The Polynomial Roots Repartition and Minimum Roots Separation

CĂLIN ALEXE MUREŞAN
Department of Mathematics, Faculty of Letters and Science,
University Petroleum and Gas of Ploiesti,
B-dul Bucuresti, No.139, Ploiesti, Prahova,
ROMANIA
acmuresan@upg-ploiesti.ro, c1a2muresan@yahoo.com

Abstract:

It is known that, if all the roots of a polynomial are real, they can be localised, using a set of intervals, which contain the arithmetic average of the roots. The aim of this paper is to present an original method for giving other distributions of the roots/ modules of the roots, on real axis, a method for evaluating and improving the "polynomial minimum root separation" results, a method for the complex polynomials and for polynomials having all roots real. We use the discriminant, Hadamard's inequality, Mahler's measure and new original inequalities. Also we will make some considerations about the cost for isolate the polynomial real roots. Our method is based on the successive splitting for the interval which contain all roots.

Key-Words: Roots repartition, Isolating the roots, Mahler's measure

1 Introduction

The roots repartition of a complex polynomial, on real axis, means to give real intervals, not necessarily disjunctive, for every polynomial root or module of the polynomial root.

Pre-isolating respectively isolating the complex roots of a polynomial with complex coefficients, means to compute separating boxes in the complex plane, which contains at most one, respectively exact one complex root of the polynomial.

The "exact algorithms" for isolating the roots are based on: Sturm's sequences, see [1]; differentiation technique, see [2] and Vincent's theorem, see [3].

Other root finding methods are the numerical methods. These methods work with a great variety of approximation errors. Recently results which generalize the univariate Hermite interpolation formula can be found in [4], [5], [6].

The numerical algorithms, compute an approximation for all the complex roots of a polynomial up to a desired accuracy and if that is smaller than "minimum roots separation" then the algorithms can be turned on the isolation algorithms. For the same approximation error, the costs of the fastest 'isolating algorithms' and 'numerical algorithms' are comparable.

The aim of this article is to give original results regarding the roots repartition on real axis, the minimum roots separation for complex polynomials and for the polynomials with all real roots. These results are necessarily in "exact algorithms" for isolating the roots based on successive splitting (see [1] and [2]) and can be useful also in numerical algorithms.

In this section we introduce the basis notations and notions and present the preliminary results in this field. We denote a complex polynomial with $P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$, $n \ge 1$, and its roots with $x_1, \ldots, x_n$.

Definition 1.1 Let be $P(x) \in C[x]$. We define:

a) $\Delta = \sum\limits_{1 \le i \le n} \left(x_i - x_j \right)^2$.

b) $sep(P) = \min\{| x_i - x_j | \ / \ x_i \ne x_j, \ 1 \le i, j \le n\}$ the minimum roots separations.

c) $\| P \| = \sqrt{|a_0|^2 + |a_1|^2 + .. + |a_n|^2}$ - the polynomial norm.

d) $L(P) = |a_0| + |a_1| + .. + |a_n|$ - the polynomial length.

e) $\bar{x} = \dfrac{x_1 + x_2 + .. + x_n}{n}$ the average of the roots.

Definition 1.2 The discriminant of a polynomial $P \in C[x]$ with leading coefficient a_n and roots $x_1, \ldots, x_n$ is defined as:

disc(P)= $a_n^{2n-2} \cdot \prod_{1 \le i < j \le n} \left(x_i - x_j\right)^2$.

Proposition 1.1 Let be $P \in C[x]$

a) The expression of $D = \prod_{1 \le i < j \le n} \left(x_i - x_j\right)$, is

$$D = (-1)^{n(n-1)/2} \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{vmatrix}.$$

b) disc(P)= $a_n^{2n-2} \cdot D^2$.

Definition 1.3 Let be
$P = P(x) = a_m x^m + a_{m-1} x^{m-1} + \dots + a_1 x + a_0$, and
$Q = Q(x) = b_n x^n + b_{n-1} x^{n-1} + \dots + b_1 x + b_0$,
$m \ge 1, a_m \ne 0, n \ge 1, b_n \ne 0$. Sylvester's matrix of P and Q is the matrix S:

$$S = \begin{pmatrix} a_m & a_{m-1} & \dots & a_0 & 0 & 0 & \dots & 0 \\ 0 & a_m & a_{m-1} & \dots & a_0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \dots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & a_m & a_{m-1} & \dots & a_0 \\ b_n & b_{n-1} & \dots & b_0 & 0 & 0 & \dots & 0 \\ 0 & b_n & b_{n-1} & \dots & b_0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \dots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & b_n & b_{n-1} & \dots & b_0 \end{pmatrix}$$

where there are n rows of the a_i followed by m rows of the b_i. $S = \left(s_{i,j}\right)$ is a matrix with $m + n$ rows and $m + n$ columns having the elements: $s_{i,j} = a_{m-j+i}$ for $1 \le i \le n$ and $s_{n+1,j} = b_{n-j+i}$ for $1 \le i \le m$.

Definition 1.4 The resultant of P and Q is the determinant of Sylvester's matrix, res (P; Q)=$det(S)$

Proposition 1.2 $a_n disc\left(P\right) = res\left(P; P'\right)$.
See [7] or [8].

Observation 1.1 The discriminant, $disc(P)$ can be expressed only by the degree and the polynomial coefficients. See Definition 1.3 and 1.4 and Proposition 1.2.

For many others results in this area see [9].

Theorem 1.1 ([7]) Let be
$P(x) = a_0 + a_1 x + a_2 x^2 + .. + a_{n-1} x^{n-1} + x^n$; $a_i \in C$; $i = \overline{0; n-1}$, $n \ge 1$, with the roots $x_1, x_2, \dots, x_n \in C$; if $r_0 = \max\{|x_1|, |x_2|, \dots, |x_n|\}$ then:

a) $r_0 \le \max\left\{1, \sum_{k=0}^{n-1} |a_k|\right\}$,

b) $r_0 \le 1 + \max_{0 \le k \le n-1} \{|a_k|\}$,

c) If $\lambda_1, \dots, \lambda_n \in (0; +\infty)$ such that $\lambda_1^{-1} + \dots + \lambda_n^{-1} = 1$ then
$r_0 \le \max\left\{1, \max_{1 \le k \le n} \{\left(\lambda_k \cdot |a_{n-k}|^{1/k}\right)\}\right\}$,

d) $r_0 \le \max\left\{1, \max_{1 \le k \le n} \left(n \cdot |a_{n-k}|\right)^{1/k}\right\}$,

e) For $a_k \ne 0$; $k \in \{0, 1, \dots, n-1\}$
we have: $r_0 \le \max\{2 \cdot \left|\frac{a_{n-2}}{a_{n-1}}\right|, \dots, 2 \cdot \left|\frac{a_1}{a_2}\right|, \left|\frac{a_0}{a_1}\right|\}$,

***f*)** $r_0 \le |1 - a_{n-1}| + |a_{n-2} - a_{n-1}| + \dots$
$\dots + |a_0 - a_1| + |a_0|$,

g) $r_0 \le \max_{1 \le k \le n} \{(\frac{|a_{n-k}|}{C_n^k})^{1/k} \cdot (2^{1/n} - 1)^{-1}\}$.

Corollary 1.1
Let be $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, $n \ge 1$. $P \in R[x]$, then $(\exists)$ r, R real, such that $r \le |x_i| \le R$, $(\forall)$ $i = \overline{1, n}$.

Proof: Obviously results from the Theorem 1.1, replacing a_i with $\frac{a_i}{a_n}$, for $i \in \{1, \dots, n-1\}$.

For $r, R \notin \{1\}$ we can take:
$R = \frac{L(P)}{|a_n|} > 1$ and $r = \frac{|a_0|}{L(P)} < 1$.

For others useful bounds see [10] and [11].

Definition 1.5 The Mahler Measure of the polynomial P, denoted by $M(P)$ is:

$$M(P) = M[P(x)] = |a_n| \cdot \prod_{j=1}^{n} \max\{1, |x_j|\}.$$

Theorem 1.2
Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, $n \ge 1$. $a_n \cdot a_0 \ne 0$, $P(x) \in C[x]$ then:

$$\ln[M(P)] = \frac{1}{2\pi} \int_0^{2\pi} \ln |P(e^{i\theta})| d\theta.$$

See [12], [13].

