the author have investigated the question of characterizing measures for which one can prove a weighted norm inequality of the form (2) on rough boundary domains, such as Lipschitz domains and Lip(1,1/2) domains, and for a wider range of second order partial differential equation solutions [25], [31], [32], [33].

Their work depends on classical topics in harmonic analysis such as A^{p} weights and Littlewood-Paley theory, [38], [36], [23], [12],[13], [39], [30] as well as the basic existence and uniqueness of kernel functions and estimates of these kernel functions for solutions to the kind of Dirichlet problem described above. The theory of A^{p} weights, as it gives weighted inequalities for maximal functions and singular integrals, was originated by Benjamin Muckenhoupt [18] and further developed by Coifman and Fefferman [3]. The main other sources for background material that is used (and often assumed especially in local estimates for the elliptic and/or parabolic solutions) are contained in [2], [4], [5], [6], [10], [11], [14], [19].

One needs other methods to handle the case of more general operators of the form $\partial/\partial t - L$, with

$$L = \sum \partial/\partial x_i (a_{i,j}(x,t)\partial/\partial x_j + b_i)$$

$$+ c_i \partial/\partial x_i + c_0$$

Here u(x,t) is a weak solution to the inhomogeneous equation with zero boundary data, i.e., for any test function $\psi(x,t)$, and any τ such that for

$\tau \in \{t : \Omega_T \cap \{(x,t) \in \mathbb{R}^{d+1}\} \neq \varnothing\}$, and

$\Omega(\tau) = \Omega_T \cap \{t : 0 < t < \tau \leq T\}$, (see Section 2 for other definitions), we have

$$\int_{T\Omega(\tau)} u\psi - \int_{B\Omega(\tau)} u\psi - \int_{\Omega(\tau)} u(\partial\psi/\partial t) +$$

$$\int_{\Omega(\tau)} \partial u/\partial x_j a^{i,j}(x,t)\partial\psi/\partial x_i +$$

$$\int_{\Omega(\tau)} b^i u(\partial\psi/\partial x_i) - c^i\psi(\partial u/\partial x_i) - c_0\psi u$$

$$= -\int_{\Omega(\tau)} \left(\vec{f} \cdot \nabla\psi - g\psi\right).$$

We also require that u(x,t) lie in the closure of $C_S^1(\Omega_T)$, the space of functions defined and continuous on the closure of Ω_T which vanish on the lateral boundary of the domain, under the norm,

$$\sup_{t_0<\tau<t_1} \int_{\omega(\tau)} |u(x,t)|^2 dx +$$

$$\int_{\Omega(\tau)} |\nabla u(x,t)|^2 dxdt \leq C_0 < \infty,$$

where

$\omega(\tau) = \{t = \tau\} \cap \Omega_T$,

$t_0 = \inf\{t : \Omega \cap \{(x,t) \in \mathbb{R}^{d+1}\} \neq \varnothing\}$, and

$t_1 := \sup\{t : \Omega \cap \{(x,t) \in \mathbb{R}^{d+1}\} \neq \varnothing\}$.

In [29] a reverse Hölder argument is employed to obtain a higher order integrability for the gradient of such a solution [8] [9]. It follows from these results that weights of the form $\delta(x,t)^\gamma$ can be introduced to the norm inequalities without difficulty. (See also [28].)

In the present paper solutions to the inhomogeneous equation Lu=divf in Ω_T, $u|_{\partial p\Omega T}=0$ will be investigated for operators

$$L = \partial/\partial t - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{i,j}(x,t)\frac{\partial}{\partial x_j})$$

as described above. In [26] the question of finding conditions on two measures μ and ν, defined on a domain Ω in R^d, to give the inequality

$$\left(\int_\Omega (|\nabla u(x)|)^q d\mu(x)\right)^{1/q} \leq$$

$$C\left(\int_\Omega \left(\left|\vec{f}(x)\right|^p + \left|div\vec{f}(x)\right|^p\right) dv(x)\right)^{1/p}$$

was introduced in the case of u(x) being a solution to an elliptic equation, Lu=divf. It was shown that a condition involving a singular potential of the measure μ gives the same kind of norm inequality for a local Hölder norm replacing $|\nabla u(x)|$. The present paper starts to investigate what can be proved for a solution to the inhomogeneous parabolic equation on rough domains in $R^{(d+1)}$. Theorem C and its proof (given in Section 3) is the major result of this paper. (The result of Theorem C was announced in a lecture given at the Fourth International Conference of Applied Mathematics in Plovdiv, Bulgaria, August 2007 [24].) Theorem C contains a condition for the measures μ and ν for solutions to

