Varadhan estimates without probability: upper bound

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Abstract: We translate in semi-group theory our proof of Varadhan estimates for subelliptic Laplacians which was using the theory of large deviations of Wentzel-Freidlin and the Malliavin Calculus of Bismut type.

Key–Words: Large deviations. Subelliptic estimates.

1 Introduction

Let us consider some vector fields X_i on R^d with bounded derivatives at each order. We consider the Hoermander's type diffusion generator

$$L = X_0 + 1/2\sum_{i=1}^{m} X_i^2 \tag{1}$$

In (1), the vector fields are considered as first order differential operators. If we consider the vector fields as smooth section of the (trivial) tangent bundle of the linear space, we can consider the horizontal differential equation associated to them: h denotes a L^2 function from $[0, 1]$ into R^m and we consider the equation

$$dx_t(h) = \sum_{i=1}^{m} X_i(x_t(h))h_t^i dt \tag{2}$$

We put $\|h\|^2 = \sum_{i=1}^m \int_0^1 |h_t^i|^2 dt$ and we introduce the Carnot-Caratheodory distance $d(x, y)$ ([6], [36]) associated to the problem

$$d^2(x, y) = \inf_{x_0(h)=x, x_1(h)=y} \|h\|^2 \tag{3}$$

In the sequel we will do the following hypothesis:

Hypothesis (H1)$(x, y) \to d(x, y)$ is continuous. We put

$$E_1(x) = \{X_1(x), .., X_m(x)\} \tag{4}$$

$$E_{l+1}(x) = E_l(x) \cup_{i>0} [X_i(x), E_l(x)] \tag{5}$$

We do the strong Hormander's hypothesis in x:

Hypothesis (H2): $\cup E_l(x) = R^d$

L generates a diffusion semi-group P_t.

$$P_t \circ P_s f = P_{t+s} f \tag{6}$$

if f is a bounded continuous function on R. Moreover when $t \to 0$

$$\lim \frac{P_t f - f}{t} = Lf \tag{7}$$

if f has bounded derivatives at each order. Moreover it is a Markovian semi-group because L satisfies the maximum principle

$$P_t[f](x) = \int_{R^d} f(y)P_t(x, dy) \tag{8}$$

where $P_t(x, dy)$ is a probability measure.

Hoermander's theorem ([2], [7], [10], [33]) states that there is a heat-kernel associated to P_t:

$$P_t f(x) = \int_{R^d} p_t(x, y)f(y)dy \tag{9}$$

for any bounded continuous function f on R^d.

The goal of this paper is to prove again the following theorem:

Theorem 1 *(Léandre [14], [15])Under hypothesis (H1) and (H2), we have*

$$\overline{\lim}_{t\to 0} 2t \log p_t(x, y) \leq -d^2(x, y) \tag{10}$$

This theorem was proved originally by using the Malliavin Calculus and the theory of large deviations of Wentzel-Freidlin in [14], [15]. Let us stress that the relationship between the Malliavin Calculus and the large deviation theory was pioneered by Bismut in [3]. Wentzel-Freidlin estimates were translated in semi-group theory by Léandre in [31] and the Malliavin Calculus of Bismut type with some applications to subelliptic estimates was translated by Léandre in semi-group theory in [21], [23], [24], [25], [26], [28], [29] and [30].

Readers interested by probabilistic methods in heat-kernels can look at the books [1], [8], [11], [38] to the review papers of Léandre ([16], [17], [19], [20]), Kusuoka ([13]) and Watanabe ([41]). Readers interested by analytical methods for heat kernels can look at the books ([4], [40]) and to the review papers of Jerison-Sanchez [9] and Kupka [12]. Let us remark that the marriage between large deviation estimates and the Malliavin Calculus can be done for others equations than the classical one (see for instance the review paper [32]). In order to be self contained, we begin by recall the scheme of our proof of Wentzel-Freidlin estimates in semi-group theory.

2 The Itô-Stratonovitch formula in semi-group theory

Let us consider some vector smooth fields $X_i, i = 0, .., m$ on the d-dimensional torus T^d and let us consider the operator

$$L = X_0 + 1/2 \sum X_i^2 \quad (11)$$

Let f be a smooth function on the torus, and let us introduce the smooth vector fields on $T^d \times R$

$$\hat{X}_i = (X_i, < df, X_i >) \quad (12)$$

and the associated operator

$$\hat{L} = \hat{X}_0 + 1/2 \sum \hat{X}_i^2 \quad (13)$$

To L is associated a semi-group P_t and to $\hat{L}$ is associated a semi-group $\hat{P}_t$. Let g be a smooth function on $T^d \times R$ bounded with bounded derivatives at each order. We get:

Theorem 2 *Let us consider the function* $\hat{g}(x) = g(x, f(x))$. *Then*

$$P_t[\hat{g}](x) = \hat{P}_t[g(.,.)](x, f(x)) \quad (14)$$

Proof: By a density result, we can suppose that f, g are finite sum of Fourier exponentials. Moreover, since P_t and $\hat{P}_t$ are limit of semi-groups where the X_i are finite sum of trigonometric function, we can suppose that the vector fields X_i are finite sums of trigonometric functions. In such a case

$$P_t[\hat{g}](x) = \sum t^n/n! L^n \hat{g}(x) \quad (15)$$

and

$$\hat{P}_t[g(.,.)](x, f(x)) = \sum t^n/n! [\hat{L}^n g(.,.)](x, f(x)) \quad (16)$$

But if we consider a function ψ which depends only from x,

$$X_i(\hat{g}\psi)(x) = \hat{X}_i[g(.,.)\psi](x, f(x)) \quad (17)$$

and is a finite sum of expressions of the same type with derivative of g involved in addition. This show us

$$L^n \hat{g}(x) = \hat{L}^n[g(.,.)](x, f(x)) \quad (18)$$

and the result follows.◇

This theorem is the translation in semi-group theory of the Stratonovitch formula in stochastic analysis. Let us recall this well-known formula: let us consider the stochastic process $x_t(x)$ on the torus associated to it. It is the solution of the Stratonovich stochastic differential equation starting of x

$$dx_t(x) = X_0(x_t(x))dt + \sum X_i(x_t(x))dB_t^i \quad (19)$$

where B_t^i is a R^m-valued Brownian motion. In the Stratonovitch Calculus,

$$f(x_t(x)) = f(x) + \int_0^t < df(x_t(x), dx_t(x) >= f(x) + \int_0^t < df(x_t(x)), X_0(x_t(x)) > dt + \int_0^t \sum_{i>0} < df(x_t(x)), X_i(x_t(x)) > dB_t^i \quad (20)$$

such that the **couple** $((x_t(x), f(x_t(x)))$ is a diffusion associated to the Laplacian $\hat{L}$ on $T^d \times R$.

It has the following important corollary.

Let $\overline{X}_i = (X_i(x), Y_i(y))$ some vector fields on $R^d \times R^{d'}$ bounded with bounded derivatives at each order. $(x, y) \in R^d \times R^{d'}$. Let g be a bilinear form on $R^d \times R^{d'}$. Let us introduce the vector fields on $R^d \times R^{d'} \times R$

$$\hat{X}_i(x, y, z) = (X_i(x), X_i(y), g(X_i(x), DY_i(y)Y_i(y)) + g(DX_i(x)X_i(x), Y_i(y))) \quad (21)$$

To the vector fields $\overline{X}_i$ is associated a generator $\overline{L}$ on $R^d \times R^{d'}$ and to the vector fields $\hat{X}_i$ is associated a generator $\hat{L}$ on $R^d \times R^{d'} \times R$. They are of the type studied in this work. Associated to $\overline{L}$ there is a semi-group $\overline{P}_t$ and to $\hat{L}$ is associated a semi-group $\hat{P}_t$. We get

Theorem 3 *(Itô-Stratonovitch). Let* f *be a smooth function on* R *with bounded derivatives at each-order. Let* $\overline{f} : (x, y) \to f(g(x, y))$ *and* $\hat{f} : (x, y, z) \to f(z)$. *Then*

$$\overline{P}_t[\overline{f}](x, y) = \hat{P}_t[\hat{f}](x, y, g(x, y)) \quad (22)$$

Proof: R is imbedded in T. The vector fields $\overline{X}_i$ by this imbedding realize smooth vector fields on $T^d \times T^{d'}$ and the vector fields $\hat{X}_i$ realize smooth vector fields with bounded derivative at each order on $T^d \times T^{d'} \times R$. The result is therefore a corollary of the previous theorem.$\diamondsuit$

3 Wentzel-Freidlin estimates in semi-group theory

Let us begin by recalling the elementary Kolmogorov lemma of the theory of stochastic processes ([34], [39]).

Let $s \to X_s$ be a family of random variables ($X_0 = 0$) parametrized by $s \in [0,1]$ with values in R^d such that

$$E[|X_t - X_s|^p] \leq C(p)|t-s|^{\alpha p} \tag{23}$$

Then there exist a continuous version of $s \to X_s$ and the L^p norm of $X_1^* = \sup_{s\leq 1}|X_s|$ is finite and can be estimated in terms of the data of (23).

We remark that all the considered Markov semi-group in this part satisfy the Burkholder-Davis-Gundy inequalities in semi-theory of [21]

$$Q_t[|. - X|^p](X) \leq C(p)t^{\alpha p} \tag{24}$$

Namely if we use the semi-group property, we remark that if p is an even integer, $\frac{\partial^r}{\partial t^r}Q_0[|. - X|^p](X) = 0$ for $r \leq p/2$. and define therefore a measure W of the involved continuous path-space by the Kolmogorov lemma.