Proposition 1.3 ([7]) With the notation from Theorem 1.1 we have

a) $M(P) = \dfrac{|a_0|}{\prod_{j=1}^{n} \min\{1, |x_j|\}}$,

b) $M[x^n \cdot P(\frac{1}{x})] = M[P(x)]$,

c) $M(P \cdot Q) = M(P) \cdot M(Q), (\forall) P, Q \in C[x]$,

d) $M[P(x^k)] = M[P(x)]$,

e) $M^2(P) + |a_0 a_n|^2 \cdot M^{-2}(P) \le \|P\|^2, \ M(P) \le \|P\|$.

Theorem 1.3 ([14])

Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$, $n \ge 1$,

and $a_n \ne 0$, $P(x) \in C[x]$ or $P(x) = a_n \cdot \prod_{j=1}^{n} (x - x_j)$,

with the roots $x_1, x_2, ..., x_n \in C$; not necessarily distinct. Introducing the polynomials

$$P_m(x) = \pm a_n^{2^m} \cdot \prod_{j=1}^{n} (x - x_j^{2^m}); \ m \ge 0 .$$

we can calculate P_m, $m \ge 0$, according to Graeffe's Method that is:

i) $P_0(x) = P(x)$.

ii) If $m \in N^*$ we can obtain $\{G_m(x), H_m(x), P_{m+1}(x)\} \in C[x]$ from the relations: $P_m(x) = G_m(x^2) - x \cdot H_m^2(x)$,

$P_{m+1}(x) = G_m^2(x) - x \cdot H_m^2(x)$.

iii) Starting to the previous points we find from recursion method: $P_m(x)$ for $m \ge 1$.

Theorem 1.4 ([14]) If $P(x) \in C[x]$; $\deg(P) \ge 1$ and P_m, $m \ge 0$, the polynomial series associated to the Graeffe's Method, then:

$$2^{-n \cdot 2^{-m}} \cdot \|P_m\|^{2^{-m}} \le M(P) \le \|P_m\|^{2^{-m}} .$$

and $\lim_{n \to \infty} \|P_m\|^{2^{-m}} = M(P)$.

Theorem 1.5 $M(P) = \inf\{\| P \cdot Q \| / Q \in C[x]$, Q is monic polynomial$\}$.

For proving, see [15] and [16].

Observation 1.2 The Mahler measure $M(P)$ can be approximated using only the degree and the polynomial coefficients. See Definition 1.5, Theorem 1.3 and 1.4.

Definition 1.6 a) A pre-isolating/isolating interval for a complex polynomial represent an open interval (a, b), having as limits two rational numbers, between which there is at most/precisely, one root (modules of the root) of the polynomial. For an isolating interval we have: $(\exists)\ i \in N$ such that $\{x_1, x_2, ..., x_n\} \cap (a,b) = \{x_i\}$ for real roots of the polynomial and $(\exists)\ i \in N$ such that $\{|x_1|, |x_2|, ..., |x_n|\} \cap (a,b) = \{|x_i|\}$ for complex roots of the polynomial.

b) Pre-isolating/isolating the real roots consists in finding for all the polynomial's roots, disjunctive pre-isolating/isolating intervals.

Definition 1.7 For a given function $g(x)$, $g: R \to R$, we denote by $O(g(x))$ the set of functions: $O(g(x))$ $=\{f(x) / f: R \to R, (\exists)\ c, x_0 \in R$ such that $0 \le f(x) \le c \cdot g(x), (\forall)\ x \ge x_0\}$. In this case for every $f(x)$ we denote: $O(g(x))=f(x)$.

We are saying that "f grows at the same rate or it may grow more slowly than g when x is very large".

Definition 1.8 For a given function $g(x)$, $g: R \to R$, we denote by $(g(x))$ the set of functions: $\Theta(g(x))=$ $\{f(x) / f: R \to R,\ O(g(x))=f(x)$ and $O(f(x))=g(x)\}$.

In this case for every $f(x)$ we denote: $\Theta(g(x))=f(x)$.

We are saying that "f grows at the same rate than g when x is very large".

For more details see [17] and [18].

2. Polynomial roots distribution

2.1 Centered intervals repartition method for polynomials with all real roots.

Definition 2.1.1 Let be b_1, b_2, real numbers. We denote by $P_n(b_1, b_2) \in R[x]$ the set of all polynomials, having the form:

$P(x) = x^n + b_1 x^{n-1} + b_2 x^{n-2} + ... + b_n$, $n \ge 2$, $n \in N$.

with all real roots and the following inequalities hold between the roots: $x_n \le x_{n-1} \le ... \le x_2 \le x_1$.

Proposition 2.1.1 If $P(x) \in P_n(b_1, b_2)$ then:

a) $\Delta = n \sum_{k=1}^{n} x_k^2 - (\sum_{k=1}^{n} x_k)^2$;

b) $\Delta = (n-1) b_1{}^2 - 2n b_2$; **c)** $\bar{x} = -\dfrac{b_1}{n}$.

See [19] and [20].

Theorem 2.1.1 Let be

$$P_j(x) = \left(x - \bar{x} + \frac{1}{n} \sqrt{\frac{j}{n-j} \Delta} \right)^{n-j} \left(x - \bar{x} - \frac{1}{n} \sqrt{\frac{n-j}{j} \Delta} \right)^{j}$$

$(\forall)$ $j \in \{1, 2, \ldots n-1\}$, then $P_j(x) \in P_n(b_1, b_2)$.

See [19] and [20].

Theorem 2.1.2 Let be $P(x) \in P_n(b_1,b_2)$ then:

a) $\bar{x}+\frac{1}{n}\sqrt{\frac{\Delta}{n-1}} \le x_1 \le \bar{x}+\frac{1}{n}\sqrt{(n-1)\cdot\Delta}$,

b) $\bar{x}-\frac{1}{n}\sqrt{(n-1)\Delta} \le x_n \le \bar{x}-\frac{1}{n}\sqrt{\frac{1}{(n-1)}\cdot\Delta}$,

c) $\bar{x}-\frac{1}{n}\sqrt{\frac{k-1}{n-k+1}\Delta} \le x_k \le \bar{x}+\frac{1}{n}\sqrt{\frac{n-k}{k}\cdot\Delta}$,

$(\forall) k \in \{2,3,...,n-1\}$.

If we denoted by $\{(y_1)_j,(y_2)_j,(y_3)_j,...,(y_n)_j\}$ the roots of $P_j(x)$ for fixed *'j'*, $j \in \{1, 2,\dots, n\text{-}1\}$ to the Theorem 2.1.1, then the previous delimitations are optimal in the set $P_n(b_1,b_2)$ so then exist j natural number, such that:

$$\min_{P\in P_n(b_1,b_2)}\{x_1\} = (y_1)_{n-1},\ \min_{P\in P_n(b_1,b_2)}\{x_i\} = (y_i)_{j-1};\ 2\le i\le n,$$

$$\max_{P\in P_n(b_1,b_2)}\{x_n\} = (y_n)_1,\ \max_{P\in P_n(b_1,b_2)}\{x_i\} = (y_i)_j;\ 1\le i\le n-1.$$

See [19] and [20].

2.2 A different distribution for the polynomials $P(x)\in P_n(b_1,b_2)$ roots.

Theorem 2.2.1 Let be $P(x)\in P_n(b_1,b_2)$. Be it:

$$\alpha_1 = \frac{1}{n}\sqrt{\frac{\Delta}{n-1}}\ ;\ \beta_1 = \frac{1}{n}\sqrt{(n-1)\Delta}\ ;$$

$$\alpha_n = -\frac{1}{n}\sqrt{(n-1)\Delta}\ ;\ \beta_n = -\frac{1}{n}\sqrt{\frac{\Delta}{n-1}};$$

$$\alpha_k = -\frac{1}{n}\sqrt{\frac{k-1}{n-k+1}\Delta};\ \ \beta_k = \frac{1}{n}\sqrt{\frac{n-k}{k}\Delta},$$

$k=\overline{2,n-1}$, $\mathrm{I}_k = [\alpha_k,\beta_k]$ for $k=\overline{1,n}$.