$$(\partial/\partial t - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{i,j}(x,t)\frac{\partial}{\partial x_j}))u(x,t) =$$

$$div\vec{f}(x,t), (x,t) \in \Omega_T,\ u|_{\partial_p\Omega_T} = 0$$

which guarantees the validity of a norm inequality similar to the one above. The condition is analogous to the condition mentioned above for solutions to the elliptic operator equation; it involves the convolution of $d\sigma=((d\nu/(dxdt))^{(1-p')}$ with a singular kernel. In the present paper we intend to present the proof of the theorem, after describing the setting in which it occurs.

The other half of the problem, namely finding conditions on weights μ on Ω_T and $\nu d\omega$ on the parabolic boundary of Ω_T, $\partial_p \Omega_T$, (this means the lateral boundary and the bottom part of the boundary) so that

$$\left(\int_{\Omega_T} \|u(x,t)\|^q d\mu(x,t)\right)^{1/q} \le$$

$$C\left(\int_{\partial_p\Omega_T} |f(z,\tau)|^p \nu(z,\tau) d\omega(z,\tau)\right)^{1/p}$$

holds for solutions to the Dirichlet problem

$$(\partial/\partial t - L)u(x,t) = 0 \quad (x,t) \in \Omega_T$$
$$u(z,\tau) = f(z,\tau) \quad (z,\tau) \in \partial_p\Omega_T.$$

will be discussed in the last section of the paper, along with open problems. The Hölder norm $\|u(x)\|_{H^\alpha}$ is defined by

$$\|u(x,t)\| = \|u(x,t)\|_{H^\alpha_{loc}} =$$

$$\sup_{(y,s)\in P_{\delta/100}(x,t)} \frac{|u(y,s)-u(x,t)|}{\left(|x-y|+|t-s|^{1/2}\right)^\alpha},$$

where

$$P_{\delta/100}(x,t) = \{(y,s) : d_p(y,s;x,t)$$
$$= |x-y|+|t-s|^{1/2} < \delta(x,t)/100\},$$

and

$$\delta(x,t) = d_p(x,t;\partial_p\Omega_T) =$$
$$\inf\{\left(|x-y|+|t-s|^{1/2}\right), (y,s) \in \partial_p\Omega_T\}.$$

2 Problem Formulation

To state Theorem C we give some background information. First W will denote the collection of certain Whitney-type parabolic dyadic cubes (these are dyadic cubes whose dimension compares with the cube's distance from the boundary of Ω_T, and whose dimension in the time direction is the square of its space dimension) that lie in Ω_T. These cubes have the property that their interiors are pairwise disjoint; a fixed dilate of any cube will also be Whitney-type with respect to Ω_T, and

$$\Omega = \bigcup_{Q_j \in W} \overline{Q_j}$$

The measures μ and ν will be taken to be Borel measures defined on Ω_T, with ν absolutely continuous with respect to Lebesgue measure.

The operators under consideration are second order divergence form whose coefficients are symmetric, bounded and measurable.

$$L = \frac{\partial}{\partial t} - L = \frac{\partial}{\partial t} - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{i,j}(x,t)\frac{\partial}{\partial x_j})$$

with the existence of a constant $\lambda>0$ such that

$$\frac{1}{\lambda}|\xi|^2 \le \sum \xi_i a_{i,j}(x,t)\xi j \le \lambda|\xi|^2$$

for all (x,t) in Ω_T.

Points in $R^{(d+1)}$ are denoted by (x',x_d,t) with x' in $R^{(d-1)}$, x_d in R^1, and t in R^1. The parabolic metric is given by

$$d_p(x,t;y,s) = \left(|x-y| + |t-s|^{\frac{1}{2}}\right).$$

The domain Ω_T lying in $R^{(d+1)}$ will be taken to be a bounded domain whose boundary, $\partial_p\Omega_T$, consists of three parts, a top part, $T\Omega_T$,