Let us consider the generator

$$L^t = tX_0 + t/2\sum X_i^2 \tag{25}$$

It generates a semi-group P_s^t. It is classical (See [5] in analysis and [35] in probability) that

$$P_1^t = P_t \tag{26}$$

We put in the sequel $\epsilon = \sqrt{t}$. We consider $(b,x,y) \in R^m \times R^d \times R^d$ and the following vector fields if $i > 0$

$$\hat{X}_i = (\epsilon, \epsilon X_i(x), 0) \tag{27}$$

$$\hat{Y}_i = (0, 0, X_i(y)h_s^i) \tag{28}$$

$$\hat{X}_0 = (0, \epsilon^2 X_0(x), 0) \tag{29}$$

We consider the generator

$$L^\epsilon = \hat{X}_0 + 1/2\sum \hat{X}_i^2 + \sum \hat{Y}_i \tag{30}$$

It generates a Markov semi-group P_s^ϵ which defines a probability measure $W_{0,x,y}^\epsilon$ of the associated path-space. We put

$$H^i(t) = \int_0^t h^i(s)ds \tag{31}$$

We get

Lemma 4 *([18], [31]))Let $R > 0$ and $K > 0$ and h_0. There exists r, C and $\epsilon_0 > 0$ such that for $\epsilon < \epsilon_0$*

$$\sup_{\|h\|\leq h_0} W_{0,x_0,x_0}^\epsilon\{(b-H)_1^* < r, |x_1 - y_1| > R\} \leq C\exp[-K/\epsilon^2] \tag{32}$$

Proof: We consider the stochastic process $(b-H)_s$, $(x_s - y_s)$. By the Itô-Stratonovitch formula of the previous part, it has the same law than the stochastic process $(b-H)_s, \sum(b_s^i - H_s^i)X_i(x_s) - X_s + Y_s$ where b_s, x_s, X_s, y_s, Y_s are the stochastic processes associated to the generator

$$\tilde{L}^\epsilon = 1/2\sum \tilde{X}^2 + \epsilon^2\tilde{X}_0 + \sum \tilde{Y}_i \tag{33}$$

with if $i > 0$

$$\tilde{X}_i = (\epsilon, \epsilon X_i(x), (b^i - H_s^i)\epsilon DX_i(x)X_i(x), 0, 0) \tag{34}$$

$$\tilde{Y}_i = (0,0,0,X_i(y)h_s^i, (X_i(x) - X_i(y))h_s^i) \tag{35}$$

$$\tilde{X}_0 = (0, \epsilon^2 X_0(x), 0, 0, 0) \tag{36}$$

with generic element on this big space (b, x, X, y, Y). By Gronwall lemma, we deduce if r is small enough that

$$W_{0,x_0,x_0}^\epsilon\{|b-H|_1^* < r; |x_1 - y_1| > R\} \leq W_{0,x_0,0,x_0,0}^\epsilon\{|b-H|_1^* < r; X_1^* > CR\} \tag{37}$$

So it remains only to estimate this last quantity. The result will follow from the next lemma:

Lemma 5 *We have for all A the estimate valid for $t > s$*

$$W_{0,x_0,0,x_0,0}^\epsilon\{|b-H|_1^* < r; |\exp[< A, X_t >] - \exp[< A, X_s >]|^p\} \leq (t-s)^{\alpha p}\exp[C(p)|A|^2\epsilon^2 r] \tag{38}$$

We postpone later the proof of this lemma, which uses the analoguous in semi-group theory of the classical exponential martingales of stochastic calculus. By the

Kolmogorov lemma, we deduce from the lemma that, if $A > 0$

$$W^{\epsilon}_{0,x_0,0,x_0,0}\{|b-H|^*_1 < r; \exp[AX^*_1]\} \leq C\exp[CA^2\epsilon^2 r] \quad (39)$$

such that

$$W^{\epsilon}_{0,x_0,0,x_0,0}\{|b-H|^*_1 < r; X^*_1 > R\} \leq C\exp[-C\frac{R^2}{\epsilon^2 r}] \quad (40)$$

by choosing $A = \frac{C_1 R}{\epsilon^2 r}$ in (39). Therefore the result arises.$\diamondsuit$.

Proof of the lemma Let us consider the vector fields for $i > 0$

$$\overline{X}_i = (\epsilon, \epsilon X_i(x), (b^i - H^i_s)\epsilon DX_i(x)X_i(x), 0) \quad (41)$$

$$\overline{X}_0 = (0, \epsilon^2 X_0(x), 0, 0) \quad (42)$$

$$\overline{Y}_0 = (0,0,0,\sum(b^i - H^i_s)^2) \quad (43)$$

with generic elements (b, x, X, z) on this big space. Let us consider the generator

$$\overline{L} = 1/2\sum(\overline{X}^i)^2 + \overline{X}_0 + \overline{Y}_0 \quad (44)$$