Then we have:

a) For $k=\overline{1,n}$, $x_k \in \mathrm{I}_k = [\bar{x}+\alpha_k,\ \bar{x}+\beta_k]$,

b) For $k\in\{2,3,..,n-2\}$: $\mathrm{I}_k \cap \mathrm{I}_{k+1} = [\alpha_k,\ \beta_{k+1}]$,

c) For $k\in\{2,3,..,n-1\}$: $\alpha_k = \beta_{n-k+1}$ and $\alpha_2=\beta_n$, $\alpha_1=\beta_{n-1}$,

d) $\alpha_n<\beta_n$, $\beta_1>\beta_2$, $\alpha_n<\alpha_{n-1}$,

e) $x_k \in [\bar{x}+\alpha_n,\ \bar{x}+\beta_1\ ]$, $k\in\{1,2,3,..,n\}$,

$\mathrm{I}_1\cap \mathrm{I}_n = \{\phi\}$, $\mathrm{I}_1\cap \mathrm{I}_{n\text{-}1} = \{\bar{x}+\alpha_1\} = \{\bar{x}+\beta_{n-1}\}$,

$\mathrm{I}_2\cap \mathrm{I}_n = \{\bar{x}+\alpha_2\} = \{\bar{x}+\beta_n\}$.

In I_2 respectively in $\mathrm{I}_{n\text{-}1}$ we can found all the polynomial roots but x_1 respectively x_n can be only at the limits of the intervals.

f) In the interval $[\bar{x}+\beta_k, \bar{x}+\beta_{k-1}]$ respectively in the interval $[\bar{x}+\alpha_{n-k},\ \bar{x}+\alpha_{n-(k-1)}]$ we can found at most the roots $\{x_k, x_{k-1},...,x_2,x_1\}$ respectively $\{x_n,x_{n-1},...,x_{n-k+2},x_{n-k+1}\}$ that is at most *'k'* roots for $k\in\{2,3,..,n\}$.

Proof: a) $x_k \in \mathrm{I}_k = [\bar{x}+\alpha_k, \bar{x}+\beta_k\,]$, $k=\overline{1,n}$ is obvious from previous theorem.

b) For $k\in\{2,3,..,n-2\}$:

$$\beta_k>\beta_{k+1} \Leftrightarrow \frac{1}{n}\sqrt{\frac{n-k}{k}\Delta} \ge \frac{1}{n}\sqrt{\frac{n-k-1}{k+1}\cdot\Delta} \Leftrightarrow$$

$$\frac{n-k}{k} > \frac{n-k-1}{k+1}\ \ /\cdot k(k+1) \Leftrightarrow n>0.$$

$$\alpha_k>\alpha_{k+1} \Leftrightarrow -\frac{1}{n}\sqrt{\frac{k-1}{n-k+1}\cdot\Delta} > -\frac{1}{n}\sqrt{\frac{k}{n-k}\cdot\Delta} \Leftrightarrow$$

$$\frac{k-1}{n-k+1} < \frac{k}{n-k}\ \ /\cdot(n-k+1)(n-k) \Leftrightarrow n>0.$$

Then both relations are true.

c) $\alpha_1=\beta_{n-1} \Leftrightarrow -\frac{1}{n}\sqrt{\frac{\Delta}{n-1}} = -\frac{1}{n}\sqrt{\frac{\Delta}{n-1}}$ obvious,

$\alpha_2=\beta_n \Leftrightarrow -\frac{1}{n}\sqrt{\frac{2-1}{n-2+1}\Delta} = -\frac{1}{n}\sqrt{\frac{\Delta}{n-1}}$ obvious.

$\alpha_k=\beta_{n-k+1}$, $k=\overline{2,n-1}$ also is obvious from the theorem notations.

d) $\alpha_n<\beta_n \Leftrightarrow$

$$\Leftrightarrow -\frac{1}{n}\sqrt{(n-1)\cdot\Delta} \le -\frac{1}{n}\sqrt{\frac{1}{n-1}\cdot\Delta} \Leftrightarrow (n-1)^2\ge 0$$

$$\beta_1>\beta_2 \Leftrightarrow \frac{1}{n}\sqrt{(n-1)\Delta} > \frac{1}{n}\sqrt{\frac{n-2}{2}\Delta} \Leftrightarrow n>0.$$

Also we can prove that:

$$\alpha_n = -\frac{1}{n}\sqrt{(n-1)\Delta} < \alpha_{n-1} = -\frac{1}{n}\sqrt{\frac{n-2}{2}\Delta}\ .$$

e) From the point b) and d): $\alpha_k>\alpha_{k+1}$, $\beta_k>\beta_{k+1}$ for $k\in\{1,2,3,..,n-1\}$. Now the first relation is immediately. The others result from point c).

f) We can observe that x_1, x_2 and only these roots, can be in $[\bar{x}+\beta_2, \bar{x}+\beta_1]$. Then x_1, x_2 and x_3 and only these roots can be in $[\bar{x}+\beta_3, \bar{x}+\beta_2]$ and from the same proceed we observe that $\{x_k, x_{k-1},\dots,x_2,x_1\}$ and only these roots can be in $[\bar{x}+\beta_k, \bar{x}+\beta_{k-1}]$. Similarly, $\{x_n, x_{n-1},\dots x_{n-k+2}, x_{n-(k-1)}\}$ and only these roots can be in the interval $[\bar{x}+\alpha_{n-k}, \bar{x}+\alpha_{n-(k-1)}]$.

Theorem 2.2.2 If $P(x) \in P_n(b_1, b_2)$, for $x_n \le x_{n-1} \le ... \le x_2 \le x_1$, the roots of P we have:

a) $sep(P) \le \frac{2}{n}\sqrt{\frac{3\Delta}{n^2-1}}$ the equality is realised for

$$P(x) = \prod_{k=1}^{n}\left(x - \bar{x} - \frac{n-2k+1}{n} \cdot \sqrt{\frac{3\Delta}{n^2-1}}\right),$$

b) $\sqrt{\frac{\Delta}{\left[\frac{n}{2}\right]\cdot\left[\frac{n+1}{2}\right]}} \le x_1 - x_n \le \sqrt{\frac{2\Delta}{n}}.$

See [19] and [20] from References.

Theorem 2.2.3

a) $(\forall)\ j \in \{2,3,..,n-1\}$ the length of the interval $[\bar{x} + \beta_j, \bar{x} + \beta_{j-1}]$ or $[\bar{x} + \alpha_{n-j}, \bar{x} + \alpha_{n-(j-1)}]$ is $\beta_{j-1} - \beta_j = \frac{\sqrt{\Delta}}{n} \cdot \frac{\sqrt{n-j+1}\cdot\sqrt{j} - \sqrt{j-1}\cdot\sqrt{n-j}}{\sqrt{j-1}\cdot\sqrt{j}}$

and $\beta_{j-1} - \beta_j < \frac{\sqrt{\Delta}}{\sqrt{j-1}\cdot\sqrt{j}}$.

b) For $n \ge 3$, is possible that $\beta_{j-1} - \beta_j \le sep(P)$.

c) A sufficiently condition for having at most one root in the interval $[\bar{x} + \beta_j, \bar{x} + \beta_{j-1}]$ is:

$$\frac{\sqrt{\Delta}}{n} \cdot \frac{\sqrt{n-j+1}\cdot\sqrt{j} - \sqrt{j-1}\cdot\sqrt{n-j}}{\sqrt{j-1}\cdot\sqrt{j}} \le sep(P).$$

Proof:

a) $\beta_j = \frac{1}{n}\cdot\sqrt{\Delta}\cdot\sqrt{\frac{n-j}{j}}$; $\beta_{j-1} = \frac{1}{n}\cdot\sqrt{\Delta}\cdot\sqrt{\frac{n-j+1}{j-1}}$, $(\forall)\ j \in \{2,3,..,n-1\}$ then:

$$\beta_{j-1} - \beta_j = \frac{\sqrt{\Delta}}{n} \cdot \frac{\sqrt{n-j+1}\cdot\sqrt{j} - \sqrt{j-1}\cdot\sqrt{n-j}}{\sqrt{j-1}\cdot\sqrt{j}}.$$

$n-j+1 \ge 1$; $j \ge 1$; $j-1 \ge 1$; $n-j \ge 1$ for $j \in \{2,3,..n-1\}$. Hence

$$\sqrt{n-j+1}\sqrt{j} - \sqrt{j-1}\sqrt{n-j} < (n-j+1)j - (j-1)(n-j) = n.$$

So $\beta_{j-1} - \beta_j < \frac{\sqrt{\Delta}}{\sqrt{j-1}\cdot\sqrt{j}}$.

