Embedding a family of 2D meshes into Möbius cubes

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Abstract: Möbius cubes are an important class of hypercube variants. This paper addresses how to embed a family of disjoint 2D meshes into a Möbius cube. Two major contributions of this paper are: **(1)** For $n \geq 1$, there exists a $2 \times 2^{n-1}$ mesh that can be embedded in the n-dimensional Möbius cube with dilation 1 and expansion 1. **(2)** For $n \geq 4$, there are two disjoint $4 \times 2^{n-3}$ meshes that can be embedded in the n-dimensional 0-type Möbius cube with dilation 1. The results are optimal in the sense that the dilations of the embeddings are 1. The result (2) mean that a family of two 2D-mesh-structured parallel algorithms can be operated on a same crossed cube efficiently and in parallel.

Key–Words: Möbius cubes, mesh embedding, dilation, expansion, interconnection network.

1 Introduction

An interconnection network plays a critical role of a multi-computer because the system performance is deeply dependent on network latency and throughput. There are a lot of mutually conflicting requirements in designing the topology of interconnection networks. It is almost impossible to design a network which is optimum in all perspectives. Processors of a multi-processor system are connected according to a given interconnection network. The topological structure of an interconnection network can be modeled by a graph whose vertices represent components of the network and whose edges represent links between components. An embedding of one guest graph, G, into another host graph, H, is a one-to-one mapping ϕ from the vertex set of G to the vertex set of H. An edge of G corresponds to a path of H under ϕ. Many application, such as architecture simulations and processor allocations, can be modeled as graph embedding [1, 2, 3, 4, 7, 8, 9, 13, 14, 15, 17, 18, 22, 23, 24, 25, 26, 29, 30].

There are two natural measures of the cost of a graph embedding, namely, the *dilation* of the embedding: the maximum distance in H between the images of vertices that are adjacent in G; and the *expansion* of the embedding: the ratio of the size of H to the size of G. For any two vertices x and y in G, let $d_G(x, y)$ denote the distance from x to y in G, i.e., the length of a shortest path between x and y in G. The dilation of embedding ϕ is defined as $dil(G, H, \phi) = max\{d_H(\phi(x), \phi(y)) \mid (x, y) \in E(G)\}$. The meaning of dilation for an embedding is the performance of communication delay when the graph H simulates the graph G. Obviously, $dil(G, H, \phi) \geq 1$. In order to measure the processor utilization of the embedding, the expansion is defined as $exp(G, H, \phi) = |V(H)|/|V(G)|$. The smaller the dilation and expansion of an embedding is that the more efficient the communication delay and processor utilization when the graph H simulates the graph G.

The hypercube is a popular interconnection network with many attractive properties such as regularity, symmetry, small diameter, strong connectivity, recursive construction, partition ability, and relatively low link complexity [19]. It has been used in a wide variety of parallel systems such as Intel iPSC, the nCUBE [10], the Connection Machine CM-2 [21], and SGI Origin 2000 [20]. A hypercube network of dimension n contains up tp 2^n nodes and has n edges per node. If unique n-bit binary address are assigned to the nodes of hypercube, then an edge connects two nodes if and only if their binary addresses differ in a single bit position. Because of its elegant topological properties and the ability to emulate a wide variety of other frequently used networks, the hypercube has been one of the most popular interconnection networks for parallel computer/communication systems. Thus, there are several variations of the hypercube have been proposed in the literature. Möbius cubes form a class of hypercube variants that give better performance with the same number of edges and vertices. The paths, cycles, trees, and meshes are the common interconnection structures used in parallel computing. Embedding of these structures into Möbius cubes have been studied in [5, 6, 11, 12, 27, 28]. However, there has been no research so far on embeddings of meshes in Möbius cubes in the literature. In this paper, we consider embedding of meshes in Möbius cubes. The main results obtained in this paper are: **(1)** For $n \geq 1$, there exists a $2 \times 2^{n-1}$ mesh that can be embedded in the n-dimensional Möbius cube with dilation 1 and expansion 1. **(2)** For $n \geq 4$, there are two disjoint $4 \times 2^{n-3}$ meshes that can be embedded in the n-dimensional 0-type Möbius cube with dilation 1. The results are optimal in the sense that the dilation 1.

The rest of this paper is organized as follows. In the next section, some fundamental definitions and notions are introduced. Section 3 shows that there exists a $2 \times 2^{n-1}$ mesh embedding in the n-dimensional Möbius cube. Section 4 proposes that two disjoint $4 \times 2^{n-3}$ meshes are embedded in n-dimensional 0-type Möbius cubes with dilation 1. The last section contains discussions and conclusions.

2 Preliminaries

Let the interconnection network be modeled by an undirected graph $G = (V, E)$ where the set of vertices $V(G)$ represents the processing elements of the network and the set of edges $E(G)$ represents the communication links. Throughout this paper, for the graph theoretic definitions and notations we follow [16]. Let $G = (V, E)$ be an undirected graph. Two vertices are adjacent when they are incident with a common edge. A *simple path* (or *path* for short) is a sequence of adjacent edges (v_0, v_1), (v_1, v_2), ..., (v_{m-1}, v_m), written as $\langle v_0, v_1, v_2, \ldots, v_m \rangle$, in which all the vertices $v_0, v_1, \ldots, v_m$ are distinct except possibly $v_0 = v_m$. The *distance* between x and y in G is denoted by $d_G(x, y)$, which is the length of a shortest path between x and y in G. A *cycle* C is a special path with at least three vertices such that the first vertex is the same as the last one. A cycle of length k is called a k-cycle. Let S be a subset of $V(G)$. The subgraph of G *induced* by S is the subgraph that has S as its vertex set and contains all edges of G having two end vertices in S. Two subgraphs of G are *node-disjoint* (or *disjoint* for short) if they have no common vertex.

The n-dimensional Möbius cube MQ_n, proposed first by Cull and Larson [5], consists of 2^n vertices and each vertex has a unique n-component binary vector for an address. Each vertex has n neighbors as follows. A vertex x denoted by a binary string of length n, $x_n x_{n-1} \ldots x_1$, connects to its ith neighbor, denoted by $N_i(x)$, for $1 \leq i \leq n-1$,

$$N_i(x) = x_n x_{n-1} \ldots x_{i+1} \overline{x}_i x_{i-1} \ldots x_1 \text{ if } x_{i+1} = 0.$$

or

$$N_i(x) = x_n x_{n-1} \ldots x_{i+1} \overline{x}_i \overline{x}_{i-1} \ldots \overline{x}_1 \text{ if } x_{i+1} = 1.$$

For $i = n$, since there is no bit on the left of x_n, $N_n(x)$ can be defined as the nth neighbor of x can be denoted as $\overline{x}_n x_{n-1} \ldots x_1$ or $\overline{x}_n \overline{x}_{n-1} \ldots \overline{x}_1$. If we assume that the $(n+1)$th bit of every vertex of MQ_n is 0, we call the network a ***0-type*** n-dimensional Möbius cube, denoted by 0-MQ_n; and if we assume that the $(n+1)$th bit of every vertex of MQ_n is 1, we call the network a ***1-type*** n-dimensional Möbius cube, denoted by 1-MQ_n. Either 0-MQ_n or 1-MQ_n may be denoted by MQ_n. The example of 0-MQ_4 and 1-MQ_4 are shown in Fig 1.

