Fast Algorithms and MATLAB Software for Solution of the Dirichlet Boundary Value Problems for Elliptic Partial Differential Equations in Domains with Complicated Geometry.

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Abstract: - New fast algorithms for solution of the Dirichlet boundary value problem for the class of elliptic Partial Differential Equations (PDE) is proposed. Algorithms are based on new version of General Ray (*GR*) method which consists in application of the Radon transform directly to the PDE and in reduction PDE to assemblage of Ordinary Differential Equations (ODE). The class of the PDE includes the Laplace, Poisson and Helmgoltz equations. *GR*-method presents the solution of the Dirichlet boundary value problem for this type of equations by explicit analytical formulas that use the direct and inverse Radon transform. Proposed version of *GR*-method is justified theoretically, realized by MATLAB software, which quality we demonstrate by numerical experiments.

Key-Words: - fast algorithms, boundary value problems, partial differential equations, Radon transform, MATLAB software

1 Introduction

There are two main approaches for solving boundary value problems for partial differential equations in analytical form: the Fourier decomposition and the Green function method [1]. The Fourier decomposition is used, as the rule, only in theoretical investigations. The Green function method is the explicit one, but it is difficult to construct the Green function for the complex geometry of the considered domain Ω. The known numerical algorithms are based on the Finite Differences method, Finite Elements (Finite Volume) method and the Boundary Integral Equation method. Numerical approaches lead to solving systems of linear algebraic equations [2] that require a lot of computer time and memory.

A new approach for the solution of boundary value problems on the base of the General Ray Principle (*GRP*) was proposed by the author in [3], [4] for the stationary waves field. *GRP* leads to explicit analytical formulas (*GR*-method) and fast algorithms, developed and illustrated by numerical experiments in [4] - [7] for solution of the direct and coefficient inverse problems for the equations of mathematical physics. But there were some difficulties with the strict theoretical foundation of that version of *GR*-method.

Here we extend the proposed approach to construct another version of *GR*-method based on application of the direct Radon transform [8] to the PDE [9] – [10]. This version of *GR*-method is justified theoretically, realized as algorithms and program package in MATLAB system, illustrated by numerical experiments.

2 General Ray Principle

The General Ray Principle (*GRP*) was proposed in [3], [4]. It gives no traditional mathematical model for considered physical field and corresponding boundary problems. *GRP* consists in the next main assumptions:

1) the physical field can be simulated mathematically by the superposition of plane vectors (general rays) that form field $\vec{V}(l)$ for some fixed straight line l; each vector of field $\vec{V}(l)$ is parallel to the direction along this line l, and the superposition corresponds to all possible lines l that intersect domain Ω;

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2) the field $\vec{V}(l)$ is characterized by some potential function $u(x,y)$;

3) we know some characteristics such as values of function $u(x,y)$ and/or flow of the vector $\vec{V}(l)$ in any boundary point $P_0 = (x^0, y^0)$ of the domain.

Application of the *GRP* to the problem under investigation means to construct for considering PDE an analogue as family of ODE describing the distribution of the function $u(x,y)$ along of "General Rays", which are presented by a straight line l with some parameterization. We use the traditional Radon parameterization with a parameter $t: x = p\cos j - t\sin j$, $y = p\sin j + t\cos j$. Here $|p|$ is a length of the perpendicular from the centre of coordinates to the line l, $j \in [0, p]$ is the angle between the axis x and this perpendicular. Using this parameterization, we considered in [5], [6] the variant of *GRP* that leads to reducing Laplace equation to the assemblage (depending of p, j) of ordinary differential equations with respect to variable t. This family of ODE was used as the local analogue of the PDE. There we constructed corresponding version of the General Ray method for the convex domain Ω. It consists in the next steps: 1) solution of boundary value problems for the obtained assemblage ODE in explicit analytical or approximate form, using well known standard formulas and numerical methods; 2) calculation of the integral average for this solution along the line l; 3) application to these solutions the inverse Radon transform that realizes desired superposition.

The numerical justification of this version of *GR*-method was given for the domain Ω as unit circle [4]. For some more complicated domains the quality of the method was illustrated by numerical examples.

The reduction of the considered PDE to the family of ODE with respect to variable t gives possibilities to satisfy directly boundary conditions, construct the effective and fast numerical algorithms. At the same time, there are problems with its realization for the complicated geometry of the region Ω, so as with its theoretical foundation even for the simple cases.