From the previous theorem, point d), $[\bar{x} + \beta_j, \bar{x} + \beta_{j-1}]$ respectively $[\bar{x} + \alpha_{n-j}, \bar{x} + \alpha_{n-(j-1)}]$ are the same length.

b) If $\beta_{j-1} - \beta_j \le sep(P)$, then from Theorem 2.2.1 we have $\beta_{j-1} - \beta_j \le sep(P) \le \frac{2}{n}\sqrt{\frac{3\Delta}{n^2-1}}$.

From the previous point, a), we have : $\beta_{j-1} - \beta_j < \frac{\sqrt{\Delta}}{\sqrt{j-1}\cdot\sqrt{j}}$. Supposing $\frac{2}{n}\sqrt{\frac{3\Delta}{n^2-1}} \le \frac{\sqrt{\Delta}}{\sqrt{j-1}\sqrt{j}}$ we can obtain $n \ge 3$.

c) If $\beta_{j-1} - \beta_j \le sep(P)$ then in the interval $[\bar{x} + \beta_j, \bar{x} + \beta_{j-1}]$ we can have at most one of the polynomial roots. Then from previous point a), the result is immediately.

Theorem 2.2.4 Let be $P(x) \in P_n(b_1, b_2)$ then we can introduce the polynomial $Q(x) = P(x + \bar{x})$ and $H(x) = x^r \cdot Q(x)$, $r>0$ natural and we have:

a) $H(x) \in P_{n+r}(c_1, c_2)$ where $c_1, c_2 \in R$,

b) If (y_i), $i \in \{1,2,3,..n+r\}$ are the roots of $H(x)$ such that: $y_{n+r} \le y_{n+r-1} \le y_{n+r-2} \le .. \le y_2 \le y_1$, then: $\bar{y} = (1-n)\bar{x}$ is the average of the roots of H. The discriminant of H is $\Delta_r = \Delta + r(b_1^2 - 2b_2)$ or $\Delta_r = (n+r-1)\cdot b_1^2 - 2(n+r)b_2$,

c) If we denote:

$$\alpha_1^r = \frac{1}{n+r}\sqrt{\frac{\Delta_r}{n+r-1}};\ \beta_1^r = \frac{1}{n+r}\sqrt{(n+r-1)\Delta_r};$$

$$\alpha_n^r = -\frac{1}{n+r}\sqrt{(n+r-1)\Delta_r};\ \beta_n^r = -\frac{1}{n+r}\sqrt{\frac{\Delta_r}{n+r-1}};$$

$$\alpha_j^r = -\frac{1}{n+r}\sqrt{\frac{j-1}{n+r-j+1}\Delta_r},$$

$$\beta_j^r = \frac{1}{n+r}\sqrt{\frac{n+r-j}{j}\Delta_r};\ j = \overline{2, n-1},$$ then

the roots of the polynomial H: $y_i \in [\alpha_n^r, \beta_1^r]$, $i = \overline{1, n+r}$.

d) In the interval $[\bar{x} + \beta_j^r, \bar{x} + \beta_{j-1}^r]$ respectively in interval $[\bar{x} + \alpha_{n-j}^r, \bar{x} + \alpha_{n-(j-1)}^r]$ we found at most the roots $\{x_j, x_{j-1}, ..., x_2, x_1\}$ respectively $\{x_{n+r}, x_{n+r-1}, ..., x_{n+r-j+2}, x_{n+r-(j-1)}\}$ so 'j' roots for $j \in \{2,3,\ldots,n+r\}$.

Proof: **a)**, **b)** Obvious **c)**, **d)** See Theorem 2.2.1 e) for $x^r \cdot P(x) \in P_{n+r}(c_1, c_2)$.

Theorem 2.2.5 With previous notations for $j \in \{2,3,.., n+r-1\}$:

i) $\beta_j^r < \beta_{j-1}^r$ and $\beta_j^{r-1} < \beta_{j-1}^{r-1}$,

ii) $\beta_{j-1}^r > \beta_j^{r-1}$ and $\beta_{j-2}^r < \beta_{j-1}^{r-1}$,

iii) $\beta_j^r > \beta_j^{r-1}$ and $\beta_{j-1}^r > \beta_{j-1}^{r-1}$ for $j \in \{2,3,...n-1\}$,

iv) For $0 < p < r$, p natural, $\beta_{j-1}^r > \beta_j^{r-p}$.

For $r > 0$ if exist $j \in \{2,3,...n+r-1\}$ fixed, such that $(\exists)\ x_{j-1} \in [\beta_j^{r-p}, \beta_{j-1}^{r-p}]$ then:

$\{x_{j-1}\} \in [\beta_j^r, \beta_{j-1}^r] \cap [\beta_j^{r-p}, \beta_{j-1}^{r-p}]$,

v) For $j \in \{2,3,..n+r-1\}$, $\beta_j^r < \sqrt{\dfrac{\Delta_r}{n+r}}$.

supposing j, n fixed and not depending at r then: $\lim_{r\to\infty} \beta_j^r = \sqrt{b_1^2 - 2b_2}$.

Proof:

i) $\beta_j^r < \beta_{j-1}^r \Leftrightarrow \dfrac{1}{n+r}\sqrt{\dfrac{n+r-j}{j}}.\sqrt{\Delta_r} <$

$\dfrac{1}{n+r}\sqrt{\dfrac{n+r-j+1}{j-1}}.\sqrt{\Delta_r}$ obvious.

Similar $\beta_j^{r-1} < \beta_{j-1}^{r-1}$. We observe the analogy with previous theorem**.**

ii) $\beta_{j-1}^r > \beta_j^{r-1} \Leftrightarrow \dfrac{1}{n+r}\sqrt{\dfrac{n+r-j+1}{j-1}}\sqrt{\Delta_r} >$

$> \dfrac{1}{n+r-1}\cdot\sqrt{\dfrac{n+r-1-j}{j}}\sqrt{\Delta_{r-1}}$.

But $\Delta_r - \Delta_{r-1} = (b_1^2 - 2b_2) > 0$ and is enough to prove that:

$$\frac{1}{n+r}\sqrt{\frac{(n+r)-(j-1)}{j-1}} > \frac{1}{(n+r)-1}\cdot\sqrt{\frac{n+r-(j+1)}{j}}.$$

For simplicity we denote $n+r=x$ and we obtain:

$$\frac{1}{x}\cdot\frac{\sqrt{x-(j-1)}}{\sqrt{j-1}} > \frac{1}{x-1}\cdot\frac{\sqrt{x-(j+1)}}{\sqrt{j}} \Leftrightarrow$$

$$\Leftrightarrow (x-1)\cdot\sqrt{x-(j-1)}\sqrt{j} > x\sqrt{x-(j+1)}\sqrt{j-1} \overset{()^2}{\Leftrightarrow}$$

$$\Leftrightarrow x^3 j - x^2(j^2-j) - 2x^2 j + 2x(j^2-j) + xj -$$

$$-(j^2-j) > x^3(j-1) - x^2(j^2-1) \Leftrightarrow$$

$$x^3 + x^2(-j-1) + x(2j^2-j) - j^2 + j > 0.$$

But $x^3 - x^2(j+1) = x^2[x-(j+1)] > 0$ because $x = r+n > j+1$, $j \in \{2,3,...,n-1\}$ and $x(2j^2-j) - j^2 > (x-1)j^2 > 0$ for $x > 1$.

$\beta_{j-2}^r < \beta_{j-1}^{r-1}$ is immediately from $\beta_j^r < \beta_{j-1}^r$ replacing j with j-1.

iii) $\beta_j^r > \beta_j^{r-1} \Leftrightarrow \dfrac{1}{n+r}\sqrt{\dfrac{n+r-j}{j}}.\sqrt{\Delta_r} >$

$\dfrac{1}{n+r-1}\cdot\sqrt{\dfrac{n+r-1-j}{j}}\sqrt{\Delta_{r-1}} \Leftrightarrow$

$$\frac{\sqrt{(n+r)-j}}{n+r} > \frac{\sqrt{(n+r)-(j+1)}}{(n+r)-1}\sqrt{\frac{\Delta_{r-1}}{\Delta_r}}.$$

If we denote $n+r=x$ we obtain:

$$\frac{\sqrt{x-j}}{x} > \frac{\sqrt{x-(j+1)}}{x-1}\cdot\sqrt{\frac{\Delta_{r-1}}{\Delta_r}}.$$

The next relation is easy to prove when we observe the positivity of the denominators.

$$\frac{\Delta_{r-1}}{\Delta_r} = \frac{(n+r-2)b_1^2 - 2(n+r-1)b_2}{(n+r-1)b_1^2 - 2(n+r)b_2} =$$

$$= \frac{(x-2)b_1^2 - 2(x-1)b_2}{(x-1)b_1^2 - 2xb_2} < \frac{x-1}{x},$$

The sufficiently relation for prove become:

$$\frac{x-j}{x^2} > \frac{x-(j+1)}{(x-1)^2}\cdot\frac{x-1}{x} \Leftrightarrow$$

$$\Leftrightarrow \frac{x-j}{x} > \frac{x-(j+1)}{x-1} \Leftrightarrow j > 0$$

Now $\beta_{j-1}^r > \beta_{j-1}^{r-1}$ is obvious from previous relation replacing j with j-1.

iv) From ii) $\beta_{j-1}^r > \beta_j^{r-1}$ and from iii)

$\beta_j^{r-1} > \beta_j^{r-p}$, for $0 < p < r$. Then $\beta_{j-1}^r < \beta_j^{r-p}$.