$$T\Omega_T = \{(x,t) \in \overline{\Omega_T} : t = T\};$$

a bottom part, $B\Omega_T$,

$$B\Omega_T = B = \{(x,t) \in \overline{\Omega_T} : t = 0\},$$

and a lateral part, $S\Omega_T$,

$$S\Omega_T = S_T = \{(x,t) \in \partial\Omega_T : 0 < t < T\}.$$

The parabolic boundary of Ω_T is denoted by

$$\partial_p\Omega_T = S_T \cup B$$

The lateral boundary can be described locally as the graph of a Lipschitz function, which may have been rotated and/or translated. More precisely $\partial_p\Omega_T$ can be covered by finitely many cylinders

$$\Psi'_R(z,\tau) = \{(x,t) : |x-z| < R, |\tau - s| < R^2\},$$

with (z,τ) lying in $\partial_p\Omega_T$; if (z,τ) is in S_T, then

$$\Psi'_R(z,\tau) \cap S_T = \{(w,\sigma) : w_d = \psi(w',\sigma)$$

where

$$|\psi(y',s) - \psi(x',t)| \le M(|y'-x'| + |t-s|^{1/2})\}.$$

The constant M>0 is called the Lipschitz constant of the domain Ω_T. It is best to think of the domain Ω_T as being a finite part of a larger domain Ω that is infinite in the time variable, so that

$$\Omega_T = \Omega \cap \{0 < t < T\}.$$

Every point P on $\partial\Omega$ satisfies the condition that there is a polygonal curve γ lying completely inside Ω. The curve starts at the point P and has a strictly increasing time coordinate. For the kinds of operators we will be considering, $\partial/\partial t - L$ as described above, Aronson [1] (see also [5]) proved the existence and uniqueness of the fundamental solution $\Gamma(x,t;y,s)$ on $R^{(d+1)}$. The Green function for the operator on a given domain Ω_T can be taken to be

$$G(x,t;y,s) =$$

$$\Gamma(x,t;y,s) - \int_{\partial_p\Omega_T} \Gamma(w,\tau;y,s)\, d\omega^{(x,t)}(w,\tau);$$

$\omega^{(x,t)}$ is the ("harmonic") measure on sets in $\partial_p\Omega_T$ induced by the operator $\partial/\partial t - L$, taken at the point (x,t). It is not hard to see that the Green function of Ω, when it is restricted to Ω_T, will be identical to the Green function on Ω_T (see [20]).

The Hölder norm of u is defined as

$$\|u\|_{H^\alpha}(x,t) = \sup_{(w,\tau)\in P_{\frac{\delta(x,t)}{100}}} \frac{|u(x,t)-u(w,\tau)|}{\left(|x-w|+|t-\tau|^{1/2}\right)^\alpha}$$

with $\delta(x,t)$ being the parabolic distance of the point (x,t) from the parabolic boundary of Ω_T. The region $P_{(\delta(x,t)/100)}$ is defined as

$$P_{\frac{\delta(x,t)}{100}} = \{(y,s) : d_p(x,t;y,s) < \tfrac{1}{100}\delta(x,t)\}$$

With W being the collection of parabolic dyadic regions in Ω_T described above, let us denote the fixed dilate of any such cube Q_j, which is again a Whitney type cube in Ω_T, to be βQ_j, with $\beta>1$.

3 Problem Solution

We will be assuming that any solution to the inhomogeneous boundary value problem,

$$(\partial/\partial t - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{i,j}(x,t)\frac{\partial}{\partial x_j}))u(x,t) = div\vec{f}(x,t)$$

$$(x,t) \in \Omega_T,\ u|_{\partial_p\Omega_T} = 0$$

has the representation

$$u(x,t) = \int_{\Omega_T} \mathrm{div}\,\vec{f}(y,s)G(x,t;y,s)dyds.$$

Theorem C: *Suppose that u(x,t) is a solution to*

$$(\partial/\partial t - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{i,j}(x,t)\frac{\partial}{\partial x_j}))u(x,t) = div\vec{f}(x,t)$$

$$(x,t) \in \Omega_T,\ u|_{\partial_p\Omega_T} = 0$$

with Ω_T as described above. Let μ and ν be Borel measures defined on Ω_T such that ν is absolutely continuous with Lebesgue measure. Let $d\sigma(x,t)=(dv)/dxdt)(x,t))^{(1-p')}dxdt$. If there is a constant $C_0>0$ such that

$$\mu(Q_j)^{1/q} \sup_{(x,t)\in\beta Q_j}\left(\int_{\Omega_T} \frac{d\sigma(y,s)}{\left(|x-y|+|t-s|^{1/2}\right)^{(d+\alpha)p'}}\right)^{1/p'}$$

$$\leq C_0|Q_j|^{1/q}$$

for all cubes Q_j in W, then for all $0<q<\infty$ and $1<p<\infty$, there is a constant $C>0$, independent of u and f, so that

$$\left(\int_{\Omega_T} \|u\|^q_{H^\alpha}(x,t)d\mu(x,t)\right)^{1/q} \leq$$

$$C\left(\int_{\Omega_T}\left|\mathrm{div}\,\vec{f}(y,s)\right|^p dv(y,s)\right)^{1/p}.$$