3 Formulation and Theoretical Foundation of *p*-Version of *GR*-Method

Let us consider the Dirichlet boundary problem for the elliptic equation:

$$\Delta u(x,y) + k^2 u(x,y) = y\,(x,y), \quad (x,y) \in \Omega; \tag{1}$$

$$u(x,y) = f(x,y), \quad (x,y) \in \Gamma. \tag{2}$$

with respect to the function $u(x,y)$ that has two continuous derivatives on bought variables inside the plane domain Ω, bounded with a continuous curve Γ. Here $y\,(x,y)$, $(x,y) \in \Omega$ and $f(x,y)$, $(x,y) \in \Gamma$ are given functions.

In works [9]-[11] there are presented investigations of the possibility to reduce solution of PDE to the family of ODE using the direct Radon transform [8]. This reduction leads to ODE with respect to variable *p* and can be interpreted in the frame of the introduced General Ray Principle. But using the variable *p* for the first point of view makes it impossible to satisfy directly to the boundary conditions expressed in (x,y) variables. Possibly by this reason the mentioned and other related investigations were concentrated only at theoretical aspect in constructing some basis of general solutions of PDE. Unfortunately, this approach was not used for construction of numerical methods and algorithms for solution of boundary value problems, except some simple examples [9]. The important new element, introduced here into this scheme, consists in satisfaction of boundary conditions by its reduction to homogeneous ones.

The *p*-version of the *GR*-method can be explained as the consequence of the next steps:

1) reduce the boundary value problem to homogeneous one;

2) describe the distribution of the potential function along the general ray (a straight line l) by its direct Radon transform $u_j(p)$;

3) construct the family of ODE on the variable *p* with respect the function $u_j(p)$;

4) solution of the constructed ODE with the zero boundary conditions;

5) calculate the inverse Radon transform of the obtained solution;

6) regress to the initial boundary conditions. We present bellow the realization of this scheme.

We suppose that the boundary Γ can be described in the polar coordinates (r, a) by some one-valued positive function that we denote $r_0(a)$, $a \in [0, 2p]$. It is always possible for the simple connected star region Ω with the centre at the coordinate origin. Let us write the boundary function as

$$\overline{f}(a) = f(r_0(a)\cos a, r_0(a)\sin a). \qquad (3)$$

Supposing that functions $r_0(a)$ and $\overline{f}(a)$ have the second derivative we introduce functions

$$f_0(a) = \overline{f}(a)/r^2{}_0(a), \quad (x,y) \in \Omega. \qquad (4)$$

$$\begin{aligned} y_0(x,y) &= y(x,y) - 4f_0(a) \\ &- f_0''(a) - k^2 r^2 f_0(a) \end{aligned} \qquad (5)$$

$$u_0(x,y) = u(x,y) - r^2 f_0(a). \qquad (6)$$

To realize the first step of the scheme we can write the boundary problem with respect the function $u_0(x,y)$ as the next two equations:

$$\begin{aligned} &\Delta u_0(x,y) + k^2 u_0(x,y) = y_0(x,y), \\ &(x,y) \in \Omega; \end{aligned} \qquad (7)$$

$$u_0(x,y) = 0, \quad (x,y) \in \Gamma. \qquad (8)$$

To make the second and the third steps we need the direct Radon transform [8]:

$$R[u](p,j) = \int_{-\infty}^{+\infty} u(p\cos j - t\sin j, p\sin j + t\cos j)dt$$

After application the Radon transform to the equation (7) we obtain, using formula (2) at the pp. 3 of [9], the family of the ODE with respect the variable p:

$$\frac{d^2 u_j(p)}{dp^2} + k^2 u_j(p) = R[y_0](p,j), \quad (p,j) \in \hat{\Omega}; \qquad (9)$$

where $\hat{\Omega}$ is the domain of the change of parameters p, j. As the rule, $j \in [0,p]$, module of the parameter p is equal to the radius in the polar coordinates and changes in the limits, determined by the boundary curve Γ. In the considered case for some fixed j parameter p is in the limits: $-r_0(j-p) < p < r_0(j)$.