$(\beta_j^r, \beta_{j-1}^r) \cap (\beta_j^{r-p}, \beta_{j-1}^{r-p}) \neq \{\phi\}$.

Then from hypothesis we have $x_{j-1} \in [\beta_j^{r-p}, \beta_{j-1}^{r-p}]$.

Supposing $x_{j-1} \notin [\beta_j^r, \beta_{j-1}^r]$ we have a contradiction with: $x_{j-1} \in [\alpha_j^r, \beta_{j-1}^r]$ and the supposition is false.

v) Results from Theorem 2.2.4 b), c):

$\Delta_r = \Delta + r(b_1^2 - 2b_2) = (n+r-1)\cdot b_1^2 - 2(n+r)b_2$,

$$\alpha_j^r = -\frac{1}{n+r}\sqrt{\frac{j-1}{n+r-j+1}\Delta_r},$$

$$\beta_j^r = \frac{1}{n+r}\sqrt{\frac{n+r-j}{j}\Delta_r};\ j = \overline{2,n-1},\ \text{then}$$

$$\beta_j^r < \frac{1}{n+r}\sqrt{\frac{n+r}{1}\Delta_r} = \sqrt{\frac{\Delta_r}{n+r}}$$

Supposing j,n fixed we obtain:

$$\lim_{r\to\infty}\beta_j^r = \lim_{r\to\infty}\frac{1}{n+r}\sqrt{\frac{n+r-j}{j}}\cdot$$

$$\cdot\sqrt{(n+r-1)\cdot b_1^2-2(n+r)b_2}=\sqrt{b_1^2-2b_2},$$

$$\lim_{r\to\infty}\beta_j^r=\sqrt{b_1^2-2b_2}.$$

Observation 2.2.1 From the last theorem **i), ii), iii)** we observe the distribution for numbers β_j^r as we can see in **fig.1**:

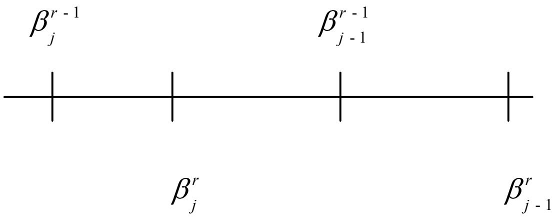

fig.1

Observation 2.2.2 If we find in one of the intervals of Theorem 2.2.4 d) a single root of the polynomial, we can redefine the interval using the relation $\{x_{j-1}\}\in[\beta_j^r,\beta_{j-1}^r]\cap[\beta_j^{r-p},\beta_{j-1}^{r-p}]$, $(\forall)\ r>0,\ (\forall)\ p>0$ naturals and taking a large r; see Theorem 2.2.5 iv) .

Application 2.2.1

For a polynomial with all real roots and with degree n=3, $x_k\in \mathrm{I}_k=[\overline{x}+\alpha_k,\overline{x}+\beta_k]$, $k=\overline{1,3}$, and the intervals are isolating intervals.
The proof is easy to make starting to the Theorem 2.2.1.

In the general case, for a real polynomial, if $(\exists)k\in\{1, 2, 3\}$ such that $P(\alpha_k)\cdot P(\beta_k)>0$ the polynomial will have only one real root and $(\exists)i\in\{1, 2, 3\}$ so that $P(\alpha_i)\cdot P(\beta_i)<0$.
Therefore we determine the interval which contains the root , $x_i\in \mathrm{I}_i$.

In both cases we can redefine the intervals containing roots, simply by dividing them and using continuous property functions, or using last Theorem 2.2.4 iv) and introducing the polynomials $Q(x)=P(x+\overline{x})$ and $H(x)=x^r\cdot Q(x)$ where r>0 natural.

Many practice processes use or can be modelled with the help of the real roots of the polynomials with small degree, see for example relation (27) in [21].

2.3 A roots distribution for complex polynomials

Theorem 2.3.1 For an arbitrary complex polynomial, $P(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$, with $n\ge 1$, $a_n\ne 0$ and for $p\in\{1, 2, \ldots n\}$ such that $|x_1|\ge|x_2|\ge\ldots\ge|x_p|\ge 1\ge|x_{p+1}|\ldots\ge|x_n|$ then

$$1\le|x_i|\le\left[\frac{M(P)}{|a_n|}\right]^{1/i}\le\left[\frac{\|P\|}{|a_n|}\right]^{1/i}\ \text{ for } i=\overline{1,p} \text{ and}$$

$$\left[\frac{|a_0|}{\|P\|}\right]^{1/n-i+1}\le\left[\frac{|a_0|}{M(P)}\right]^{1/n-i+1}\le|x_i|\le 1,$$

for $i=\overline{p+1,n}.$

Proof: From Definition 1.5 and Proposition 1.3:

$$|a_n|\cdot\prod_{i=1}^{n}\max\{1,|x_i|\}=M(P)\le\|P\|,$$

$$\frac{|a_0|}{\prod_{i=1}^{n}\min\{1,|x_i|\}}=M(P)\le\|P\| \qquad (1)$$

$$\text{Then } |x_1|\cdot|x_2|\cdot\ldots\cdot|x_p|=\frac{M(P)}{|a_n|}\ge 1 \qquad (2)$$

Because $|x_i|\ge 1$ for $i=\overline{1,p}$ then $|x_1|\le\dfrac{M(P)}{|a_n|}.$

Supposing

$$|x_2|>\left[\frac{M(P)}{|a_n|}\right]^{1/2} \text{ then } |x_1|\ge|x_2|>\left[\frac{M(P)}{|a_n|}\right]^{1/2},$$

But $|x_1|\ge\ldots\ge|x_p|\ge 1$ and then

$$|x_1|\cdot|x_2|\ldots\cdot|x_p|\ge|x_1|\cdot|x_2|>\frac{M(P)}{|a_n|},$$

contradiction with (2).

Then the supposition is false and $|x_2|\le\left[\dfrac{M(P)}{|a_n|}\right]^{1/2}.$

Using the induction method then

$$1\le|x_i|\le\left[\frac{M(P)}{|a_n|}\right]^{1/i},\ i=\overline{1,p} \qquad (3)$$

Starting for (2): $|x_{p+1}|\cdot\ldots\cdot|x_n|=\dfrac{|a_0|}{M(P)}\le 1.$

Because $|x_i|\le 1$ for $i=\overline{p+1,n}$ then $|x_n|\ge\dfrac{|a_0|}{M(P)}$

Supposing

$|x_{n-1}| < \left[\frac{|a_0|}{M(P)}\right]^{1/2}$ then $|x_n| \le |x_{n+1}| < \left[\frac{|a_0|}{M(P)}\right]^{1/2}$ and then

$|x_{n-1}| \cdot |x_n| < \frac{|a_0|}{M(P)}$ and $|x_{p+1}| \cdot ... \cdot |x_{n-1}| \cdot |x_n| \le$

$\le |x_{n-1}||x_n| < \frac{|a_0|}{M(P)}$ contradiction with (2).

The supposition is false and $|x_{n-1}| \ge \left[\frac{|a_0|}{M(P)}\right]^{1/2}$.

By induction we can prove that

$\left[\frac{|a_0|}{M(P)}\right]^{1/i} \le |x_{n-i+1}| \le 1$ for $i = \overline{1, n-p}$ or

$$\left[\frac{|a_0|}{M(P)}\right]^{1/n-i+1} \le |x_i| \le 1, \text{ for } i = \overline{p+1, n} \quad (4)$$

Now because $M(P) \le \|P\|$ we have the result.