For example, let $u = 01011$ be a vertex of 0-MQ_5. The 4-,3-,2-,1-, and 0-neighbors of u are given by 11011, 00011, 01100, 01001, and 01010, respectively. The symbol $N(u)$ is used to denote the set of neighbors of u and $N(01011) = \{11011, 00011, 01100, 01001, 01010\}$. Similarly, let $u = 01011$ be a vertex of 1-MQ_5. The 4-,3-,2-,1-, and 0-neighbors of u are given by 10100, 00011, 01100, 01001, and 01010, respectively.

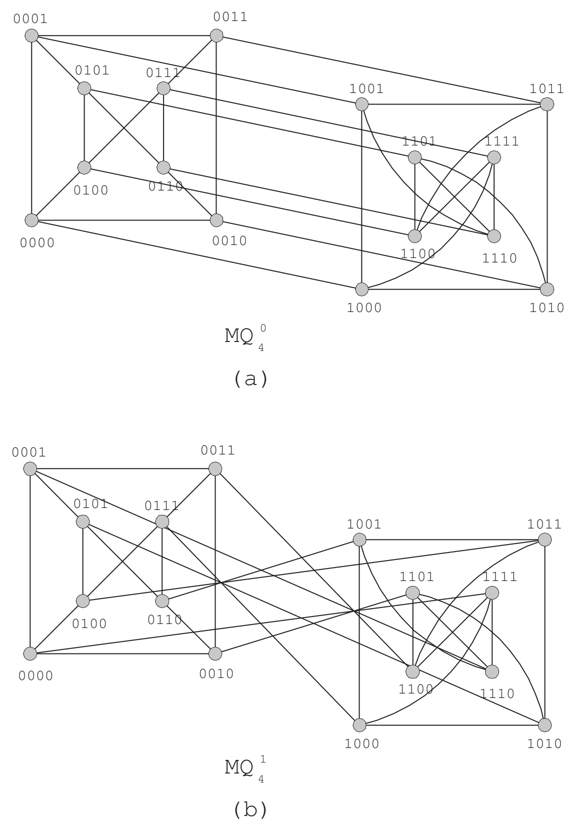

Figure 1: (a) A 0-type 4-dimensional Möbius cube. (b) A 1-type 4-dimensional Möbius cube.

Therefore, MQ_n is an n-regular graph and can be recursively defined as follows: Both 0-MQ_1 and 1-MQ_1 are complete graph K_2 with one vertex labeled 0 and the other 1. 0-MQ_n and 1-MQ_n are both composed of a sub-Möbius cube MQ^0_{n-1} and a sub-Möbius cube MQ^1_{n-1}. Each vertex $X = 0x_{n-1}x_{n-2}\dots x_2x_1 \in V(MQ^0_{n-1})$ connects to $1x_{n-1}x_{n-2}\dots x_2x_1 \in MQ^1_{n-1}$ in 0-MQ_n and to $1\overline{x}_{n-1}\overline{x}_{n-2}\dots\overline{x}_2\overline{x}_1$ in 1-MQ_n. For convenience, we say that MQ^0_{n-1} and MQ^1_{n-1} are two *sub-Möbius cubes* of MQ_n, where MQ^0_{n-1} (respectively, MQ^1_{n-1}) is an $(n-1)$-dimensional 0-type Möbius cube (respectively, 1-type Möbius cube) which includes all vetices $0x_{n-1}x_{n-2}\dots x_2x_1$ (respectively, $1x_{n-1}x_{n-2}\dots x_2x_1$), $x_i \in \{0,1\}$. An edge (u,v) in $E(MQ_n)$ is of *dimension* i if $u = N_i(v)$. In addition, we define the edge set of dimension i of MQ_n to be $E_i(MQ_n) = \{(x,y) \in E(MQ_n) \mid y = N_i(x)\}$. Indeed, there are 2^{n-1} elements in $E_i(MQ_n)$ for all $1 \le i \le n$. Every n-dimension edge is called to be a *crossing edge* between MQ^0_{n-1} and MQ^1_{n-1} of MQ_n.

Lemma 1 *Let x and y be two vertices of an n-dimensional 0-type Möbius cube 0-MQ_n with $n \ge 3$, and $y = N_i(x)$. Then $d_{0-MQ_n}(N_n(x), N_n(y)) = 1$ if $1 \le i \le n-2$ and $d_{0-MQ_n}(N_n(x), N_n(y)) = 2$ if $i = n-1$.*

Proof. Let $x = x_nx_{n-1}\dots x_{i+1}x_ix_{i-1}\dots x_1$ where $x_j \in \{0,1\}$ for $1 \le j \le n$. Since y is an ith neighbor of x, $y = x_nx_{n-1}\dots x_{i+1}\overline{x}_ix_{i-1}\dots x_1$ if $x_{i+1} = 0$ or $y = x_nx_{n-1}\dots x_{i+1}\overline{x}_i\overline{x}_{i-1}\dots\overline{x}_1$ if $x_{i+1} = 1$.

Case 1: $i = n-1$.

Suppose that $x_n = 0$. Then, $N_n(x) = 1x_{n-1}x_{n-2}\dots x_1$ and $N_n(y) = 1\overline{x}_{n-1}x_{n-2}\dots x_1$. By definition, $d_{0-MQ_n}(N_n(x), N_n(y)) > 1$ for $n \ge 3$. If $x_{n-1} = 0$, $N_{n-2}(N_n(y)) = 1\overline{x}_{n-1}\,\overline{x}_{n-2}\dots\overline{x}_1$. Thus, $N_{n-1}(N_{n-2}(N_n(y))) = 1x_{n-1}x_{n-2}\dots x_1$. Hence $d_{0-MQ_n}(N_n(x), N_n(y)) = 2$. If $x_{n-1} = 1$, $N_{n-2}(N_n(x)) = 1x_{n-1}\overline{x}_{n-2}\dots\overline{x}_1$. Hence $N_{n-1}(N_{n-2}(N_{n-1}(x))) = 1\overline{x}_{n-1}x_{n-2}\dots x_1$. Therefore, $d_{0-MQ_n}(N_n(x), N_n(y)) = 2$.

Suppose that $x_n = 1$. Then, $N_n(x) = 0x_{n-1}x_{n-2}\dots x_1$ and $N_n(y) = 0\overline{x}_{n-1}\overline{x}_{n-2}\dots\overline{x}_1$. By definition, $d_{0-MQ_n}(N_n(x), N_n(y)) > 1$ for $n \ge 3$. If $x_{n-1} = 0$, $N_{n-2}(N_n(y)) = 0\overline{x}_{n-1}x_{n-2}\dots x_1$. It is observed that $N_{n-1}(N_{n-2}(N_n(y))) = 0x_{n-1}x_{n-2}\dots x_1$. As a result, $d_{0-MQ_n}(N_n(x), N_n(y)) = 2$. If $x_{n-1} = 1$, $N_{n-2}(N_n(x)) = 0x_{n-1}\overline{x}_{n-2}\dots\overline{x}_1$. Hence $N_{n-1}(N_{n-2}(N_n(x))) = 0\overline{x}_{n-1}\overline{x}_{n-2}\dots\overline{x}_1$. Therefore, $d_{0-MQ_n}(N_n(x), N_n(y)) = 2$.