There is associated a Markov semi-group $\overline{P}_t$. Let p be an even integer and k an even bigger integer. Following [21], we introduce the auxiliary function

$$F_C(X) = \frac{|X - X_0|^p}{1 + |X - X_0|^k / C} \quad (45)$$

Let us put

$$u_t = \overline{P}[F_C](0, x_0, X_0, 0) \quad (46)$$

u_t is finite and $u_0 = 0$. Moreover, by applying the semi-group property, we deduce that

$$|d/dt u_t| \leq A + Bu_t \quad (47)$$

where A, B don't depend on C and X_0. Therefore we get a uniform estimate of u_t and by using the Fatou lemma, we deduce that:

$$\overline{P}_t[|X - X_0|^p](0, x_0, X_0, 0) \leq C(p) \quad (48)$$

By doing as in the beginning of this part, we deduce that the derivatives $\frac{\partial}{\partial t^r}\overline{P}_0[|. - X_0|^p](0, x_0, X_0, 0) = 0$ for $r \leq p/3$. Let us convert the generator $\overline{L}$ in Itô form. The X part can be written as

$$\sum < (b^i - H^i_s)\epsilon DX_i(x)X_i(x), D^2F(X), (b^i - H^i_s)\epsilon DX_i(x)X_i(x) > + < A^i(\epsilon, x, f), DF(X) > \quad (49)$$

where

$$|A^i(\epsilon, x, ; h)| \leq C\epsilon^3 + |b - H_s|\epsilon^2 \quad (50)$$

We add an extra-variable u to the space and we modify $\overline{L}$ in R_A:

$$R_A F = \overline{L}F + \sum < (b^i - H^i_s)\epsilon DX_i(x)X_i(x), A >^2 D_u F + \sum < A^i(\epsilon, x, h), A > D_u F \quad (51)$$

where F depends on (b, x, X, z, u). R_A generates a semi-group Q^A_t. We consider a smooth decreasing function g equals to 1 if $z \leq r^2$ and equals to 0 if $z > 2r^2$ and we consider the function

$$\psi(b, x, X, z, u) = g(z)\exp[< A, X > -u] \quad (52)$$

We remark that $R_A\psi \leq 0$. Therefore

$$Q^A_t\psi(0, x_0, 0, 0, 1) \leq 1 \quad (53)$$

By Cauchy-Schwartz inequality, we deduce that

$$\overline{P}_t[g(z)\exp[< A, X >]](0, x_0, 0, 0) \leq (Q^A_t[g(z)\exp[2u]](0, x_0, 0, 0, 1))^{1/2} \quad (54)$$

But this last quantity is smaller than $\exp[C|A|^2\epsilon^2 r]$. The result arises then by replacing in (45) $F_C(X)$ by

$$\frac{(\exp[< A, X - X_0 >] - 1)^p}{1 + (\exp[< A, X - X_0 >] - 1)^k / C} \quad (55)$$

$\diamondsuit$.

Theorem 6 *(Wentzel-Freidlin)When* $t \to 0$*, and if* $d^2(x, O) = \inf_{y \in O} d^2(x, y)$

$$\overline{\lim}_{t \to 0} 2t \log P_t[O](x) \leq -d^2(x, O) \quad (56)$$

if O *is an open suset of* R^d*.*

Proof: Let b_t be the process associated to the first component in L^ϵ. It is the Brownian motion. Its transition heat-kernel is $C(t\epsilon)^{-m/2}\exp[-\frac{|x-y|^2}{2\epsilon^2 t}]$. Therefore if $t > s$

$$W^{\epsilon}_{0,x_0,0,x_0}\{\exp[< A, b_t - b_s >]\} \leq (t-s)^{\alpha p}\exp[C|A|^2(t-s)\epsilon^2] \quad (57)$$

By doing as in the end of the previous part, we deduce that:

$$W^{\epsilon}_{0,x_0,0,x_0}\{b^*_t > R\} \leq C\exp[-C\frac{R^2}{t\epsilon^2}] \quad (58)$$

Therefore in order to estimat $W^{\epsilon}_{0,x_0,0,x_0}\{x_1 \in O\}$, we can by the previous estimate do the restriction that $b_1^* < R$ for some big R.

We choose a very small r_0 and a lattice on R^m with smesh r_0 and a lattice on $[0,1]$ with a very small smesh t_0. We consider the space R of polygonal curves H on the previous lattice on R^m with smesh t_0. We can choose r_0 very small and t_0 very small such that $\inf_{H\in R, x_1(d/dtH)\in 0} \|d/dtH\|^2$ is very close of $\inf_{z\in O} d^2(x,z)$.