Corollary 2.3.1
If $1 > |x_1| \ge ... \ge |x_n|$ then $M(P) = a_n$ for $i = \overline{1, n}$:

$$\left[\frac{|a_0|}{\|P\|}\right]^{1/n-i+1} \le \left[\frac{|a_0|}{|M(P)|}\right]^{1/n-i+1} \le |x_i| < 1 = \left[\frac{M(P)}{|a_n|}\right]^{1/i},$$

Proof:

$$M(P) = |a_n| \cdot \prod_{i=1}^{n} \max\{1, |x_i|\} = |a_n|, \frac{M(P)}{|a_n|} = 1.$$

We can repeat the procedure above and then

$$\left[\frac{|a_0|}{|M(P)|}\right]^{1/n-i+1} = \left[\frac{|a_0|}{|a_n|}\right]^{1/n-i+1} \le |x_i| < 1, \text{ for } i = \overline{1, n}.$$

Proposition 2.3.1 For $n \ge 1$,

$$(\ln n) + \gamma < \sum_{i=1}^{n} \frac{1}{i} < (\ln n) + \gamma + \frac{1}{2n} \text{ where}$$

$\gamma \approx 0,577...$ is Euler constant. (5)

Proof:
It is well known, see for that and for other similar inequalities [22], that

$$\lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{1}{i} - \ln n + c_n\right) = \gamma, \text{ where } (c_n)_{n\ge 1}, \text{real,}$$

$$\lim_{n\to\infty} c_n = 0, \gamma \approx 0,577... \text{ is Euler constant.} \quad (6)$$

$$u_n = \sum_{i=1}^{n} \frac{1}{i} - \frac{1}{2n} - \ln n \to \gamma, \; v_n = \sum_{i=1}^{n} \frac{1}{i} - \ln n \to \gamma,$$

are obtained from the relation above. (7)

Then $u_{n+1} - u_n = \frac{2n+1}{2(n+1)n} + \ln\frac{n}{n+1}$,

$$v_{n+1} - v_n = \frac{1}{n+1} + \ln\frac{n}{n+1} \quad (8)$$

For the functions $u(x) = \frac{2x+1}{2(x+1)\cdot x} + \ln\frac{x}{x+1}$,

$v(x) = \frac{1}{x+1} + \ln\frac{x}{x+1}$, $x > 0$.

$$u'(x) = \frac{-1}{2\cdot(x+1)^2\cdot x^2} < 0, \; v'(x) = \frac{1}{x\cdot(x+1)^2} > 0.$$

Because $\lim_{x\to\infty} u(x) = 0$, $\lim_{x\to\infty} v(x) = 0$. For $n > 0$,

then $u(n) = u_{n+1} - u_n > 0$, $v(n) = v_{n+1} - v_n < 0$.

$(u_n)_{n\ge 1}$ is strictly increasing, $u_n < \gamma$,

$(v_n)_{n\ge 1}$ is strictly decreasing, $v_n > \gamma$ (9)

From (7), and (9) we have the result.

Observation 2.3.1 In the previous results we generalise the inequality: $1 \le |x_i| \le (\|P\|/|a_n|)^{1/i}$ for $i = \overline{1, p}$, see [23] from references.

Theorem 2.3.2 For an arbitrary complex polynomial with degree $n \ge 1$, and the leading coefficient a_n, for $p \in \{1, 2, ...n\}$ so that $|x_1| \ge |x_2| \ge ... \ge |x_p| \ge 1 \ge |x_{p+1}| ... \ge |x_n|$ then:

$$\left[\ln(n-p) + \gamma + \frac{1}{2(n-p)}\right] \cdot \ln\left[\frac{a_0}{M(P)}\right] \le$$

$$\le \ln\left|\frac{a_0}{a_n}\right| \le \left(\ln\frac{n}{p} + \frac{1}{2n}\right) \cdot \ln\left[\frac{M(P)}{|a_n|}\right].$$

where $\gamma \approx 0,577...$ is Euler constant.

Proof: From theorem 2.3.1 we have:

$$\left[\frac{|a_0|}{M(P)}\right]^{\sum_{i=p+1}^{n} \frac{1}{n-i+1}} \le |x_1| \cdot |x_2| \cdot ... \cdot |x_n| =$$

$$= \left|\frac{a_0}{a_n}\right| \le \left[\frac{M(P)}{|a_n|}\right]^{\sum_{i=p+1}^{n} \frac{1}{i}}. \quad (10)$$

But $\ln n + \gamma < \sum_{i=1}^{n} \frac{1}{i} < \ln n + \gamma + \frac{1}{2n}$ where

$\gamma \approx 0,577...$ is Euler constant. (11)

$$\sum_{i=p+1}^{n} \frac{1}{n-i+1} = \sum_{i=1}^{n-p} \frac{1}{i} < \ln(n-p) + \gamma + \frac{1}{2(n-p)}, \quad (12)$$

$$\sum_{i=p+1}^{n} \frac{1}{i} = \sum_{i=1}^{n} \frac{1}{i} - \sum_{i=1}^{p} \frac{1}{i} < \ln n + \gamma + \frac{1}{2n} - \ln p - \gamma$$

$$\sum_{i=p+1}^{n} \frac{1}{i} > \ln \frac{n}{p} + \frac{1}{2n}. \quad (13)$$

Then from (10), (11) and (13) we have the result.

Observation 2.3.2 The previous theorem can be useful when we try to evaluate $p \in N$ the number of the roots with modules bigger or equal with one. Another similar theorem is the next one.

Theorem 2.3.3 For a complex polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$, $n \geq 1$, with $a_0 \cdot a_n \neq 0$; where $p \in N^*$ such that: $|x_1| \geq |x_2| \geq ... \geq |x_p| \geq 1 > |x_{p+1}| ... \geq |x_n|$, then

$$p \leq 2\sqrt{n \cdot \ln\left(\frac{L(P)}{|a_n|}\right)},$$ see [24], [25] .

3. Minimum Roots Separation

Theorem 3.1

If $P(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$; $n \geq 1$, $a_n \neq 0$, $P \in C[x]$ is square free then:

a) $sep(P) > n^{-\frac{n+2}{2}} \cdot |disc(P)|^{1/2} \cdot \| P \|^{1-n}$

and for $P \in Z[x]$: $sep(P) > n^{-\frac{n+2}{2}} \cdot \| P \|^{1-n}$,

b) For $P \in Z[x]$: $sep(P) > \frac{1}{2} e^{-n/2} n^{-3n/2} \|P\|^{-n}$.

c) $sep(P) > min(1, |a_n|)^{n(\ln n+1)} \cdot |\operatorname{disc}(P)| \cdot \cdot \{(2n)^{n-1} L(P)^{n \ln(n+3)}\}^{-1}$,

$sep(P) > \{2 \cdot n^{\frac{n}{2}+2} [L(P)+1]^n\}^{-1}$.

To prove **a)** see [7], for **b)** see [26] for **c)** see [23] from references. For others inequalities in this area, see [14] .

Proposition 3.1 Let be $n \geq 2$, $f, g, h : (0, +\infty) \to (0, +\infty)$,

$f(x) = 1 + x + x^2 + ... + x^{n-1}$,

$g(x) = 1 + 2^2 x + 3^2 x^2 + ...(n-1)^2 x^{n-2}$,

$h(x) = \frac{f(x)}{g(x)}$. Then f, g are monotonically increasing functions, $g(x) = \left(x \cdot f'(x)\right)'$ and

$$f(x) = \begin{cases} \frac{x^n - 1}{x-1}, & x \neq 1 \\ n, & x = 1 \end{cases},$$

$$g(x) = \begin{cases} \frac{\left(n^2 - 2n + 1\right) x^{n+1} - (2n^2 - 2n - 1) x^n}{(x-1)^3} + \\ + \frac{n^2 x^{n-1} - x - 1}{(x-1)^3} & \text{for } x \neq 1 \\ \frac{n(n-1)(2n-1)}{6}, & \text{for } x = 1 \end{cases}$$

The proof can be done from calculation.

Corollary 3.1 With the previous notations: $h(x) \geq \frac{1}{(n-1)^2}$, $h(1) = \frac{6}{(n-1)(2n-1)}$.