Unfortunately, boundary condition (8) can not be modified directly by Radon transform to the corresponding boundary conditions for the every equation of the family (9). For the fourth step we propose to use the next boundary conditions for every fixed $j \in [0,p]$:

$$u_j(-r_0(j-p)) = 0; \quad u_j(r_0(j)) = 0. \qquad (10)$$

Let us designate $\hat{u}_j(p)$ solution of the problem (9)-(10) that can be univocally determined as function of variable p for every $j \in [0,p]$, $p \in (-r_0(j-p), r_0(j))$, and out of this interval we extend $\hat{u}_j(p) \equiv 0$ for all j with continuity on p.

Let us denote the inverse Radon transform as operator R^{-1}, which for any function $z(p,j)$ can be presented by formula [9]:

$$R^{-1}[z] = \frac{1}{2p^2} \int_{-p}^{p} \int_{-\infty}^{+\infty} \frac{z_p'(x\cos j + y\sin j, j)}{(x\cos j + y\sin j) - t} dt dj$$

The foundation of the fifth and sixth steps of the scheme is concentrated in the next theorem.

Theorem. *The following formula for the solution of boundary value problems (1)-(2) is true*

$$\begin{aligned} &\overline{u}(x,y) = R^{-1}[\hat{u}_j(p)] + r^2 f_0(a), \\ &(x,y) \in \Omega \end{aligned} \qquad (11)$$

Proof. Substituting function defined by (11) into left hand side of equation (7) we obtain from Lemma 2.1 at the pp. 3 of [9] the next relations:

$$\Delta\overline{u}_0(x,y) = R^{-1}[\frac{d^2\hat{u}_j(p)}{dp^2} + k^2 u_j(p)]$$
$$= R^{-1}[R[y_0](p,j)] = y_0(x,y) \quad (12)$$

that convinces us of the satisfaction of the equation (7) (see also [7], p. 40). From the condition $\hat{u}_j(p) \equiv 0$, $p \notin (-r_0(j-p), r_0(j))$, $j \in [0,p]$ and Theorem 2.6 (the support theorem) at page 10 of the book [9] it follows that $\overline{u}_0(x,y) \equiv 0$ for $(x,y) \notin \Omega$ and, due its continuity, it satisfies boundary conditions (8). Then using formula (6) we obtain (11). This finishes the proof.

In the case of the Laplace equation the solution $\overline{u}(x,y)$ of boundary value problems (1), (2) is presented by the next formulas

$$\overline{u}(x,y) = R^{-1}[(\hat{y}_2(p,j) - \frac{(p+r_0(j-p))}{(r_0(j)+r_0(j-p))}\hat{y}_2(r_0(j),j))] + r^2 f_0(a) \quad (13)$$

$$\hat{y}_2(p,j) = \int_{-r_0(j-p)}^{p} \int_{-r_0(j-p)}^{p} \hat{y}_0(p,j)dp,$$
$$\hat{y}_0(p,j) = R[y_0(x,y)]. \quad (14)$$

The inverse Radon transforms in explicit formula (11) can be realized numerically by fast Fourier discrete transformation (FFDT) that guarantees the rapidity of proposed method and developed algorithmic realization.

4 Description of the Computer Program Package

We have constructed the fast program realization of developed algorithms for *GR*-method as a program package in MATLAB system. The package consists of the set of programs, the main of which *GRELL2(k,N)* realizes the solution of the problem (9)-(10) and then calculation of the function $\overline{u}(x,y)$ by formula (11) on the bidimensional rectangular red with *NxN* nodes in the minimal rectangle that contains the domain Ω. Parameter k is equal to the corresponding coefficient in the formula (1). The main program presents also graphic illustration of the approximate solution using corresponding MATLAB programs. Program *GRELL2* calls programs *R (AL)* and *F(AL)* that realize functions $r_0(a)$, $\overline{f}(a)$ and must be constructed by user. Calculation of the direct and inverse Radon transform in formulae (9), (11) is realized with the original modification of the MATLAB programs *radon, iradon* [12].