Proof: (see [26] Theorem 3): In the following argument we use the estimate on the Green function:

$$|G(x,t;y,s) - G(w,\tau;y,s)| \leq C \cdot$$

$$d_p(x,t;w,\tau)^\alpha\left(\frac{1}{d_p(x,t;y,s)^{d+\alpha}} + \frac{1}{d_p(w,\tau;y,s)^{d+\alpha}}\right).$$

This estimate can be proved by using Moser iteration and geometric decay of the Green function if (x,t) and (w,τ) are in a Whitney-type region whose dimension is also comparable to its distance from the pole (y,s). In this paper we only need the estimate if

$$(x,t) \in Q_j, (w,\tau) \in P_{\delta(x,t)}(x,t).$$

So if

$$(y,s) \in \Omega_T \backslash \beta' Q_j$$

($\beta'>\beta$ by a fixed amount) this will be the case. Also if (y,s) lies in $\beta'Q_j$, but if $d_p(x,t;y,s)\geq 0.05\delta(x,t)$ (and this implies that $d_p(w,\tau;y,s)\geq 0.04\delta(x,t)$ and that $d_p(x,t;y,s)> d_p(x,t;w,\tau)$), again Moser iteration gives the estimate. Lastly if $d_p(x,t;y,s)\leq C\ d_p(x,t;w,\tau)$, then $d_p(x,t;w,\tau)/ d_p(x,t;y,s))^\alpha \geq C^{-\alpha}$. Assuming that $G(x,t;y,s)\geq G(w,\tau;y,s)$ means that

$$|G(x,t;y,s) - G(w,\tau;y,s)| \leq G(x,t;y,s) \leq$$

$$C(\alpha,d,\lambda)\left(\frac{d_p(x,t;w,\tau)}{d_p(x,t;y,s)}\right)^\alpha\left(\frac{1}{d_p(x,t;y,s)^d}\right) \leq$$

$$C \cdot (d_p(x,t;w,\tau))^\alpha\left(\frac{1}{d_p(x,t;y,s)^{d+\alpha}} + \frac{1}{d_p(w,\tau;y,s)^{d+\alpha}}\right).$$

If $G(w,\tau;y,s)\geq G(x,t;y,s)$, a symmetric argument gives the same upper bound. (Other cases can be dealt with by using elementary estimates and the adaptation of Moser's techniques [17] to regions on the boundary of Ω_T introduced by Fabes and Safonov, and Nystrom [7], [20].)

Writing

$$\int_{\Omega_T} \|u\|^q_{H^\alpha}(x,t)d\mu(x,t) =$$

$$\sum_{Q_j \in \mathcal{W}} \int_{Q_j} \left(\sup_{(w,\tau)\in P_{\frac{\delta(x,t)}{100}}} \frac{|u(x,t)-u(w,\tau)|}{\left(|x-w|+|t-\tau|^{1/2}\right)^{\alpha}} \right)^q d\mu(x,t) \le$$

$$\sum_{Q_j \in \mathcal{W}} \mu(Q_j) \cdot$$

$$\sup_{\substack{(x,t)\in Q_j \\ (w,\tau)\in\beta Q_j}} \left(\left| \int_{\Omega_T} \operatorname{div} \vec{f}(y,s) \frac{G(x,t;y,s)-G(w,\tau;y,s)}{\left(|x-w|+|t-\tau|^{1/2}\right)^{\alpha}} dyds \right|^q \right)$$

$$\le C' \sum_{Q_j \in \mathcal{W}} \mu(Q_j) \cdot$$

$$\sup_{(x,t)\in\beta Q_j} \left(\int_{\Omega_T} \left| \operatorname{div} \vec{f}(y,s) \right| \frac{2dyds}{\left(|x-y|+|t-s|^{1/2}\right)^{d+\alpha}} \right)^q$$

$$\le C'' \sum_{Q_j \in \mathcal{W}} \mu(Q_j) \left(\int_{\Omega_T} \left| \operatorname{div} \vec{f}(y,s) \right|^p d\nu(y,s) \right)^{\frac{q}{p}} \cdot$$

$$\sup_{(x,t)\in\beta Q_j} \left(\int_{\Omega_T} \left(\frac{d\sigma(y,s)}{\left(|x-y|+|t-s|^{1/2}\right)^{(d+\alpha)p'}} \right) \right)^{\frac{q}{p'}}.$$

