

The Polynomial Roots Repartition and Minimum Roots Separation

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Abstract:

It is known that, if all the roots of a polynomial are real, they can be localised, using a set of intervals, which contain the arithmetic average of the roots. The aim of this paper is to present an original method for giving other distributions of the roots/ modules of the roots, on real axis, a method for evaluating and improving the “polynomial minimum root separation” results, a method for the complex polynomials and for polynomials having all roots real. We use the discriminant, Hadamard’s inequality, Mahler’s measure and new original inequalities. Also we will make some considerations about the cost for isolate the polynomial real roots. Our method is based on the successive splitting for the interval which contain all roots.

Key-Words: Roots repartition, Isolating the roots, Mahler’s measure

1 Introduction

The roots repartition of a complex polynomial, on real axis, means to give real intervals, not necessarily disjunctive, for every polynomial root or module of the polynomial root.

Pre-isolating respectively isolating the complex roots of a polynomial with complex coefficients, means to compute separating boxes in the complex plane, which contains at most one, respectively exact one complex root of the polynomial.

The “exact algorithms” for isolating the roots are based on: Sturm’s sequences, see [1]; differentiation technique, see [2] and Vincent’s theorem, see [3].

Other root finding methods are the numerical methods. These methods work with a great variety of approximation errors. Recently results which generalize the univariate Hermite interpolation formula can be found in [4], [5], [6].

The numerical algorithms, compute an approximation for all the complex roots of a polynomial up to a desired accuracy and if that is smaller than “minimum roots separation” then the algorithms can be turned on the isolation algorithms. For the same approximation error, the costs of the fastest ‘isolating algorithms’ and ‘numerical algorithms’ are comparable.

The aim of this article is to give original results regarding the roots repartition on real axis, the minimum roots separation for complex polynomials

and for the polynomials with all real roots. These results are necessarily in “exact algorithms” for isolating the roots based on successive splitting (see [1] and [2]) and can be useful also in numerical algorithms.

In this section we introduce the basis notations and notions and present the preliminary results in this field. We denote a complex polynomial with

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0, \quad n \geq 1,$$

and its roots with x_1, \dots, x_n .

Definition 1.1 Let be $P(x) \in C[x]$. We define:

a) $\Delta = \sum_{1 \leq i < j \leq n} (x_i - x_j)^2$.

b) $sep(P) = \min\{|x_i - x_j| \mid x_i \neq x_j, 1 \leq i, j \leq n\}$

the minimum roots separations.

c) $\|P\| = \sqrt{|a_0|^2 + |a_1|^2 + \dots + |a_n|^2}$ - the polynomial norm.

d) $L(P) = |a_0| + |a_1| + \dots + |a_n|$ - the polynomial length.

e) $\bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n}$ the average of the roots.

Definition 1.2 The discriminant of a polynomial $P \in C[x]$ with leading coefficient a_n and roots

x_1, \dots, x_n is defined as:

$$\text{disc}(P) = a_n^{2n-2} \cdot \prod_{1 \leq i < j \leq n} (x_i - x_j)^2.$$

Proposition 1.1 Let be $P \in C[x]$

a) The expression of $D = \prod_{1 \leq i < j \leq n} (x_i - x_j)$, is

$$D = (-1)^{n(n-1)/2} \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{vmatrix}.$$

b) $\text{disc}(P) = a_n^{2n-2} \cdot D^2$.

Definition 1.3 Let be

$$P = P(x) = a_m x^m + a_{m-1} x^{m-1} + \dots + a_1 x + a_0, \quad \text{and}$$

$$Q = Q(x) = b_n x^n + b_{n-1} x^{n-1} + \dots + b_1 x + b_0,$$

$m \geq 1, a_m \neq 0, n \geq 1, b_n \neq 0$. Sylvester's matrix of P and Q is the matrix S :