5 Results of Numerical Experiments

We have constructed the fast algorithmic and program realization of *GR*-method for considered problem in MATLAB system. We used the uniform discretization of variables $p \in [-1,1]$, $j \in [0,p]$, so as for variables x, y, with ***N*** nodes. We made testes on mathematically simulated model examples with known exact functions $u(x,y)$, $f(x,y)$, $y(x,y)$. Graphic illustration of solution of numerical example by *GR*-method for the Laplace equation are presented at Fig. 1, 2, 3 (bellow always upper graphic – exact solution, bottom graphic – approximate solution) for domains with smooth boundary curve Γ, determined for two first figures by formula

$$r_0(a) = 0.8+0.2*\sin(k*a).$$

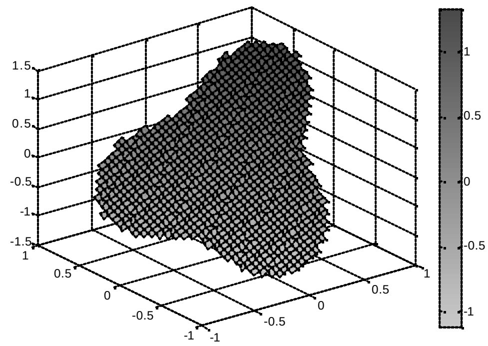
Parameter k=3 for Fig. 1 and k=5 for Fig. 2.

At Fig. 4 and 5, 6 illustrations of numerical examples for solutions of the Poisson and of the Helmgoltz equations correspondently are presented. As we can see at graphics, the method gives a good approximation also for no differentiable curve Γ, for which we must use formulas (5) in piecewise form .

Some numerical experiments for solution of the Laplace equation by *GR*-method are presented in comparison with the results obtained by program *pdemodel* from PDE toolbox of MATLAB system. As domain, we have in this case the symmetrical cross and exact bilinear function $u = x + y$.

PDE toolbox of MATLAB system realizes solution of problem using some triangulation of the domains, which are illustrated at Fig. 7, 10. At Fig. 8, 11 there are presented correspondent approximate solutions by program *pdemodel* , at Fig. 9, 12 – approximate solutions by program *GRELL2*. with corresponding number N of discretization The time of calculation is indicated under corresponding figures.

a)



b)

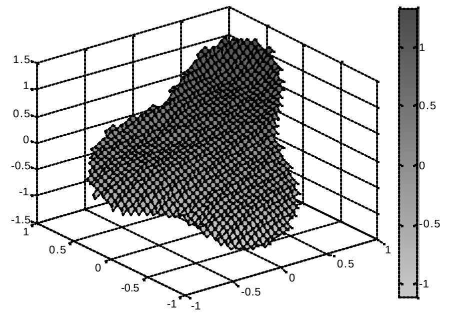
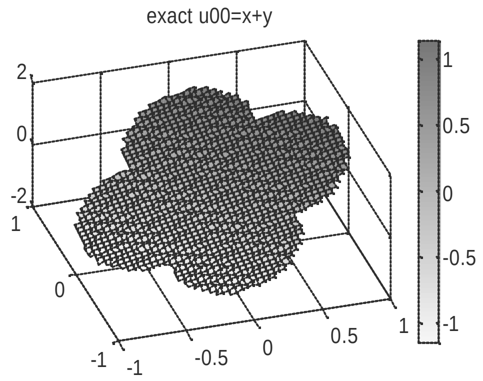


Fig. 1



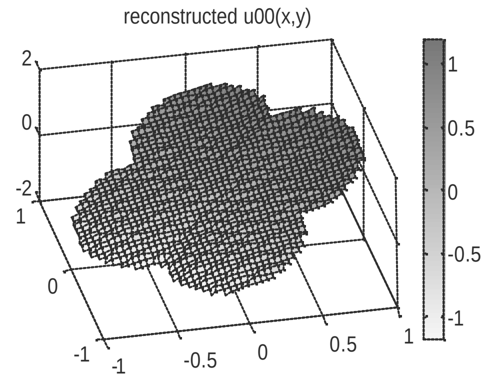
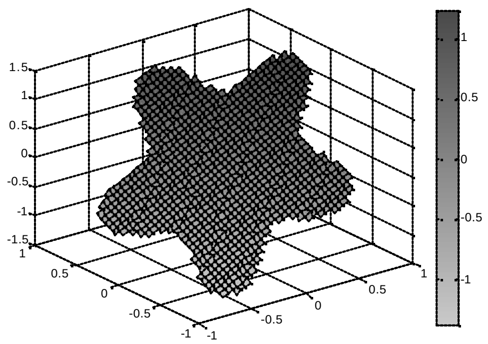


Fig. 3.

a)