Consequently

$$\left(\int_{\Omega_T} \|u\|^q_{H^\alpha}(x,t) d\mu(x,t) \right)^{\frac{1}{q}} \le$$

$$C \left(\int_{\Omega_T} \left| \operatorname{div} \vec{f}(y,s) \right|^p d\nu(y,s) \right)^{\frac{1}{p}} \cdot (\mathcal{S})^{\frac{1}{q}}$$

With $\mathcal{S}$ equal to the sum

$$\sum_{Q_j \in \mathcal{W}} \mu(Q_j) \cdot$$

$$\sup_{(x,t)\in\beta Q_j} \left(\int_{\Omega_T} \frac{d\sigma(y,s)}{\left(|x-y|+|t-s|^{1/2}\right)^{(d+\alpha)p'}} \right)^{\frac{q}{p'}}$$

The condition given in Theorem C implies that

$$\mu(Q_j) \cdot \sup_{(x,t)\in\beta Q_j} \left(\int_{\Omega_T} \frac{d\sigma(y,s)}{\left(|x-y|+|t-s|^{1/2}\right)^{(d+\alpha)p'}} \right)^{\frac{q}{p'}}$$

$$\le |Q_j|,$$

and this means that the q[th] root of the expression

$$\sum_{Q_j \in \mathcal{W}} \mu(Q_j) \sup_{(x,t)\in\beta Q_j} \left(\int_{\Omega_T} \frac{d\sigma(y,s)}{\left(|x-y|+|t-s|^{1/2}\right)^{(d+\alpha)p'}} \right)^{\frac{q}{p'}}$$

is less than or equal to

$$C|\Omega_T|^{\frac{1}{q}}.$$

4 Conclusion

The question of what conditions on two weights will allow one to prove a norm inequality of the form stated in the Introduction has been extensively studied for solutions to the Dirichlet problem over the past 30 years. The companion result to Theorem C was proved in [27]. It is given below as Theorem A. Prior to proving Theorem A, the author and Wilson had proved a norm inequality [32] for the space gradient of the solution u(x,t) to

$$(\partial/\partial t - L)u(x,t) = 0 \quad (x,t) \in \Omega_T$$
$$u(z,\tau) = f(z,\tau) \quad (z,\tau) \in \partial_p\Omega_T.$$

Obtaining the norm inequality for $\|u\|_{H^\alpha}$ instead of $|\nabla u|$ was originally due to a suggestion of R.L.Wheeden. There are several advantages to using the Hölder norm defined in the Introduction instead of trying to deal with $|\nabla_x u(x,t)|$.One important reason is that with a Hölder norm one can gain control of the rate of change of the temperature function as it changes in time as well as its rate of change with respect to the space variable, x. Another reason is that, for the most general kind of operator whose solutions are amenable to our methods, namely a strictly parabolic, divergence form operator, L as described above, we were obliged to put additional restrictions on the range of the exponents, p and q, for which we could prove an inequality of the nature of (2) for $\|u(x,t)\| = |\nabla_x u(x,t)|$. We also had to assume an extra condition on the measure μ. (Remark: the additional restriction on p and q and the extra condition on μ are not necessary in dealing with solutions to the heat operator, ∂/∂t-Δ.)

Theorem A: *For Ω_T and ∂/∂t-L as described above, assume that u(x,t) is a weak solution of*

$$(\partial/\partial t - L)u(x,t) = 0 \quad (x,t) \in \Omega_T$$
$$u(z,\tau) = f(z,\tau) \quad (z,\tau) \in \partial_p\Omega_T.$$