$$S = \begin{pmatrix} a_m & a_{m-1} & \dots & a_0 & 0 & 0 & \dots & 0 \\ 0 & a_m & a_{m-1} & \dots & a_0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \dots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & a_m & a_{m-1} & \dots & a_0 \\ b_n & b_{n-1} & \dots & b_0 & 0 & 0 & \dots & 0 \\ 0 & b_n & b_{n-1} & \dots & b_0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \dots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & b_n & b_{n-1} & \dots & b_0 \end{pmatrix}$$

where there are n rows of the a_i followed by m rows of the b_j . $S = (s_{i,j})$ is a matrix with $m + n$ rows and $m + n$ columns having the elements:
 $s_{i,j} = a_{m-j+i}$ for $1 \leq i \leq n$ and $s_{n+1,j} = b_{n-j+i}$ for $1 \leq i \leq m$.

Definition 1.4 The resultant of P and Q is the determinant of Sylvester's matrix, $\text{res}(P; Q) = \det(S)$

Proposition 1.2 $a_n \text{disc}(P) = \text{res}(P; P')$.

See [7] or [8].

Observation 1.1 The discriminant, $\text{disc}(P)$ can be expressed only by the degree and the polynomial coefficients. See Definition 1.3 and 1.4 and Proposition 1.2.

For many others results in this area see [9].

Theorem 1.1 ([7]) Let be

$$P(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1} + x^n; \quad a_i \in C;$$

$i = 0; n-1, n \geq 1$, with the roots $x_1, x_2, \dots, x_n \in C$;

if $r_0 = \max\{|x_1|, |x_2|, \dots, |x_n|\}$ then:

a) $r_0 \leq \max\left\{1, \sum_{k=0}^{n-1} |a_k|\right\},$

b) $r_0 \leq 1 + \max_{0 \leq k \leq n-1} \{|a_k|\},$

c) If $\lambda_1, \dots, \lambda_n \in (0; +\infty)$ such that

$$\lambda_1^{-1} + \dots + \lambda_n^{-1} = 1 \text{ then}$$

$$r_0 \leq \max\left\{1, \max_{1 \leq k \leq n} \left\{ \lambda_k \cdot |a_{n-k}|^{1/k} \right\}\right\},$$

d) $r_0 \leq \max\left\{1, \max_{1 \leq k \leq n} (n \cdot |a_{n-k}|)^{1/k}\right\},$

e) For $a_k \neq 0; k \in \{0, 1, \dots, n-1\}$

$$\text{we have: } r_0 \leq \max\left\{2 \cdot \left|\frac{a_{n-2}}{a_{n-1}}\right|, \dots, 2 \cdot \left|\frac{a_1}{a_2}\right|, \left|\frac{a_0}{a_1}\right|\right\},$$

f) $r_0 \leq |1 - a_{n-1}| + |a_{n-2} - a_{n-1}| + \dots$

$$\dots + |a_0 - a_1| + |a_0|,$$

g) $r_0 \leq \max_{1 \leq k \leq n} \left\{ \left(\frac{|a_{n-k}|}{C_n^k} \right)^{1/k} \cdot (2^{1/n} - 1)^{-1} \right\}.$

Corollary 1.1

Let be $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0, n \geq 1$.

$P \in R[x]$, then $(\exists) r, R$ real, such that

$$r \leq |x_i| \leq R, (\forall) i = \overline{1, n}.$$

Proof: Obviously results from the Theorem 1.1,

replacing a_i with $\frac{a_i}{a_n}$, for $i \in \{1, \dots, n-1\}$.

For $r, R \notin \{1\}$ we can take:

$$R = \frac{L(P)}{|a_n|} > 1 \text{ and } r = \frac{|a_0|}{L(P)} < 1.$$

For others useful bounds see [10] and [11].

Definition 1.5 The Mahler Measure of the polynomial P , denoted by $M(P)$ is:

$$M(P) = M[P(x)] = |a_n| \cdot \prod_{j=1}^n \max\{1, |x_j|\}.$$

Theorem 1.2

Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0, n \geq 1$.

$a_n \cdot a_0 \neq 0, P(x) \in C[x]$ then:

$$\ln[M(P)] = \frac{1}{2\pi} \int_0^{2\pi} \ln |P(e^{i\theta})| d\theta.$$