Proof: We can observe that: $f, g, h : (0, +\infty) \to (0, +\infty)$,

$$h(x) = \frac{1 + x + x^2 + ...x^{n-1}}{1 + 2^2 x + 3^2 x^2 + ...(n-1)^2 x^{n-2}}, \quad (14)$$

$$h(x) \geq \frac{1 + x + x^2 + ...x^{n-1}}{(n-1)^2 + (n-1)^2 x + ... + (n-1)^2 x^{n-2}},$$

$$h(x) \geq \frac{1}{(n-1)^2}, \quad h(1) = \frac{f(1)}{g(1)} = \frac{6}{n(2n-1)}. \quad (15)$$

Theorem 3.2

For $P(x) = a_0 + a_1 x + .. + a_n x^n$, $a_n \neq 0$; R a real numbers such that $x_i \leq R$, and for $1 < x_i$, x_i real for $i = \overline{1,n}$, $n \geq 2$, $l = \left[\frac{n}{2}\right]$,

then $(\exists)$ $c \in [1, R^2]$, such that:

$$\sqrt{disc(P) \cdot h(c)} \cdot \sqrt{\frac{\left|\sum_{k=0}^{n} a_k \cdot \sum_{k=0}^{n} (-1)^k a_k\right|}{a_0^{2n} + a_2^{2n} + ...a_{2l}^{2n}}} \leq sep(P).$$

Proof: Supposing $|x_n| \leq |x_{n-1}| \leq ... \leq |x_1|$ and using the functions *f*, *g*, *h*, the Hadamard's inequality and Proposition 1.1 we can obtain:

$$D=(-1)^{n(n-1)/2}\begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{vmatrix},$$

$|D| \le \sqrt{S\cdot T}$ where $S=\left(\sum_{k=1}^{n-1}\left|x_j{}^k - x_i{}^k\right|^2\right)$ and

$$T=\prod_{j=1,j\ne i}^{n}\left(\sum_{k=1}^{n-1}\left|x_j{}^k\right|^2\right). \tag{16}$$

$$S=\left|x_j-x_i\right|^2\cdot(1+\left|x_j+x_i\right|^2+\dots$$

$$\dots+\left|x_j{}^{n-2}+x_j{}^{n-3}x_i+\dots x_i{}^{n-3}x_j+x_i{}^{n-2}\right|^2),$$

For $1\le i<j\le n$, $|x_i|\ge|x_j|$ and $|x_j+x_i|\le 2\cdot|x_i|$,

and using the triangle inequality we can write:

$$S\le\left|x_j-x_i\right|^2\cdot g\left(\left|x_i\right|^2\right),\ T=\frac{\prod_{i=1}^{n} f\left(\left|x_i\right|^2\right)}{f\left(\left|x_i\right|^2\right)} \tag{17}$$

From (16) and (17):

$$|D|\cdot\frac{\sqrt{h\left(\left|x_i\right|^2\right)}}{\sqrt{\prod_{i=1}^{n} f\left(\left|x_i\right|^2\right)}}\le\left|x_j-x_i\right|, \tag{18}$$

$\sqrt{h(x)}$ is a continuous function on $(1,R^2]$ and exist $c\in[1,R^2]$ so that:

$$\min\left\{\sqrt{h\left(\left|x_i\right|^2\right)}/\ i=\overline{1,n}\right\}=\sqrt{h(c)} \tag{19}$$

Then, because $x_k\notin\{-1,1\}, k=\overline{1,n}$,

$$\sum_{k=0}^{n}a_k\ne 0,\ \sum_{k=0}^{n}(-1)^k a_k\ne 0.$$

$$\prod_{i=1}^{n} f\left(\left|x_i\right|^2\right)=\frac{\left|\prod_{i=1}^{n}\left(x_i{}^{2n}-1\right)\right|}{\left|\prod_{i=1}^{n}\left(x_i{}^2-1\right)\right|} \tag{20}$$

Using the Viete's relations we can observe that

$$\left|\prod_{i=1}^{n}\left(x_i{}^2-1\right)\right|=\left|\prod_{i=1}^{n}(x_i-1)(x_i+1)\right|,$$

$$\left|\prod_{i=1}^{n}\left(x_i{}^2-1\right)\right|=\left|\prod_{i=1}^{n}(x_i-1)\right|\left|\prod_{i=1}^{n}(x_i+1)\right|, \tag{21}$$

$$\left|\prod_{i=1}^{n}\left(x_i{}^2-1\right)\right|=\frac{\left|\sum_{k=0}^{n}a_k\cdot\sum_{k=0}^{n}(-1)^k a_k\right|}{a_n^2}. \tag{22}$$

$$\left|\prod_{i=1}^{n}\left(x_i{}^{2n}-1\right)\right|=\left|(x_1\dots x_i\dots x_n)^{2n}-\sum_{j=1}^{n}\left(\prod_{\substack{i=1\\ i\ne j}}^{n}x_i\right)^{2n}+\right.$$

$$+\sum_{\substack{j,k=1\\ j<k}}^{n}\left(\prod_{\substack{i=1\\ j<k,i\notin\{j,k\}}}^{n}x_i\right)^{2n}+\dots+$$

$$\left.+(-1)^{n-1}\left(x_1{}^{2n}+\dots x_i{}^{2n}\dots+x_n{}^{2n}\right)+(-1)^n\right|\le$$

$$\le(x_1\dots x_i\dots x_n)^{2n}+\left(\sum_{j,k=1}^{n}\prod_{\substack{i=1\\ j<k,i\notin\{j,k\}}}^{n}x_i\right)^{2n}+\dots$$

$$\left|\prod_{i=1}^{n}\left(x_i{}^{2n}-1\right)\right|\le\frac{a_0^{2n}+a_2^{2n}+\dots a_{2l}^{2n}}{a_n^{2n}}$$

where $l=\left[\frac{n}{2}\right]$. (23)

From (20), (22), (23):

$$\prod_{i=1}^{n} f\left(\left|x_i\right|^2\right)\le\frac{a_0^{2n}+a_2^{2n}+\dots a_{2l}^{2n}}{a_n^{2n-2}\left|\sum_{k=0}^{n}a_k\cdot\sum_{k=0}^{n}(-1)^k a_k\right|} \tag{24}$$

From (18), (19), (24):

$$|D|\cdot\sqrt{h(c)}\cdot\sqrt{\frac{a_n^{2n-2}\left|\sum_{k=0}^{n}a_k\cdot\sum_{k=0}^{n}(-1)^k a_k\right|}{a_0^{2n}+a_2^{2n}+\dots a_{2l}^{2n}}}\le\left|x_j-x_i\right|$$

where $1\le i<j\le n$. (25)

$\dfrac{\sqrt{disc(P)}}{\left|a_n^{n-1}\right|}=|D|$ (see Proposition 1.1**).**

Using (25) we obtain the result.

Theorem 3.3 For $n\ge 1$, $a_n\ne 0$,

$P(x)=a_n x^n+a_{n-1}x^{n-1}+\dots+a_1 x+a_0\in C[x]$, if exist $p \in \{1, 2, ...n\}$ such that $|x_1| \geq |x_2| \geq ... \geq |x_p| \geq 1 \geq |x_{p+1}| ... \geq |x_n|$ then:

a) $$sep(P) \geq \frac{\sqrt{|disc(P)|}}{|a_n|^{n-1} n^{n/2}(n-1)} \cdot \left[\frac{|a_n|}{\|P\|}\right]^{\left(\gamma + \ln p + \frac{1}{2p}\right)\cdot(n-1)}.$$

b) If $P(x) \in P_n(b_1, b_2)$ then

$$sep(P) \geq \frac{\sqrt{|disc(P)|}}{|a_n|^{n-1} \cdot n^{n/2}(n-1)} \left(\prod_{i=1}^{n} \gamma_i\right)^{1-n}.$$

where $\gamma_i = \mathbf{max}\{\left|\frac{-a_{n-1}}{na_n} + \alpha_i\right|, \left|\frac{-a_{n-1}}{na_n} + \beta_i\right|, 1\}$.