By the inequality (58), we have if r' is small enough for a big K:

$$W^{\epsilon}_{0,x_0,0,x_0}\{(b-H)_1^* > r' \, for\, all\, h \in R\} \leq \exp[-K/\epsilon^2] \quad (59)$$

So it remains only to do the estimate for $h \in R$ of $W^{\epsilon}_{0,x_0,0,x_0}\{(b-H)_1^* \leq r'$. But this quantity is smaller than $W^{\epsilon}_{0,x_0,0,x_0}\{for\, all\, t_i\, |b_{t_i} - H_{t_i}| \leq r'\}$ where the t_i run on the subdivision considered of $[0,1]$. But the heat kernel associated to the Brownian motion is classically known and this last quantity is smaller than $\exp[\frac{-\|d/dtH\|^2 + r"}{2\epsilon^2}]$ for a convenient small $r"$.$\diamondsuit$

Remark: Let us motivate this theorem by using the heuristic formulas of path integrals. We put $\epsilon = \sqrt{t}$ and we consider the Stratonovitch stochastic differential equation

$$dx_s^{\epsilon}(x) = \epsilon^2 X_0(x_s^{\epsilon}(x))ds + \epsilon \sum_{i>0} X_i(x_s^{\epsilon}(x))dB_s^i \quad (60)$$

We write formally

$$dB_s^i = d/dsB_s^i ds \quad (61)$$

where d/dsB_s^i is the white-noise. Formally, the white noise follows the infinite dimensional Gaussian measure

$$d\mu = 1/Z \exp[-\|d/dsB_.\|^2/2]dD \quad (62)$$

where dD is the formal Lebesgue measure of the path space (We refer to [22], [27] for a rigorous approach of this formal Lebesgue measure as a distribution in infinite dimension in the Hida-Streit approach of functional integrals). Therefore, $\epsilon d/dsB_.$ follows formally the Law

$$d\mu(\epsilon) = 1/Z(\epsilon) \exp[-\|d/dsB_.\|^2/2\epsilon^2]dD \quad (63)$$

and the result (56) goes as if we were in finite dimension.

4 Proof of the main theorem

We consider the Malliavin generator $\hat{L}$ on $R^d \times G^d \times M^d$ where G^d denotes the set of invertible matrices on R^d and M^d denotes the set of symmetric matrices on R^d. (x,U,V) denotes the generic element of this space (V is called the Malliavin matrix). We consider the vector fields

$$\hat{X}_i = (X_i, DX_iU, 0) \quad (64)$$

$$\hat{Y} = \sum_{i=1}^{m} < U^{-1}X_i, . >^2 \quad (65)$$

and the Malliavin generator is defined by

$$\hat{L} = \hat{X}_0 + \hat{Y} + \sum_{i=1}^{m} \hat{X}_i^2 \quad (66)$$

It generates a semi-group $\hat{P}_t$.

We use the integration by parts formula of [16]: if (α) is a multi-index on R^d, we have by [21] and (56)

$$|P_t[g\frac{\partial^{(\alpha)}}{\partial y^{(\alpha)}}f](x)| \leq \exp[\frac{-d^2(x,y)+\eta}{2t}](\hat{P}_t[|V^{-1}|^{r(\alpha)}](x,I,0))^{1/p}\|f\|_\infty \quad (67)$$

for a convenient $r(\alpha)$, a big p ($\|f\|_\infty$ denotes the uniform norm of the test function f).

This comes from the basical tools of the Malliavin Calculus of Bismut type without probability of [21]. Let us give some details on that.

We consider some convenient step by step constructed vector fields:

$$X_i^k(x_1,..,x_k) = X_{1,i}^k(x_1,..,x_{k-1})x^k + X_{2,i}^k(x_1,..,x_k) + X_{3,i}^k(x_1,..,x_{k-1}) \quad (68)$$

where $X_{i,1}^k$ have bounded derivatives at all order as well as $X_{2,i}^k$ and $X_{3,i}^k(x_1,..,x_{k-1})$ has derivatives with polynomial growth.

$(x_1,..,x_k)$ is the generic element of the big space $R^{d_1} \times R^{d_2} \times R^{d_3} .. \times R^{d_k}$ (In order to simplify the exposition, we omit to describle the details coming from the fact some times R^{d_j} is replaced by a set of invertibles matrices). We consider the vector fields on the big space

$$X_i^{tot} = (X_i^1(x_1),..,X_i^k(x_1,..,x_k)) \quad (69)$$

and the big generator

$$L^{tot} = 1/2\sum_{i>0}(X_i^{tot})^2 + X_0^{tot} \quad (70)$$

L^{tot} spans a Markovian semi-group P_t^{tot}, which enlarges modulo some nice choices of X_i^2 and of X_i^3 the Malliavin semi-group $\hat{P}_t$. The main remark is the following: if f has a polynomial growth, $\hat{P}_t[f](x, I, 0, ..)$ is finite ([21]). We denote this enlarged semi-group Q_t^{tot}. Let us consider a multi-index (α). There exists a semi-group of this type Q_t^{tot} so that

$$P_t[\frac{\partial^\alpha}{\partial y^{(\alpha)}}(gf)](x) = Q_t^{tot}[(gf)R](x, I, 0, ..) \quad (71)$$

where R contains polynomial in the enlarged variables and some inverse of the Malliavin matrix V. By distributing the derivatives in gf and proceding inductively on the length of the multi-index (α), we find that

$$P_t[g\frac{\partial^\alpha}{\partial y^\alpha}f](x) = Q_t^{tot}[\tilde{g}Rf](x, I, 0, ..) \quad (72)$$

where $\tilde{g}R$ is an algebraic expressions containing some derivatives of g of length smaller than the length of (α), in the extra variables and in the inverse of the Malliavin matrix. We apply Cauchy-Schwartz inequality and we deduce that

$$|P_t[g\frac{\partial^\alpha}{\partial y^\alpha}f](x)| \leq \|f\|_\infty \hat{P}_t[|V^{-1}|^{r(\alpha)}]^{1/\beta'}(x, I, 0)P_t[|\tilde{g}|^\beta]^{1/\beta}(x, 0) \quad (73)$$

for a big $r(\alpha)$, a convenient β' and a β close from 1. The results comes then by (56). The result comes from the following proposition:

Theorem 7 *The following estimate is valid for* $t \leq 1$*:*

$$\hat{P}_t[|V^{-1}|^p](x, I, 0) \leq C(p)t^{-r(p)} \quad (74)$$

for a convenient $r(p)$ *associated to the positive integer* p.

Namely from (73) and Theorem 7, we deduce the bound valid for $t \leq 1$

$$|P_t[g\frac{\partial^\alpha}{\partial y^\alpha}f](x,)| \leq t^{-r(\alpha)} \exp[\frac{-d^2(x, y) + \eta}{2t}]\|f\|_\infty \quad (75)$$

We apply this estimate when f is a Fourier exponential in order to deduce that the density of the measure $f \to P_t[gf](x)$ is bounded by $t^{-r} \exp[\frac{-d^2(x,y)+\eta}{2t}]$. The conclusions comes from the fact the density of this measure in y is $g(y)p_t(x, y) = p_t(x, y)$.

5 Estimation of the Malliavin matrix in small time

In this part we prove the Theorem 2. This follows closely the appendix of [23]. We remark that $t\hat{L}$ generates a semi-group $\hat{P}_s^t$ and that $\hat{P}_t = \hat{P}_1^t$. We put

$$F_l^t(x, U, \xi) = \sum_{E_l(x)} t < U^{-1}Y(x), \xi >^2 \quad (76)$$

where ξ is a bounded element of R^d. We get

Lemma 8 *: Let us suppose that for arbitrarly small* s_0 *we have*

$$\hat{P}_{s_0}^t[F_l^t(., ., \xi) > t^\beta s_0^\alpha](x, U_0, 0) > C > 0 \quad (77)$$

Then (77) remains true on an interval starting from s_0 *and of length* $t^{\beta_1}s_0^{\alpha_1}$ *if* U_0*,* U_0^{-1} *and* ξ *remain bounded.*

Proof: We introduce a function g from R^+ into $[0, 1]$ with bounded derivatives, equals to 1 at a neighborhood of the infinity and equals to 0 in 0. We introduce the auxiliary function

$$s \to h(s) = \hat{P}_s^t[g(\frac{F_l^t(., ., \xi)}{t^\beta s_0^\alpha})](x_0, U_0, 0) \quad (78)$$

It has derivative bouded by $t^{-2\beta}s_0^{-2\alpha}$. This comes from the fact that

$$h'(s) = \hat{P}_s^t[t\hat{L}(g(\frac{F_l^t(., ., \xi)}{t^\beta s_0^\alpha}))] \quad (79)$$

When we compute $\hat{L}(g(\frac{F_l^t(.,.,\xi)}{t^\beta s_0^\alpha}))$, there is a polynomial in U_0^{-1} which appears, and some derivatives of Y, $Y \in E_l(x)$ which appear, two derivatives of g and a coefficient in $t^{-2\beta}s_0^{-2\alpha}$ which comes from the rule of derivation of composition of functions. By [21], for all p, $\hat{P}_s^t[|U^{-1}|^p](x, U_0, 0)$ remains bounded if U_0 and U_0^{-1} remain bounded. We deduce since $h(s_0) = 1$ that

$$h(s) > 1 - C\frac{t^{2\beta}s_0^{2\alpha}}{t^{2\beta}s_0^{2\alpha}} > C > O \quad (80)$$

if $s \in [s_0, s_0 + C_1t^{2\beta}s_0^{2\alpha}]$ for C_1 small enough. Therefore the result.◇

We recall the result of [23]

$$P_s^t[|y - x| > C](x) + \hat{P}_s^t[(|U| + |U^{-1}|) > C](x, I, 0) \leq C(p)t^ps^p \quad (81)$$

for all $s \leq 1, t \leq 1$ if C is big enough.

Let us proof this result. We consider a smooth and positive function g equals to 1 outside a neighborhood of x and equals to 0 in a neighborhood of x. We get

$$P_s^t[|y-x| > C](x) \leq P_s^t[g(y)](x) \quad (82)$$

Moreover

$$\frac{d^r}{ds^r} P_0^t[g(u)](0) = 0 \quad (83)$$

and

$$\frac{d^r}{ds^r} P_s^t[g(y)](x) = P_s^t[t^r L^r g(y)](x) \quad (84)$$

This last quantity is obviously bounded bt $C(r)t^r$. The result goes by Taylor formula.