Case 2: $1 \le i \le n-2$.

Suppose that $x_{i+1} = 0$. $N_n(x) = \overline{x}_nx_{n-1}\dots x_{i+2}0x_i\dots x_1$ and $N_n(y) = \overline{x}_nx_{n-1}\dots x_{i+2}0\overline{x}_ix_{i-1}\dots x_1$. It is obvious that $N_i(N_n(y)) = N_n(x)$. Hence $d_{0-MQ_n}(N_n(x), N_n(y)) = 1$.

Suppose that $x_{i+1} = 1$. $N_n(x) = \overline{x}_nx_{n-1}\dots x_{i+2}1x_i\dots x_1$ and $N_n(y) = \overline{x}_nx_{n-1}\dots x_{i+2}1\overline{x}_i\dots\overline{x}_1$. It is obvious that $N_i(N_n(y)) = N_n(x)$. Hence $d_{0-MQ_n}(N_n(x), N_n(y)) = 1$. The lemma is proved. □

Lemma 2 *Let x and y be two vertices of an n-dimensional 1-type Möbius cube 1-MQ_n with $n \ge 3$, and $y = N_i(x)$. Then $d_{1-MQ_n}(N_n(x), N_n(y)) = 1$ if $i = 1$ and $d_{1-MQ_n}(N_n(x), N_n(y)) = 2$ if $2 \le i \le n-1$.*

Proof. Let $x = x_nx_{n-1}\dots x_{i+1}x_ix_{i-1}\dots x_1$ where $x_j \in \{0,1\}$ for $1 \le j \le n$. Since y is an ith neighbor of x, $y = x_nx_{n-1}\dots x_{i+1}\overline{x}_ix_{i-1}\dots x_1$ if $x_{i+1} = 0$ or $y = x_nx_{n-1}\dots x_{i+1}\overline{x}_i\overline{x}_{i-1}\dots\overline{x}_1$ if $x_{i+1} = 1$.

Case 1: $2 \le i \le n-1$.

Suppose that $x_{i+1} = 0$. $N_n(x) = \overline{x}_n\overline{x}_{n-1}\dots\overline{x}_{i+2}1\overline{x}_i\dots\overline{x}_1$ and $N_n(y) = \overline{x}_n\overline{x}_{n-1}\dots\overline{x}_{i+2}1x_i\overline{x}_{i-1}\dots\overline{x}_1$. By definition, $d_{1-MQ_n}(N_n(x), N_n(y)) > 1$ for $n \ge 3$. If $x_i = 0$, $N_{i-1}(N_n(x)) = \overline{x}_n\overline{x}_{n-1}\dots\overline{x}_{i+2}1\overline{x}_ix_{i-1}\dots x_1$. Hence $N_i(N_{i-1}(N_n(x))) = \overline{x}_n\overline{x}_{n-1}\dots\overline{x}_{i+2}1x_i\overline{x}_{i-1}\dots\overline{x}_1$. Hence $d_{1-MQ_n}(N_n(x), N_n(y)) = 2$. If $x_i = 1$, $N_{i-1}(N_n(y)) = \overline{x}_n\overline{x}_{n-1}\overline{x}_{n-2}\dots\overline{x}_{i+2}1x_ix_{i-1}\dots x_1$. Hence $N_i(N_{i-1}(N_n(y))) = N_n(x)$. Therefore, $d_{1-MQ_n}(N_n(x), N_n(y)) = 2$.

Suppose that $x_{i+1} = 1$. $N_n(x) = \overline{x}_n\overline{x}_{n-1}\dots\overline{x}_{i+2}0\overline{x}_i\dots\overline{x}_1$ and $N_n(y) = \overline{x}_n\overline{x}_{n-1}\dots\overline{x}_{i+2}0x_ix_{i-1}\dots x_1$. By definition, $d_{1-MQ_n}(N_n(x), N_n(y)) > 1$ for $n \ge 3$. If $x_i = 0$, $N_{i-1}(N_n(x)) = \overline{x}_n\overline{x}_{n-1}\dots\overline{x}_{i+2}0\overline{x}_ix_{i-1}\dots x_1$. Hence $N_i(N_{i-1}(N_n(x))) = N_n(y)$. Hence $d_{1-MQ_n}(N_n(x), N_n(y)) = 2$. If $x_i = 1$, $N_{i-1}(N_n(y)) = \overline{x}_n\overline{x}_{n-1}\dots\overline{x}_{i+2}0x_i\overline{x}_{i-1}\dots\overline{x}_1$. Hence $N_i(N_{i-1}(N_n(y))) = N_n(x)$. Therefore, $d_{1-MQ_n}(N_n(x), N_n(y)) = 2$.

Case 2: $i = 1$.

$N_n(x) = \overline{x}_n\overline{x}_{n-1}\dots\overline{x}_3\overline{x}_2\overline{x}_1$ and $N_n(y) = \overline{x}_n\overline{x}_{n-1}\dots\overline{x}_3\overline{x}_2x_1$. It is obvious that $N_1(N_n(y)) = N_n(x)$. Hence $d_{1-MQ_n}(N_n(x), N_n(y)) = 1$. □

According to Lemma 1 and Lemma 2, there exists a 4-cycle of $\langle x, N_1(x), N_1(N_n(x)), N_n(x), x\rangle$ for any vertex x in MQ_n. However, not every 4-cycle in one sub-Möbius cube MQ^i_{n-1} of MQ_n is corresponding to a 4-cycle in the other sub-Möbius cube MQ^{1-i}_{n-1}. Finding a 4-cycle in one sub-Möbius cube MQ^i_{n-1} of MQ_n such that it is corresponding to a 4-cycle in MQ^{1-i}_{n-1} is important for embedding of $2 \times 2^{n-1}$ mesh in MQ_n. The following lemma discusses how to find that 4-cycle.

Lemma 3 *For $n \geq 3$, assume that $\langle a, b, c, d, a \rangle$ is a 4-cycle in MQ_{n-1}^i of MQ_n satisfying (a, b), $(c, d) \in E_1(MQ_n)$ and (b, c), $(a, d) \in E_2(MQ_n)$. Then $N_n(a)$, $N_n(b)$, $N_n(c)$, and $N_n(d)$ forms a 4-cycle in MQ_{n-1}^{1-i} of MQ_n. Moreover, $(N_n(a), N_n(b))$, $(N_n(c), N_n(d)) \in E_1(MQ_n)$, and $(N_n(a), N_n(c))$, $(N_n(b), N_n(d)) \in E_2(MQ_n)$ or $(N_n(a), N_n(d))$, $(N_n(b), N_n(c)) \in E_2(MQ_n)$.*

Proof. It is clearly that the lemma holds for $n = 3$. Assume that $n \geq 4$. Let $a = a_n a_{n-1} \ldots a_3 a_2 a_1$. Thus $b = a_n a_{n-1} \ldots a_3 a_2 \overline{a}_1$. According to the value of a_3, the proof is divided into two cases: **(1)** $a_3 = 0$ and **(2)** $a_3 = 1$.