with f(z,τ) in $L^\infty(\partial_p\Omega_T, d\omega)$, and $\omega = \omega^{(X_0,T)}$ being the parabolic measure on $\partial_p\Omega_T$ generated by the operator ∂/∂t-L, measured from the fixed point (X_0,T). Let μ be a Borel measure defined on Ω_T, and let ν be a non-negative weight defined on $\partial_p\Omega_T$ so that ν is locally integrable on $\partial_p\Omega_T$ with respect to the measure dω. Further assume that for $\sigma(z,\tau) \equiv (\nu(z,\tau))^{(1-p')}$, then σdω is an A^∞ measure with respect to dω. Suppose for all parabolic cubes Q_b on $\partial_p\Omega_T$, with T_{Qb} denoting the top half of the Carleson-type region associated to Q_b a boundary cube, with

$$\Psi_{Q_b}(z,\tau) = \omega(Q_b)\sum_{j=0}^{\infty} \frac{2^{-j(2\beta-\eta)}}{\omega(2^j Q_b))} \chi_{R_j(Q_b)}(z,\tau)$$

the following inequality is valid:

$$\mu(T(Q_b))^{1/q} \cdot$$
$$\left(\int_{\partial_p\Omega_T} (\Psi_{Q_b}(z,\tau))^{p'/2} \sigma(z,\tau) d\omega(z,\tau)\right)^{1/p'}$$
$$\leq \omega(Q_b) l(Q_b)^{2\alpha}.$$

Then there is a constant

$C = C(d,\lambda,\alpha,\beta,\delta,\eta,\Omega_T,r_0,p,q)$ so that for $1 < p \leq q < \infty$, $q \geq 2$, and $\Omega_{\{T,\delta\}} \equiv \{(x,t) \text{in } \Omega_T, \delta(x,t) < \delta\}$, the following inequality is valid:

$$\left(\int_{\Omega_{T,\delta}} \|u(x,t)\|^q_{H^\alpha_{loc}} d\mu(x,t)\right)^{1/q} \leq$$
$$C\left(\int_{\partial_p\Omega_T} |f(z,\tau)|^p \nu(z,\tau) d\omega(z,\tau)\right)^{1/p}.$$

Remark: It will be shown below that an analogous condition on $\Omega_T \setminus \Omega_{T,\delta}$ is sufficient to prove that

$$\left(\int_{\Omega_T \setminus \Omega_{T,\delta}} \|u(x,t)\|^q_{H^\alpha_{loc}} d\mu(x,t)\right)^{1/q} \leq$$
$$C\left(\int_{\partial_p\Omega_T} |f(z,\tau)|^p \nu(z,\tau) d\omega(z,\tau)\right)^{1/p};$$

Thus givng the norm inequality for the entire domain Ω_T.

To prove Theorem A one must first establish a Littlewood-Paley type norm inequality for functions of the form

$f(z,\tau) = \sum_{Q_b \in F} \lambda_{Q_b} \phi_{(Q_b)}(z,\tau)$. F is a finite family of "dyadic" parabolic boundary cubes (the ones mentioned in Theorem A) Q_b. The functions $\varphi_{Qb}(z,\tau)$ have certain decay, smoothness (Hölder continuity is enough) and cancellation properties that are essential to obtaining the square function result by the method employed in [27]. The φ_{Qb} depend on the kernel function for the operator L in the case of the Dirichlet problem or on the Green function of L and the domain in the case of the inhomogeneous equation.

Recently the author began to investigate what kinds of results could be obtained for solutions to Poisson's equation for the same kinds of second order operators on rough boundary domains. Results obtained for strictly elliptic operators on Lipschitz domains ([24], [25]) have indicated that it may be possible to prove similar theorems for solutions to parabolic operators on rough boundary domains. Theorem C is the first (and simplest) finding in this direction.

Future work will involve finding conditions on two measures so that one can prove a weighted norm inequality and a semi-discreet Littlewood-Paley type inequality in the setting that is appropriate for the generalized heat equation for solutions to the inhomogeneous parabolic boundary value problem stated at the beginning of Section 3. To prove sufficient conditions on μ and ν for a norm inequality that depends on a dual operator argument, for parabolic u, along the lines of what is known to work for elliptic operator solutions, one must establish estimates for the Green function. These estimates are proved in Gruter and Widman for the elliptic Green's function on a non-smooth domain [GW]. However, it is well-known that the capacity arguments used by Gruter and Widman are not valid in the case of parabolic operators of the type considered here. One can, however, obtain geometric estimates on the parabolic Green's function that are needed for a Littlewood-Paley type inequality from results proved by Kaj Nystrom [20]; so it is probable that a similar result can be established for parabolic Hölder norms on non-smooth domains. This is work in progress.

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