We introduce a positive smooth function equals to 1 outside a neighborhood of I and equals to 0 in a neighborhood of I. We get

$$\hat{P}_s^t[|U| > C](x, I, 0) \leq \hat{P}_s^t[g(U)](x, I, 0) \quad (85)$$

Clearly,

$$\frac{d^r}{ds^r} \hat{P}_0^t[g(U)](x, I, 0) = 0 \quad (86)$$

On the other hand

$$\frac{d^r}{ds^r} \hat{P}_s^t[g(U)](x, I, 0) = t^r \hat{P}_s^t[\hat{L}^r g(U)](x, I, 0) \quad (87)$$

In $\hat{L}^r g(U)$ some polynomials in U appear. But $\hat{P}_s^t[|U|^p](x, I, 0)$ remains bounded ([21]). The result goes as before.

We do the same to study $\hat{P}_s^t[|U^{-1}| > C](x, I, 0)$: we have only to remark that the vector fields DX_iU are transformed in $-UDX_i$ under the transformation $U \to U^{-1}$.

In the sequel U_0 and U_0^{-1} will remain bounded.

Lemma 9 *Let us suppose that*

$$\hat{P}_{s_0}^t[F_l^t(.,.,\xi) > t^\beta s_0^\alpha](x_0, U_0, 0) > C > 0 \quad (88)$$

on an interval $I(x_0, U_0)$ starting from s_0 and of length $t^{\beta_1} s_0^{\alpha_1}$. Then there exists β_2 and α_2 depending on the previous data and a s_1 belonging to the previous interval such that

$$\hat{P}_{s_1}^t[F_{l-1}^t(.,.,\xi) > t^{\beta_2} s_0^{\alpha_2}](x_0, U_0, 0) > C > 0 \quad (89)$$

Proof: Either

$$\hat{P}_{s_0}^t[F_{l-1}^t(.,.,\xi) > t^{\beta_2} s_0^{\alpha_2}](x_0, U_0, 0) \quad (90)$$

and the proof is finished or not. Let us suppose that we are in the second situation. We consider

$$G_{l-1}(x", U", \xi) = \sum_{E_{l-1}} (< (U")^{-1} Y(x"), \xi > - < (U_0)^{-1} Y(x_0), \xi >)^2 \quad (91)$$

We consider a increasing function g from R^+ into $[0, 1]$ equals to 1 on a neighborhood on the infinity and such that $g(t) = t$ on a neighborhood of 0. We consider the auxiliary function $s \to h(s)$

$$s \to \hat{P}_s^t[g(\frac{G_{l-1}}{s_0^{\alpha_3} t^{\beta_3}})](x, U, 0) \quad (92)$$

for some big α_3, β_3 and x, U being chosen according the law of $\hat{P}_{s_0}^t(x_0, U_0, 0)$. By the consideration done before this lemma, we can suppose that x, U and U^{-1} remain bounded. This function is equal to 0 in s_0, has a first derivative in s_0 in $Ct^{-\alpha_4}\ s_0^{-\beta_4}$ ($C > 0$), and has a second derivative bounded by $Ct^{-2\alpha_4}\ s_0^{-2\beta_4}$ for some big α_4 and β_4.

Let us give the details of this statement. The main remark is that

$$\hat{X}_i < U^{-1}Y, \xi > = < U^{-1}[X_i, Y], \xi > \quad (93)$$

Therefore

$$\hat{L}G_{l-1}(x", U", \xi) = \sum_{E_l} < (U")^{-1} Y(x"), \xi >^2 + \sum_{E_{l-1}} < (U")^{-1}[X_0, Y], \xi > (< (U")^{-1} Y(x"), \xi > - < U_0^{-1} Y(x_0), \xi >) + \sum_{E_l, i>0} < (U")^{-1}[X_i, [X_i, Y]], \xi > (< (U")^{-1} Y(x"), \xi > - < U_0^{-1} Y(x_0), \xi >) \quad (94)$$

This shows the following inequality:

$$|t\hat{L}[g(\frac{G_{l-1}}{s_0^{\alpha_3} t^{\beta_3}})](x_0, U_0, 0)| \geq \frac{t^\beta s_0^\alpha - t^{\beta_2/2} s_0^{\alpha_2/2}}{s_0^{\alpha_3} t^{\beta_3}} \quad (95)$$

We choose $\beta_2/2$, $\alpha_2/2$, α_3 and β_3 very big and we take $\alpha_4 = \alpha_3 - \alpha$ and $\beta_4 = \beta_3 - \beta$. We would like to estimate

$$t^2 \hat{L}^2[g(\frac{G_{l-1}}{s_0^{\alpha_3} t^{\beta_3}})](x, U, 0) \quad (96)$$

For that we iterate (93). We have

$$\hat{X}_i[g(\frac{G_{l-1}}{s_0^{\alpha_3} t^{\beta_3}})](x, U, 0) = \frac{1}{s_0^{\alpha_3} t^{\beta_3}} g'(\frac{G_{l-1}}{s_0^{\alpha_3} t^{\beta_3}}) \sum_{E_{l-1}} < U^{-1}[X_i, Y], \xi > (< U^{-1} Y(x), \xi > - < U_0^{-1} Y(x_0), \xi >) \quad (97)$$