Case 1: $a_3 = 0$.

Note that $a = a_n a_{n-1} \ldots 0 a_2 a_1$, $b = a_n a_{n-1} \ldots 0 a_2 \overline{a}_1$, $c = a_n a_{n-1} \ldots 0 \overline{a}_2 \overline{a}_1$, and $d = a_n a_{n-1} \ldots 0 \overline{a}_2 a_1$. Suppose that the MQ_n is a 0-type Möbius cube. Since $n \geq 4$, $N_n(a) = \overline{a}_n a_{n-1} \ldots 0 a_2 a_1$, $N_n(b) = \overline{a}_n a_{n-1} \ldots 0 a_2 \overline{a}_1$, $N_n(c) = \overline{a}_n a_{n-1} \ldots 0 \overline{a}_2 \overline{a}_1$, and $N_n(d) = \overline{a}_n a_{n-1} \ldots 0 \overline{a}_2 a_1$. Therefore, $(N_n(a), N_n(b)), (N_n(c), N_n(d)) \in E_1(MQ_n)$ and $(N_n(b), N_n(c)), (N_n(a), N_n(d)) \in E_2(MQ_n)$. Consequently, $\langle N_n(a), N_n(b), N_n(c), N_n(d), N_n(a) \rangle$ is a 4-cycle in the sub-Möbius cube MQ_{n-1}^{1-i} of MQ_n.

Suppose that the MQ_n is a 1-type Möbius cube. Hence $N_n(a) = \overline{a}_n \overline{a}_{n-1} \ldots \overline{a}_4 1 \overline{a}_2 \overline{a}_1$, $N_n(b) = \overline{a}_n \overline{a}_{n-1} \ldots \overline{a}_4 1 \overline{a}_2 a_1$, $N_n(c) = \overline{a}_n \overline{a}_{n-1} \ldots \overline{a}_4 1 a_2 a_1$, and $N_n(d) = \overline{a}_n \overline{a}_{n-1} \ldots \overline{a}_4 1 a_2 \overline{a}_1$. Therefore, $(N_n(a), N_n(b)), (N_n(c), N_n(d)) \in E_1(MQ_n)$ and $(N_n(a), N_n(c)), (N_n(b), N_n(d)) \in E_2(MQ_n)$. Consequently, $\langle N_n(a), N_n(c), N_n(d), N_n(b), N_n(a) \rangle$ is a 4-cycle in the sub-Möbius cube MQ_{n-1}^{1-i} of MQ_n.

Case 2: $a_3 = 1$.

Note that $a = a_n a_{n-1} \ldots 1 a_2 a_1$, $b = a_n a_{n-1} \ldots 1 a_2 \overline{a}_1$, $c = a_n a_{n-1} \ldots 1 \overline{a}_2 \overline{a}_1$, and $d = a_n a_{n-1} \ldots 1 \overline{a}_2 a_1$. Suppose that the MQ_n is a 0-type Möbius cube. Since $n \geq 4$, $N_n(a) = \overline{a}_n a_{n-1} \ldots 1 a_2 a_1$, $N_n(b) = \overline{a}_n a_{n-1} \ldots 1 a_2 \overline{a}_1$, $N_n(c) = \overline{a}_n a_{n-1} \ldots 1 \overline{a}_2 \overline{a}_1$, and $N_n(d) = \overline{a}_n a_{n-1} \ldots 1 \overline{a}_2 a_1$. Therefore, $(N_n(a), N_n(b)), (N_n(c), N_n(d)) \in E_1(MQ_n)$ and $(N_n(a), N_n(c)), (N_n(b), N_n(d)) \in E_2(MQ_n)$. Consequently, $\langle N_n(a), N_n(c), N_n(d), N_n(b), N_n(a) \rangle$ is a 4-cycle in the sub-Möbius cube MQ_{n-1}^{1-i} of MQ_n.

Suppose that the MQ_n is a 1-type Möbius cube. Hence $N_n(a) = \overline{a}_n \overline{a}_{n-1} \ldots \overline{a}_4 0 \overline{a}_2 \overline{a}_1$, $N_n(b) = \overline{a}_n \overline{a}_{n-1} \ldots \overline{a}_4 0 \overline{a}_2 a_1$, $N_n(c) = \overline{a}_n \overline{a}_{n-1} \ldots \overline{a}_4 0 a_2 a_1$, and $N_n(d) = \overline{a}_n \overline{a}_{n-1} \ldots \overline{a}_4 0 a_2 \overline{a}_1$. Therefore, $(N_n(a), N_n(b)), (N_n(c), N_n(d)) \in E_1(MQ_n)$ and $(N_n(b), N_n(c)), (N_n(a), N_n(d)) \in E_2(MQ_n)$. Consequently, $\langle N_n(a), N_n(b), N_n(c), N_n(d), N_n(a) \rangle$ is a 4-cycle in the sub-Möbius cube MQ_{n-1}^{1-i} of MQ_n. $\square$

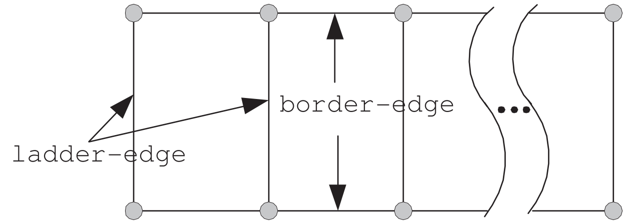

Figure 2: Illustration for ladder-edges and border-edges of 2D mesh with size $2 \times m$.

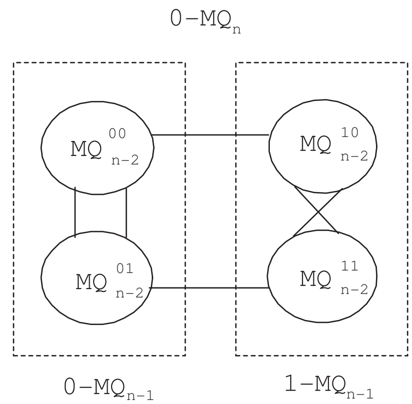

Figure 3: Illustration for 0-MQ_n.

3 Embedding of $2 \times 2^{n-1}$ meshes in MQ_n

Definition 1 *A $n \times m$ mesh $M_{n\times m}$ can be denoted by an $n \times m$ matrix*

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1m} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nm} \end{pmatrix}$$

where $V(M_{n\times m}) = \{\alpha_{ij} \mid 1 \leq i \leq n$, and $1 \leq j \leq m\}$, $(\alpha_{ij}, \alpha_{i,j+1}) \in E(M_{n\times m})$ for $1 \leq i \leq n$ and $1 \leq j \leq m-1$, and $(\alpha_{kl}, \alpha_{k+1,l}) \in E(M_{n\times m})$ for $1 \leq k \leq n-1$ and $1 \leq l \leq m$.