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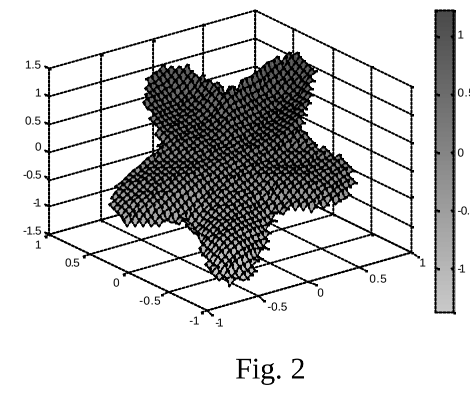
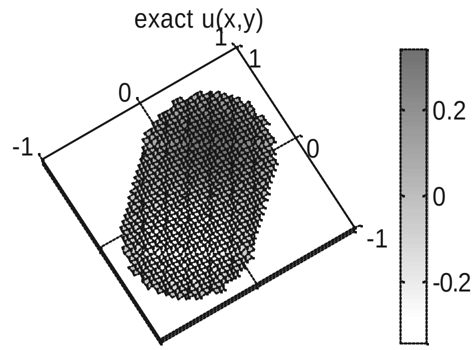


Fig. 2



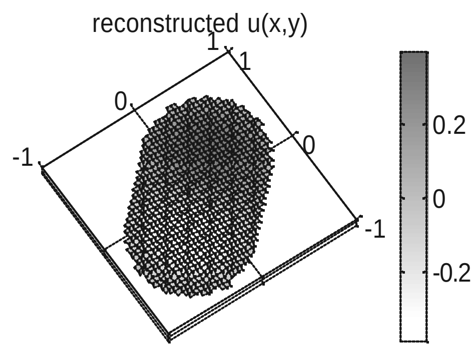


Fig. 4.

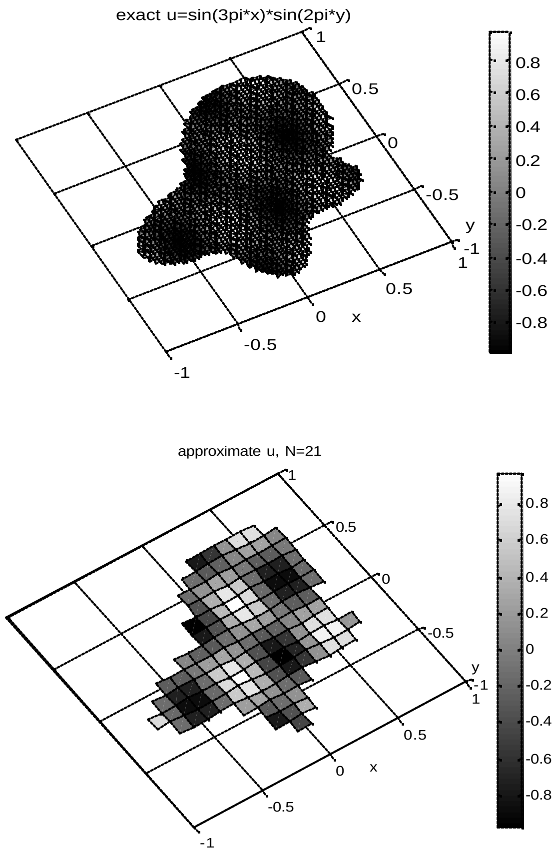


Fig. 5.

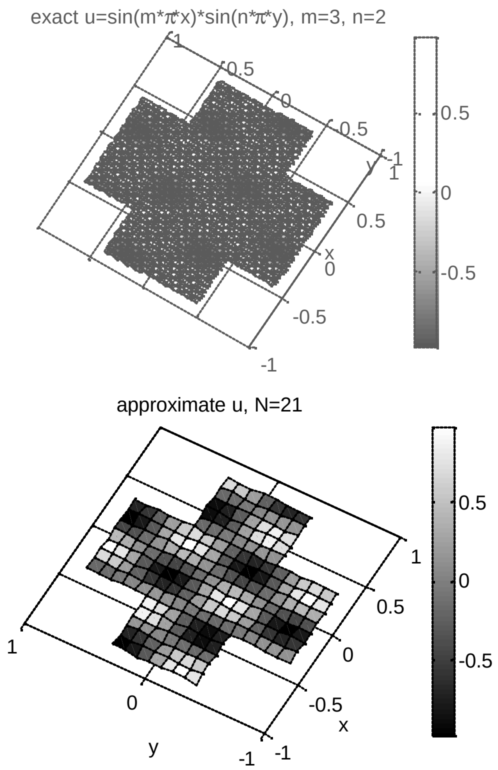


Fig. 6.