We have moreover

$$\hat{X}_i^2[g(\frac{G_{l-1}}{s_0^{\alpha_3}t^{\beta_3}})](x,U,0) = \frac{1}{s_0^{2\alpha_3}t^{2\beta_3}}g''(\frac{G_{l-1}}{s_0^{\alpha_3}t^{\beta_3}})\sum_{E_{l-1}} < U^{-1}[X_i,Y],\xi > (< U^{-1}Y(x),\xi > - < U_0^{-1}Y(x_0),\xi >)^2 + \frac{1}{s_0^{\alpha_3}t^{\beta_3}}g'(\frac{G_{l-1}}{s_0^{\alpha_3}t^{\beta_3}}) \{\sum_{E_l} < U^{-1}Y(x),\xi >^2 + \sum_{E_{l-1}} < U^{-1}[X_i,[X_i,Y]],\xi > (< U^{-1}Y,\xi > - < U_0^{-1}Y(x_0),\xi >)\} \quad (98)$$

We distinguish if $t\sum_{E_{l-1}}(< U^{-1}Y(x),\xi > - < U_0^{-1}(x_0),\xi >)^2$ is larger of $Ct^{\beta_2}s_0^{\alpha_2}$ or not. If it the case, we do as in the previous lemma. If it not the case, we remark that

$$\frac{G_{l-1}}{s_0^{\alpha_3}t^{\beta_3}} < \frac{t^{\beta_2-1}s_0^{\alpha_2}}{s_0^{\alpha_3}t^{\beta_3}} < C \quad (99)$$

is very small because β_2 and α_2 are very big. Therefore in $t^2\hat{L}^2[g(\frac{G_{l-1}}{s_0^{\alpha_3}t^{\beta_3}})](x,U,0)$ there is only one derivative of g which appears. Therefore the leading exponent which appears in this expression is $s_0^{-\alpha_3}t^{-\beta_3}$ which is smaller than $s_0^{-2\alpha_4}t^{-2\beta_4}$ because α_3 and β_3 are much more bigger than β and α.

Therefore the result. Namely, the first derivative of $h(s)$ on a time interval starting from s_0 of length $Ct^{\beta_4}s_0^{\alpha_4}$ are larger than $Ct^{-\beta_4}s_0^{-\alpha_4}$. This shows there exists C_1 and C_2 such that

$$h(s_0 + C_1t^{\beta_4}s_0^{\alpha_4}) > C_2 > 0 \quad (100)$$

◇

By using the strong Hoermander's hypothesis in x, we deduce if x_0 remains in a small neighborhood of x and if U_0, U_0^{-1} remain bounded that

$$\hat{P}_s^t[F_1^t(.,.,\xi) > s^\alpha t^\beta](x_0,U_0,0) > C > 0 \quad (101)$$

on an interval $I(x_0,U_0)$ starting from s_0 and of length $s_0^{\alpha_0}t^{\beta_0}$.

By doing as in Lemma 5 of [23], we deduce that for any x_0 in a small neighborhood of x, if U_0 and U_0^{-1} remain bounded, there exists an interval $I(x_0,U_0)$ starting from s_0 small of length $s_0^\alpha t^\beta$ and α_0 and β_0 such that

$$\hat{P}_s^t[V(\xi) < t^{\beta_0}s^{\alpha_0}](x_0,U_0,0) < C < 1 \quad (102)$$

for $s \in I(x_0,U_0)$.

We slice the time interval $[0,1]$ into $s^{-\alpha}t^{-\beta}$ small intervals, we apply the semi-group property and we deduce as in [23], Theorem 2 that

$$\hat{P}_1^t[V(\xi) < t^{\beta_0}s^{\alpha_0}](x,I,0) < CC_1^{s^{-\alpha}}C_1^{t^{-\beta}} \quad (103)$$

where C_1 is smaller than 1.

We deduce that

$$\hat{P}_1^t[V(\xi) < t^\beta\epsilon](x,I,0) \le C(p)\epsilon^p t^p \quad (104)$$

for all p. We choose $t^{-\beta_r}\epsilon^{-r}$ points ξ_i on the unit sphere of R^d. We deduce that

$$\hat{P}_1^t[|V^{-1}| > t^{-\beta}\epsilon^{-1}](x,I,0) \le \sum \hat{P}_1^t[V(\xi_i) < t^\beta\epsilon](x,I,0) + \hat{P}_1^t[|V| > t^{-\gamma}\epsilon^{-\gamma_1}](x,I,0) \le C(p)\epsilon^p \quad (105)$$

Therefore (74) in Theorem 7 holds.◇

6 Conclusion

We translated in this work in semi-group theory our proof of Varadhan estimates for subelliptic heat-kernels which says that the estimates of large deviation theory are still true for heat kernels.

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