The edge $(\alpha_{1i}, \alpha_{2i})$ in a mesh $M_{2\times m}$ is called to be the ith *ladder-edge* for $1 \leq i \leq m$; two edges $(\alpha_{1j}, \alpha_{1,j+1})$ and $(\alpha_{2j}, \alpha_{2,j+1})$ are called to be the jth pair of *border-edge* for $1 \leq j \leq m-1$. Let $M_{2\times m}(i, j; MQ_n) = \{M_{2\times m} \mid (\alpha_{1k}, \alpha_{2k}) \in E_i(MQ_n)$ for $1 \leq k \leq m$ and there exists an integer $1 \leq l \leq m-1$ such that $(\alpha_{1l}, \alpha_{1,l+1})$ and $(\alpha_{2l}, \alpha_{2,l+1})$ are in $E_j(MQ_n)$. $\}$, i.e., if $M_{2\times m} \in M_{2\times m}(i, j; MQ_n)$, all ladder-edges of $M_{2\times m}$ are in $E_i(MQ_n)$ and there exists a pair of border-edges,

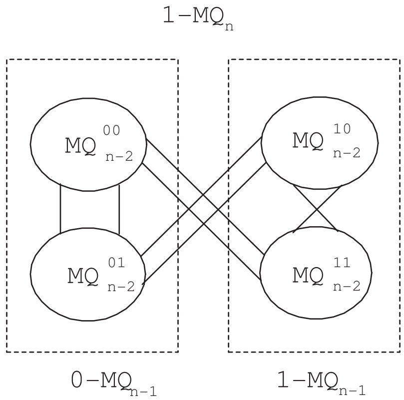

Figure 4: Illustration for 1-MQ_n.

$(\alpha_{1l}, \alpha_{1,l+1})$ and $(\alpha_{2l}, \alpha_{2,l+1})$ for some $1 \le l \le m-1$, such that $(\alpha_{1l}, \alpha_{1,l+1}), (\alpha_{2l}, \alpha_{2,l+1}) \in E_j(MQ_n)$. In this section, we propose that a $2 \times 2^{n-1}$ mesh can be embedded with dilation 1 and expansion 1 in an n-dimensional Möbius cube. According this result, we show that a $4 \times 2^{n-2}$ mesh cab be embedded with dilation 2 and expansion 1 in MQ_n.

Lemma 4 *For any two dimension 1 edges e_1 and e_2 that form a 4-cycle in MQ_3, there exists a 2×4 mesh in $M_{2\times 4}(1, 2; MQ_3)$ where e_1 is the first ladder-edge and e_2 is the last ladder-edge of the mesh, or e_1 is the last ladder-edge and e_2 is the first ladder-edge of the mesh.*

Proof. Since 0-MQ_3 and 1-MQ_3 are isomorphic, we only consider 0-MQ_3. Note that E_1(0-MQ_3) = { $(000, 001)$, $(010, 011)$, $(100, 101)$, $(110, 111)$ } and E_2(0-MQ_3) = { $(000, 010)$, $(001, 011)$, $(100, 111)$, $(101, 110)$ }. Let e_1 and e_2 be in E_1(0-MQ_3) and both of them lie on the same 4-cycle in 0-MQ_3. Hence $\{e_1, e_2\} \subset \{\{(000, 001), (101, 100)\}, \{(000, 001), (011, 010)\}, \{(100, 101), (110, 111)\}, \{(010, 011), (111, 110)\}\}$. Let M_1, M_2, M_3, and M_4 be four 2×4 meshes in 0-MQ_3 as follows.

$$M_1 = \begin{pmatrix} 000 & 010 & 110 & 101 \\ 001 & 011 & 111 & 100 \end{pmatrix}$$

,

$$M_2 = \begin{pmatrix} 000 & 100 & 111 & 011 \\ 001 & 101 & 110 & 010 \end{pmatrix}$$

,

$$M_3 = \begin{pmatrix} 100 & 000 & 010 & 110 \\ 101 & 001 & 011 & 111 \end{pmatrix}$$

, and

$$M_4 = \begin{pmatrix} 010 & 000 & 100 & 111 \\ 011 & 001 & 101 & 110 \end{pmatrix}$$

One can see that all ladder-edges of M_i are in E_1(0-MQ_3) for $1 \le i \le 4$. It is observed that the 3th pair of border-edges in M_1 and M_4 are in the set E_2(0-MQ_3), and the 2th pair of border-edges in M_2 and M_3 are in the set E_2(0-MQ_3). □

Lemma 5 *Assume that $n \ge 3$. For any two dimension 1 edges e_1 and e_2 that form a 4-cycle in MQ_n, there exists a $2 \times 2^{n-1}$ mesh in $M_{2\times 2^{n-1}}(1, 2; MQ_n)$ such that e_1 is the first ladder-edge and e_2 is the last ladder-edge of the mesh, or e_1 is the last ladder-edge and e_2 is the first ladder-edge of the mesh.*

Proof. The proof is by induction on n. By Lemma 4, the lemma holds for $n = 3$. Assume that the lemma is true for every integer $3 \le m < n$. We now consider $m = n$ as follows. Let $e_1 = (a, b)$ and $e_2 = (c, d)$ be two dimension 1 edges and $\langle a, b, c, d, a\rangle$ is a 4-cycle in MQ_n. By the relative position of e_1 and e_2, the proof is divided into two parts: **(1)** e_1 and e_2 are in the same sub-Möbius cube MQ^i_{n-1} and **(2)** $e_1 \in E(MQ^i_{n-1})$ and $e_2 \in E(MQ^{1-i}_{n-1})$ for $i = 0, 1$.

Case 1: $e_1, e_2 \in E(MQ^i_{n-1})$ for $i = 0, 1$.

By the induction hypothesis, there exists a $2 \times 2^{n-2}$ mesh $M^0_{2\times 2^{n-2}} \in M_{2\times 2^{n-2}}(1, 2; MQ^i_{n-1})$ such that e_1 is the first ladder-edge and e_2 is the last ladder-edge of $M^0_{2\times 2^{n-2}}$, or e_1 is the last ladder-edge and e_2 is the first ladder-edge of $M^0_{2\times 2^{n-2}}$. Without loss of generality, we may assume that $M^0_{2\times 2^{n-2}} =$

$$\begin{pmatrix} a = \alpha_1 & \cdots & \alpha_j & \alpha_{j+1} & \cdots & \alpha_{2^{n-2}} = c \\ b = \beta_1 & \cdots & \beta_j & \beta_{j+1} & \cdots & \beta_{2^{n-2}} = d \end{pmatrix}$$

where $(\alpha_k, \beta_k) \in E_1(MQ_n)$ for all $1 \le k \le 2^{n-2}$ and (α_j, α_{j+1}), $(\beta_j, \beta_{j+1}) \in E_2(MQ_n)$ for some $1 \le j \le 2^{n-2} - 1$.