Proof : a) From the previous theorem relation (12):

$$|D| \leq |x_i - x_j| \cdot \left[\prod_{i=1}^{n} f(|x_i|^2) \cdot \frac{f(|x_i|^2)}{g(|x_i|^2)}\right]^{1/2} \quad (26)$$

Then $$|x_i - x_j| \geq \frac{|D|}{\sqrt{\prod_{i=1}^{n} f(|x_i|^2)}} \cdot \sqrt{\frac{f(|x_i|^2)}{g(|x_i|^2)}} \quad (27)$$

$$\frac{f(|x_i|^2)}{g(|x_i|^2)} \geq h(c),\ h(c) = \min\{h(|x_i|^2) / i = \overline{1, n}\}, \quad (28)$$

From Corollary 3.1

$$h(x) \geq \frac{1}{(n-1)^2},\ \sqrt{h(c)} \geq \frac{1}{n-1}. \quad (29)$$

Then $$\frac{\sqrt{disc(P)}}{|a_n^{n-1}|} = |D| \quad (30)$$

$$\prod_{i=1}^{n} f(|x_i|^2) = \prod_{i=1}^{p} f(|x_i|^2) \cdot \prod_{i=p+1}^{n} f(|x_i|^2)$$

$$\prod_{i=1}^{n} f(|x_i|^2) \leq \prod_{i=1}^{p} n|x_i|^{2n-2} \cdot \prod_{i=p+1}^{n} f(1),$$

$$\prod_{i=1}^{n} f(|x_i|^2) \leq n^n \prod_{i=1}^{p} |x_i|^{2n-2}. \quad (31)$$

Now using Theorem 2.3.1 we obtain:

$$\prod_{i=1}^{n} f(|x_i|^2) \leq n^n \left[\prod_{i=1}^{p} \left(\frac{M(P)}{|a_n|}\right)^{2n-2/i}\right],$$

$$\prod_{i=1}^{n} f(|x_i|^2) \leq n^n \left[\frac{M(P)}{|a_n|}\right]^{2(n-1)\cdot \sum_{i=1}^{p} \frac{1}{i}}.$$

$$\prod_{i=1}^{n} f(|x_i|^2) \leq n^n \left[\frac{M(P)}{|a_n|}\right]^{2(n-1)\cdot \sum_{i=1}^{p} \frac{1}{i}}.$$

$$\prod_{i=1}^{n} f(|x_i|^2) \leq n^n \cdot \left(\frac{M(P)}{|a_n|}\right)^{2\left(\gamma + \ln p + \frac{1}{2p}\right)\cdot(n-1)} \quad (32)$$

Then from (27), (29), (30) and (32) we obtain the result.

b) $$\prod_{i=1}^{n} f(|x_i|^2) \leq \prod_{i=1}^{p} n|x_i|^{2n-2} \cdot \prod_{i=p+1}^{n} n \cdot 1^2,$$

$$\prod_{i=1}^{n} f(|x_i|^2) \leq n^n \cdot \prod_{i=1}^{p} |x_i|^{2n-2} \leq n^n \cdot \prod_{i=1}^{n} |\gamma_i|^{2n-2}$$

for $$\gamma_i = \mathbf{max}\left\{\left|\frac{-a_{n-1}}{na_n} + \alpha_i\right|, \left|\frac{-a_{n-1}}{na_n} + \beta_i\right|, 1\right\} \quad (33)$$

see Theorem 2.2.1. Replacing (32) with (33) in the demonstration to the first point, we have the result.

Theorem 3.4 For $n \geq 1$, $a_n \neq 0$,

$P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 \in C[x]$,

If $1 > |x_1| \geq |x_2| \geq ... \geq |x_i| \geq |x_{i-1}| \geq .... \geq |x_{n-1}| \geq |x_n|$ then

$$sep(P) \geq \frac{\sqrt{|disc(P)|}}{|a_n|^{n-1} n^{n/2}(n-1)}.$$

Proof:

We follow the steps to the previous theorem, replacing (32) with the relation

$$\prod_{i=1}^{n} f(|x_i|^2) \leq n^n. \quad (34)$$

Observation 3.1

In the previous theorems we can use the relation: $disc(P) \geq 1$, for $P \in Z[x]$ we obtain similar results for integers polynomials.

4 Isolating the roots. Conclusions

Remark 4.1 If we compare the results from the sections **2.1** with our result from **2.2** one of the advantage of the second approach, as we can see in **Theorem 2.2.1 f)**, is that we can predict the roots and the maximum number of the roots which can be in the intervals $[\overline{x} + \beta_k, \overline{x} + \beta_{k-1}]$ respectively in the intervals $[\overline{x} + \alpha_{n-k}, \overline{x} + \alpha_{n-(k-1)}]$ and we can give the lengths of the intervals. Also we can redefine the intervals containing roots. See **Theorem 2.2.5 iv), v)**, creating another polynomial and knowing that for $Q(x) = P(x + \overline{x})$ and

$H(x) = x^r \cdot Q(x),$ r>0 natural exist $p>0$ natural so that $(\exists)\ x_{j-1} \in [\beta_j^{r-p}, \beta_{j-1}^{r-p}]$. Then we have:

$\beta_{j-1}^r > \beta_j^{r-p}$, $\{x_{j-1}\} \in [\beta_j^r, \beta_{j-1}^r] \cap [\beta_j^{r-p}, \beta_{j-1}^{r-p}]$.

Our results, **Theorem 2.3.1,** using Mahler's Measure, represent natural inequalities for bounding every module's root of the polynomial and giving the roots repartition for complex polynomials.

Using these, in a natural way, we obtain a new theorem with the best inequalities from the method presented, **Theorem 2.3.2,** about the numbers of the roots that are outside of the unit circle. We can compare the theorem with one of Szego's theorem.

Remark 4.2

a) For comparing our result from **Theorem 3.2,** of the minimum roots separation, for the polynomial having the roots $1 < x_i$, for $i = \overline{1,n}$, with the others, we can take as we can see in **Corollary 3.1**,

$$\sqrt{h(c)} \geq \frac{1}{n-1}. \tag{35}$$

Then

$$\frac{\left|\sum_{k=0}^{n} a_k \cdot \sum_{k=0}^{n} (-1)^k a_k\right|}{a_0^{2n} + a_2^{2n} + \ldots a_{2l}^{2n}} \geq \frac{C}{L(P)^{2n}} = \frac{C}{L(P)^{2n}},$$

$$l = \left[\frac{n}{2}\right], \ C>0 \text{ particularly, convenient}. \tag{36}$$

From the theorem, we can prove for n, a natural great number:

$$\sqrt{disc(P)} \cdot \frac{C}{\sqrt{n} \cdot (n-1)} \cdot \frac{1}{L(P)^n} \leq sep(P). \tag{37}$$

Our result contains $n^{-3/2}$ while in all the others papers appear, $n^{-s(n)}$ where $s(x)$ is a real continuous function. But the polynomial have all roots real and positive and $1 < x_i \leq R, (\forall)\ i = \overline{1,n}$.

To obtain a result where the roots are not positive, we can apply the theorem for the polynomial $Q(x) = P(R-x),\ R > x_1$, where sep (Q)=sep(P).

b) Now we can observe from the previous relation For the integer polynomials we have $\sqrt{disc(P)} \geq 1$, taking r and R as we can see in **Corollary 1.1** then

$$O\left[\log_2 \frac{R-r}{sep(P)}\right] = O\left(n \ln L(P)\right) \tag{38}$$

is the order for the number of successive splitting of the interval $[-R,-r] \cup [r,R]$ until we accomplish the root pre-isolation.

We can precise the evaluation known in the general case for the successive number of splitting:

$$O\left[\log_2 \frac{R-r}{sep(P)}\right] = O\left(n \ln\left[n \cdot L(P)\right]\right) \tag{39}$$

for more details see [8], [27], [28], [29].

c) The cost for isolating the roots, in the case of the complex polynomials, which is the number of the arithmetic operations needed, is dominated to the number of successive divisions multiplied by the cost of Sturm's series assessment see [1], [8], [27], [29], [30] or by the cost of polynomial evaluation, in a point, see [28], or by others numbers of operations see [2]. For the polynomials with all roots real, we make, the divisions of the intervals: $[\bar{x} + \beta_j, \bar{x} + \beta_{j-1}]$ and $[\bar{x} + \alpha_{n-j}, \bar{x} + \alpha_{n-(j-1)}]$ to the previous section and we obtain the minimum operations of splitting, then we apply Sturm's Theorem or other methods, for isolating the roots.

Remark 4.3

In **Theorem 3.3**, **Theorem 3.4** we give new results about minimum roots separation for complex polynomials and for polynomials with all real roots. One of idea is to use the bounds for modules of the roots, given in **Theorem 2.3.1**.

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