Since (α_j, β_j), $(\alpha_{j+1}, \beta_{j+1}) \in E_1(MQ_n)$, (α_j, α_{j+1}), $(\beta_j, \beta_{j+1}) \in E_2(MQ_n)$, and $\langle\alpha_j, \alpha_{j+1}, \beta_{j+1}, \beta_j, \alpha_j\rangle$ is a 4-cycle in MQ^i_{n-1}. By Lemma 3, $N_n(\alpha_j)$, $N_n(\beta_j)$, $N_n(\alpha_{j+1})$, and $N_n(\beta_{j+1})$ forms a 4-cycle in MQ^{1-i}_{n-1}. Let $e_3 = (N_n(\alpha_j), N_n(\beta_j))$ and $e_4 = (N_n(\alpha_{j+1}), N_n(\beta_{j+1}))$. Subsequently, $e_3, e_4 \in E_1(MQ_n)$ and they form a 4-cycle in MQ^{1-i}_{n-1}. By the induction hypothesis, there exists a $2 \times 2^{n-2}$ mesh $M^1_{2\times 2^{n-2}} \in M_{2\times 2^{n-2}}(1, 2; MQ^{1-i}_{n-1})$ such that e_3 is the first ladder-edge and e_4 is the last ladder-edge of $M^1_{2\times 2^{n-2}}$, or e_3 is the last ladder-edge and e_4 is the

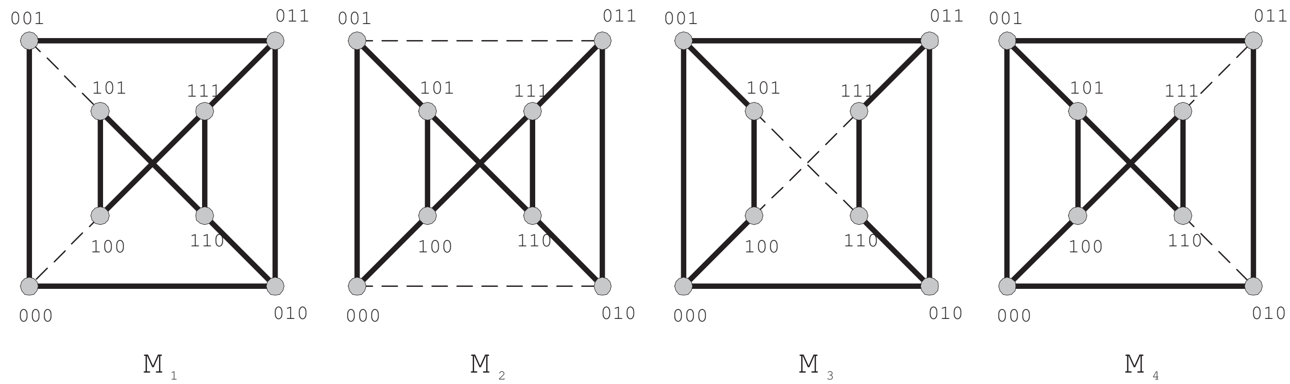

Figure 5: Illustration of Lemma 4

first ladder-edge of $M^1_{2\times 2^{n-2}}$. Without loss of generality, one may assume that $M^1_{2\times 2^{n-2}} =$

$$\begin{pmatrix} \alpha_j = \mu_1 & \cdots & \mu_k & \mu_{k+1} & \cdots & \mu_{2^{n-2}} = \alpha_{j+1} \\ \beta_j = \nu_1 & \cdots & \nu_k & \nu_{k+1} & \cdots & \nu_{2^{n-2}} = \beta_{j+1} \end{pmatrix}$$

where all ladder-edges are in $E_1(MQ_n)$ and $(\mu_k, \mu_{k+1}), (\nu_k, \nu_{k+1}) \in E_2(MQ_n)$ for some $1 \leq k \leq 2^{n-2} - 1$.

Next, replace the 4-cycle of $\langle \alpha_j$,α_{j+1}, β_{j+1}, α_{j+1}, $\alpha_j \rangle$ in $M^0_{2\times 2^{n-2}}$ with the mesh $M^1_{2\times 2^{n-2}}$. We have a disered $2\times 2^{n-1}$ mesh in $M_{2\times 2^{n-1}}(1, 2; MQ_n)$ such that (a, b) is the first ladder-edge and (c, d) is the last ladder-edge of the mesh.

Case 2: $e_1 \in E(MQ^i_{n-1})$ and $e_2 \in E(MQ^{1-i}_{n-1})$ for $i = 0, 1$.

Note that $e_1 = (a, b)$ and $e_2 = (c, d)$, and $\langle a, b, c, d, a\rangle$ is a 4-cycle. Since e_1 and e_2 are in different sub-Möbius cubes of MQ_n, $N_n(a) = d$ and $N_n(b) = c$. Let $u = N_2(a)$ and $v = N_2(b)$. Hence $(u, v) \in E_1(MQ^i_{n-1})$ because $(a, b) \in E_1(MQ_n)$ and $\langle a, u, v, b, a\rangle$ is a 4-cycle in MQ^i_{n-1}. By the induction hypothesis, there exists a $2 \times 2^{n-2}$ mesh $M^0_{2\times 2^{n-2}} \in M_{2\times 2^{n-2}}(1, 2; MQ^i_{n-1})$ such that (a, b) is the first ladder-edge and (u, v) is the last ladder-edge of the mesh, or (a, b) is the last ladder-edge and (u, v) is the first ladder-edge of the mesh. Without loss of generality, we may assume that $M^0_{2\times 2^{n-2}} =$

$$\begin{pmatrix} a = \alpha_1 & \cdots & \alpha_j & \alpha_{j+1} & \cdots & \alpha_{2^{n-2}} = u \\ b = \beta_1 & \cdots & \beta_j & \beta_{j+1} & \cdots & \beta_{2^{n-2}} = v \end{pmatrix}$$

where all ladder-edges are in $E_1(MQ_n)$ and $(\alpha_j, \alpha_{j+1}), (\beta_j, \beta_{j+1}) \in E_2(MQ_n)$ for some $1 \leq j \leq 2^{n-2} - 1$.

Since $\langle a, u, v, b, a\rangle$ is a 4-cycle in MQ^i_{n-1}, and $(u, v), (a, b) \in E_1(MQ^i_{n-1})$ and $(a, u), (b, v) \in E_2(MQ^i_{n-1})$, by Lemma 3, c, d, $N_n(u)$, and $N_n(v)$ forms a 4-cycle in MQ^{1-i}_{n-1} of MQ_n. In addition, (c, d), $(N_n(u), N_n(v)) \in E_1(MQ_n)$. By the induction hypothesis, there exists a $2\times 2^{n-2}$ mesh $M^1_{2\times 2^{n-2}}$ $\in M_{2\times 2^{n-2}}(1, 2; MQ^{1-i}_{n-1})$ such that $(N_n(u), N_n(v))$ is the first ladder-edge and (c, d) is the last ladder-edge of $M^1_{2\times 2^{n-2}}$, or (c, d) is the first ladder-edge and $(N_n(u), N_n(v))$ is the last ladder-edge of $M^1_{2\times 2^{n-2}}$. Without loss of generality, we may assume that $M^1_{2\times 2^{n-2}} =$

$$\begin{pmatrix} N_n(u) = \mu_1 & \cdots & \mu_k & \mu_{k+1} & \cdots & \mu_{2^{n-2}} = c \\ N_n(v) = \nu_1 & \cdots & \nu_k & \nu_{k+1} & \cdots & \nu_{2^{n-2}} = d \end{pmatrix}$$

where all ladder-edges are in $E_1(MQ^{i-1}_{n-1})$ and $(\mu_k, \mu_{k+1}), (\nu_k, \nu_{k+1}) \in E_2(MQ^{1-i}_{n-1})$ for some $1 \leq k \leq 2^{n-2} - 1$.

Therefore, we have a desired $2 \times 2^{n-1}$ mesh as follows.

$$\begin{pmatrix} a = \alpha_1 & \cdots & u & N_n(u) = \mu_1 & \cdots & \mu_{2^{n-2}} = c \\ b = \beta_1 & \cdots & v & N_n(v) = \nu_1 & \cdots & \nu_{2^{n-2}} = d \end{pmatrix}$$

The proof is completed. □

Theorem 1 *For any integer $n \geq 1$, there exists a $2 \times 2^{n-1}$ mesh in MQ_n.*

Proof. It is trivial that the theorem holds for $n = 1, 2$. By Lemma 5, the theorem holds for $n \geq 3$. Hence, the proof is completed. □

As a result, we have the following corollary.

Corollary 1 *For any integer $n \geq 1$, a $2 \times 2^{n-1}$ mesh can be embedded in MQ_n with dilation 1 and expansion 1.*

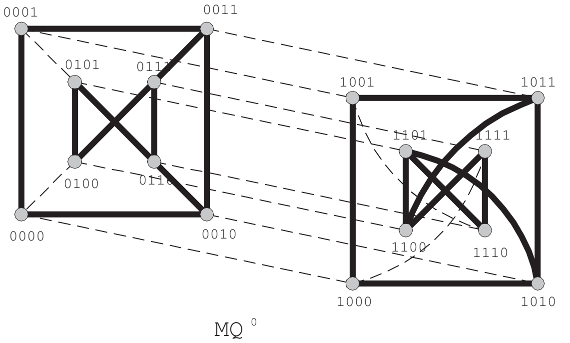

Figure 6: Illustration for two 4×2 meshes in $0-MQ_4$

4 Embedding two disjoint $4 \times 2^{n-3}$ meshes

Applying the $2 \times 2^{n-3}$ mesh embedding of MQ_{n-2} in the last section, we provide $4 \times 2^{n-3}$ mesh embeddings in the 0-MQ_n. As $n = 4$, one can observe the following lemma.

Lemma 6

$$M_1 = \begin{pmatrix} 0000 & 0001 \\ 0010 & 0011 \\ 0110 & 0111 \\ 0101 & 0100 \end{pmatrix}$$

and

$$M_2 = \begin{pmatrix} 1000 & 1001 \\ 1010 & 1011 \\ 1101 & 1100 \\ 1110 & 1111 \end{pmatrix}$$

are two 4×2 meshes in MQ_4.

Let $V_{i,j} = \{a_n a_{n-1} \dots a_1 \mid a_n = i, a_{n-1} = j\}$ where $i, j \in \{0,1\}$. Hence $V(MQ_n) = V_{0,0} \cup V_{0,1} \cup V_{1,0} \cup V_{1,1}$ and $V_{i,j} \cap V_{k,l} = \emptyset$ if $V_{i,j} \neq V_{k,l}$. It is without difficult to prove that the induced subgraph $MQ^{i,j}_{n-2}$ of MQ_n is isomorphic to j-MQ_{n-2} where $i, j \in \{0,1\}$. According to the definition of MQ_n, we have that each vertex $00a_{n-3} \dots a_1$ in the subgraph $MQ^{0,0}_{n-2}$ of 0-MQ_n connects to $10a_{n-3} \dots a_1$ in the subgraph $MQ^{1,0}_{n-2}$ by a dimension n edge; and each vertex $01a_{n-3} \dots a_1$ in the subgraph $MQ^{0,1}_{n-2}$ of 0-MQ_n connects to $11a_{n-3} \dots a_1$ in the subgraph MQ^{11}_{n-2} by a dimension n edge. With these properties we propose the following two lemmas.

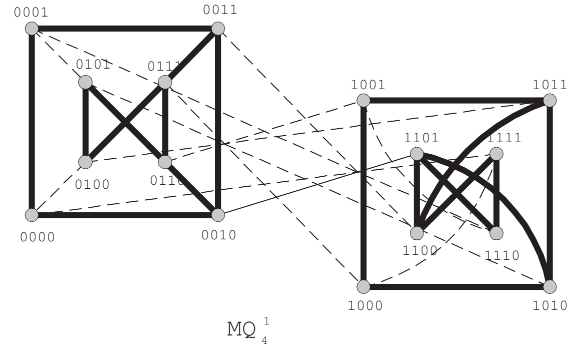

Figure 7: Illustration for two 4×2 meshes in $1-MQ_4$

Lemma 7 *For $n \geq 5$, there is a $4 \times 2^{n-3}$ mesh*

$$M = \begin{pmatrix} 00\alpha_1 & 00\alpha_2 & \cdots & 00\alpha_{2^{n-3}} \\ 00\beta_1 & 00\beta_2 & \cdots & 00\beta_{2^{n-3}} \\ 10\beta_1 & 10\beta_2 & \cdots & 10\beta_{2^{n-3}} \\ 10\alpha_1 & 10\alpha_2 & \cdots & 10\alpha_{2^{n-3}} \end{pmatrix}$$

in the 0-MQ_n where

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{2^{n-3}} \\ \beta_1 & \beta_2 & \cdots & \beta_{2^{n-3}} \end{pmatrix}$$

is a $2 \times 2^{n-3}$ mesh of an 0-MQ_{n-2}.

Proof. For $n \geq 5$, let $MQ^{i,j}_{n-2}$ be a subgraph of MQ_n induced by $V_{i,j}$ for $i, j \in \{0,1\}$. Note that $MQ^{0,0}_{n-2}$ and $MQ^{1,0}_{n-2}$ are both isomorphic to 0-MQ_{n-2}, and $MQ^{0,1}_{n-2}$ and $MQ^{1,1}_{n-2}$ are both isomorphic to 1-MQ_{n-2}. By Lemma 1, there exists a $2 \times 2^{n-3}$ mesh of

$$M = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{2^{n-3}} \\ \beta_1 & \beta_2 & \cdots & \beta_{2^{n-3}} \end{pmatrix}$$

in the 0-MQ_{n-2}. Then,

$$M_{00} = \begin{pmatrix} 00\alpha_1 & 00\alpha_2 & \cdots & 00\alpha_{2^{n-3}} \\ 00\beta_1 & 00\beta_2 & \cdots & 00\beta_{2^{n-3}} \end{pmatrix}$$

and

$$M_{10} = \begin{pmatrix} 10\alpha_1 & 10\alpha_2 & \cdots & 10\alpha_{2^{n-3}} \\ 10\beta_1 & 10\beta_2 & \cdots & 10\beta_{2^{n-3}} \end{pmatrix}$$

are $2 \times 2^{n-3}$ meshes in $MQ^{0,0}_{n-2}$ and $MQ^{1,0}_{n-2}$ of 0-MQ_n, respectively. Since each vertex of M_{00} are in $V_{0,0}$ and each vertex of M_{10} are in $V_{1,0}$, $V(M_{00}) \cap V(M_{10}) = \emptyset$. Indeed, $(00\alpha_j, 00\alpha_{j+1})$, $(00\beta_j, 00\beta_{j+1})$, $(10\alpha_j, 10\alpha_{j+1})$, and $(10\beta_j, 10\beta_{j+1})$

are edges of 0-MQ_n for all $1 \le j \le 2^{n-3}-1$. Furthermore, $N_n(00\alpha_k) = 10\alpha_k$ and $N_n(00\beta_k) = 10\beta_k$ for all $1 \le k \le 2^{n-3}$. Thus,

$$M' = \begin{pmatrix} 00\alpha_1 & 00\alpha_2 & \cdots & 00\alpha_{2^{n-3}} \\ 00\beta_1 & 00\beta_2 & \cdots & 00\beta_{2^{n-3}} \\ 10\beta_1 & 10\beta_2 & \cdots & 10\beta_{2^{n-3}} \\ 10\alpha_1 & 10\alpha_2 & \cdots & 10\alpha_{2^{n-3}} \end{pmatrix}$$

is a $4 \times 2^{n-3}$ mesh in the 0-MQ_n. □

Lemma 8 *For $n \ge 5$, there is a $4 \times 2^{n-3}$ mesh*

$$M = \begin{pmatrix} 01\gamma_1 & 01\gamma_2 & \cdots & 01\gamma_{2^{n-3}} \\ 01\delta_1 & 01\delta_2 & \cdots & 01\delta_{2^{n-3}} \\ 11\delta_1 & 11\delta_2 & \cdots & 11\delta_{2^{n-3}} \\ 11\gamma_1 & 11\gamma_2 & \cdots & 11\gamma_{2^{n-3}} \end{pmatrix}$$

in the 0-MQ_n where

$$\begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_{2^{n-3}} \\ \delta_1 & \delta_2 & \cdots & \delta_{2^{n-3}} \end{pmatrix}$$

is a $2 \times 2^{n-3}$ mesh of a 1-MQ_{n-2}.

Proof. For $n \ge 5$, let $MQ_{n-2}^{i,j}$ be a subgraph of MQ_n induced by $V_{i,j}$ for $i, j \in \{0,1\}$. Note that $MQ_{n-2}^{0,0}$ and $MQ_{n-2}^{1,0}$ are both isomorphic to 0-MQ_{n-2}, and $MQ_{n-2}^{0,1}$ and $MQ_{n-2}^{1,1}$ are both isomorphic to 1-MQ_{n-2}. By Lemma 1, there exists a $2 \times 2^{n-3}$ mesh of

$$M = \begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_{2^{n-3}} \\ \delta_1 & \delta_2 & \cdots & \delta_{2^{n-3}} \end{pmatrix}$$

in the 1-MQ_{n-2}. Then,

$$M_{01} = \begin{pmatrix} 01\gamma_1 & 01\gamma_2 & \cdots & 01\gamma_{2^{n-3}} \\ 01\delta_1 & 01\delta_2 & \cdots & 01\delta_{2^{n-3}} \end{pmatrix}$$

and

$$M_{11} = \begin{pmatrix} 11\gamma_1 & 11\gamma_2 & \cdots & 11\gamma_{2^{n-3}} \\ 11\delta_1 & 11\delta_2 & \cdots & 11\delta_{2^{n-3}} \end{pmatrix}$$

are $2 \times 2^{n-3}$ meshes in $MQ_{n-2}^{0,1}$ and $MQ_{n-2}^{1,1}$ of 0-MQ_n, respectively. Since each vertex of M_{01} are in $V_{0,1}$ and each vertex of M_{11} are in $V_{1,1}$, $V(M_{01}) \cap V(M_{11}) = \emptyset$. Indeed, $(01\gamma_j, 01\gamma_{j+1})$, $(01\delta_j, 01\delta_{j+1})$, $(11\gamma_j, 11\gamma_{j+1})$, and $(11\delta_j, 11\delta_{j+1})$ are edges of 0-MQ_n for all $1 \le j \le 2^{n-3}-1$. Furthermore, $N_n(01\gamma_k) = 11\gamma_k$ and $N_n(01\delta_k) = 11\delta_k$ for all $1 \le k \le 2^{n-3}$. Thus,

$$M' = \begin{pmatrix} 01\gamma_1 & 01\gamma_2 & \cdots & 01\gamma_{2^{n-3}} \\ 01\delta_1 & 01\delta_2 & \cdots & 01\delta_{2^{n-3}} \\ 11\delta_1 & 11\delta_2 & \cdots & 11\delta_{2^{n-3}} \\ 01\gamma_1 & 01\gamma_2 & \cdots & 01\gamma_{2^{n-3}} \end{pmatrix}$$

is a $4 \times 2^{n-3}$ mesh in the 0-MQ_n. □

Theorem 2 *For any integer $n \ge 4$, there are two disjoint $4 \times 2^{n-3}$ meshes in an 0-MQ_n.*

Proof. By Lemma 6, the theorem holds for $n = 4$. For $n \ge 5$, by Lemma 7, 8, there are two $4 \times 2^{n-3}$ meshes M_1 and M_2 in 0-MQ_n. One can observe that for any vertices $a_n a_{n-1} \ldots a_1$ in M_1, $a_n a_{n-1} = 00$ or $a_n a_{n-1} = 10$ and for any vertices $b_n b_{n-1} \ldots b_1$ in M_2, $b_n b_{n-1} = 01$ or $b_n b_{n-1} = 11$. Therefore, M_1 and M_2 are disjoint, i.e., $V(M_1) \cap V(M_2) = \emptyset$. □

As a result, we have the following corollary.

Corollary 2 *For $n \ge 4$, there exists a $4 \times 2^{n-3}$ mesh that can be embedded with dilation 1 and expansion 2 in the 0-type n-dimensional Möbius cube 0-MQ_n. In addition, two node-disjoint $4 \times 2^{n-3}$ meshes can be embedded in an 0-MQ_n covering all vertices of the 0-MQ_n.*

5 Conclusions

Möbius cubes are important variants of hypercubes. The n-dimensional Möbius cube, MQ_n, has several better properties than the n-dimensional hypercube, Q_n, for example, the diameter of MQ_n is about one half that of Q_n and graph embedding capability of MQ_n is better than Q_n. Embedding of paths and cycles in Möbius cubes have been studied by several researchers. However, there has been no research so far as we known to study meshes embedding of Möbius cubes. In this paper, we focus on the issue for meshes embedding of Möbius cubes. The major findings in this paper are follows:

(1) For $n \ge 1$, a $2 \times 2^{n-1}$ mesh can be embedded in the n-dimensional Möbius cube with dilation 1 and expansion 1.

(2) For $n \ge 4$, two disjoint $4 \times 2^{n-3}$ meshes can be embedded into n-dimensional 0-type Möbius cubes with dilation 1.

The results are optimal because the dilations of the embeddings are equal to 1.

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