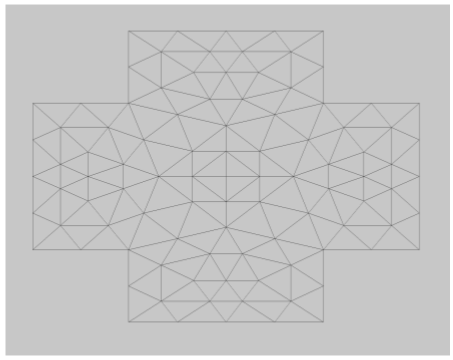


Fig. 7

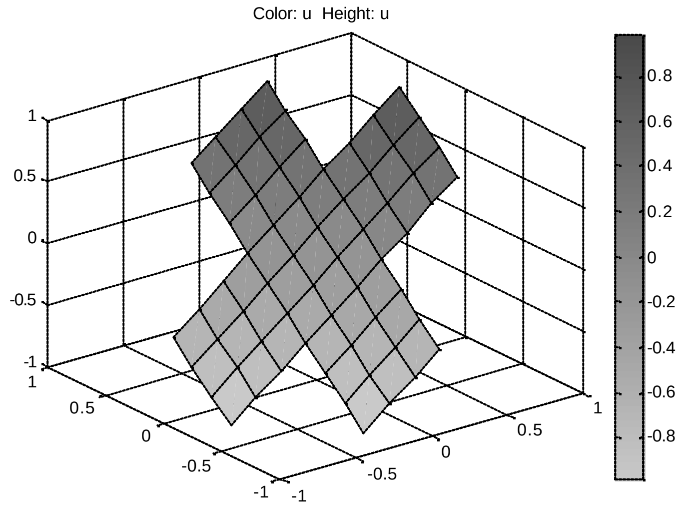


Fig. 8

Time *pdemodel* = 4.4370 sec

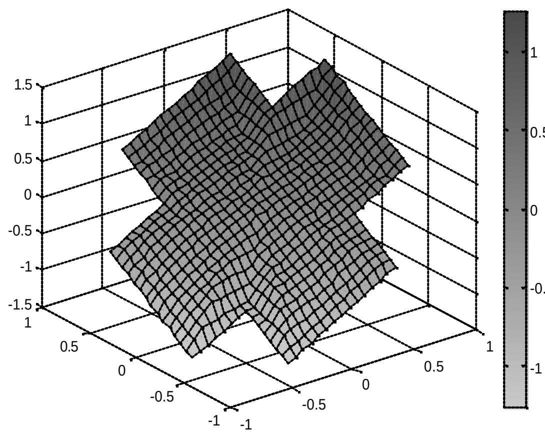


Fig. 9

Time of *GRELL2* = 0.3910 sec

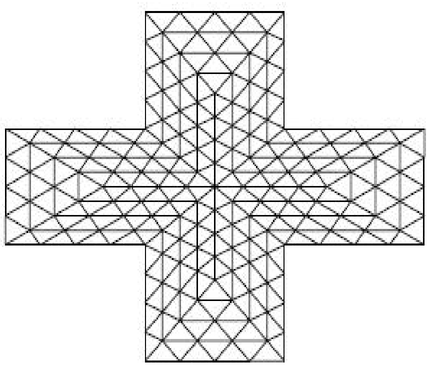


Fig. 10

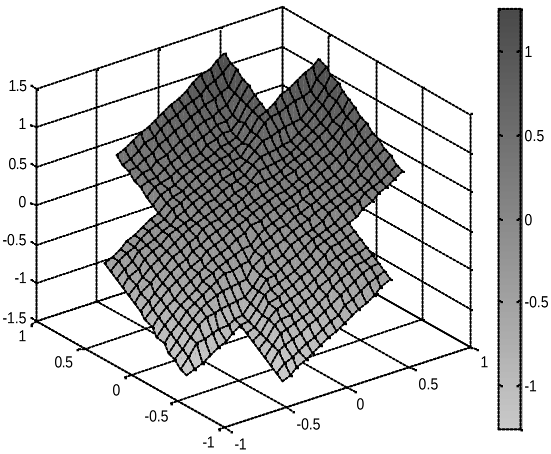


Fig. 11

Time *pdemodel* = 12.8750 sec

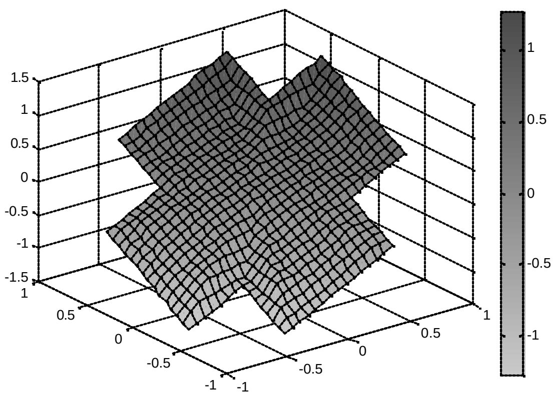


Fig. 12

Time of *GRELL2* = 1.2970

6 Regularization of GR-algorithm with Recursive Spline Smoothing method

Function $\overline{f}\left(a\right)$ in formula (3) present the input data. In the case of the noised input data the second derivative, that appears in formula (5), can provoke instability in the final result. The same effect can give unboundedness of the inverse Radon transform R^{-1} in certain spaces, which is conditioned by its explicit form (11) that contains differentiation and singular integration. It causes the same instability of *GR*-method that the Boundary Integral Equation method has [5]. So, we can use the Recursive Spline Smoothing (*RSS*) [5], [6] for regularization of the *GR*-method. The *RSS* is presented by explicit formulas that are realized in considered case with cubic explicit spline approximation. We suppose that boundary conditions are presented by noised function $\tilde{f}$ and this provokes perturbation of the right hand side of the equation (5), which we denote $\tilde{y}_0$.

We apply *RSS* for smoothing of the $\tilde{f}$ using the explicit formulas for one-dimensional spline on the regular uniform grid $\left\{x_i\right\}$, $x_i = -1 + h\left(i-1\right)$, $i = -2, \ldots, n+2$;,; $h = 2/(n-1)$,. Let $s_i(x)$ be a local basic cubic spline, constructed on the units $x_{i-2}, \ldots x_{i+2}$; $i = 0, \ldots, n+1$.

Mentioned formulas are the next ones:

$$S_k\left(x\right) = \sum_{i=1}^{n} S_{k-1}\left(x_i\right) s_i\left(x\right),$$
$$k = 1,\ 2, \ldots, \overline{K};$$
$$S_0\left(x_i\right) = \tilde{f}\left(x_i\right) \ .$$

The number of smoothes $\overline{K}$ is the regularization parameter, which can be chosen here in accordance with residual principle using the discrete estimation d of the errors. It means, if the values of the exact function $\overline{f}$ and the noised function $\tilde{f}$ satisfy to the conditions

$$\left| f\left(x_i\right) - \tilde{f}\left(x_i\right) \right| \le d, i = 1,...,n$$

,

then $\overline{K}$ is chosen as maximum among all k, for which the inequality is fulfilled:

$$\sum_{i=1}^{n}\left|S_k\left(x_i\right)-\tilde{f}\left(x_i\right)\right|^2 \le cd^2n^2 ,$$

where $c = \text{conts} > 1$.Theoretical and numerical justification of the regularization properties of this type of smoothing is presented in [5]. We use *RSS* in matrix form, which presents the resulting matrix F_K by formula

$$F_K = V^K F;$$

where F is the vector columna of given noised values,

$$V = \begin{pmatrix} \frac{2}{3} & \frac{1}{6} & 0 & . & . & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & . & . & 0 \\ 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & . & 0 \\ . & . & . & . & . & . \\ . & . & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & . & . & 0 & \frac{1}{6} & \frac{2}{3} \end{pmatrix}.$$

Graphic illustrations of solution of numerical examples with regularization of *GR*-method for the Laplace equation in the case of the noised data with the relative error over 10% are presented at figures 13 - 20. At Fig. 13, 17 the noised functions $\tilde{y}_0$ are presented, at Fig. 14, 18 – approximate solutions for noised data without regularization; at Fig. 15, 19 – smoothed function $\tilde{y}_0$; at Fig. 16, 20 – regularized solutions.

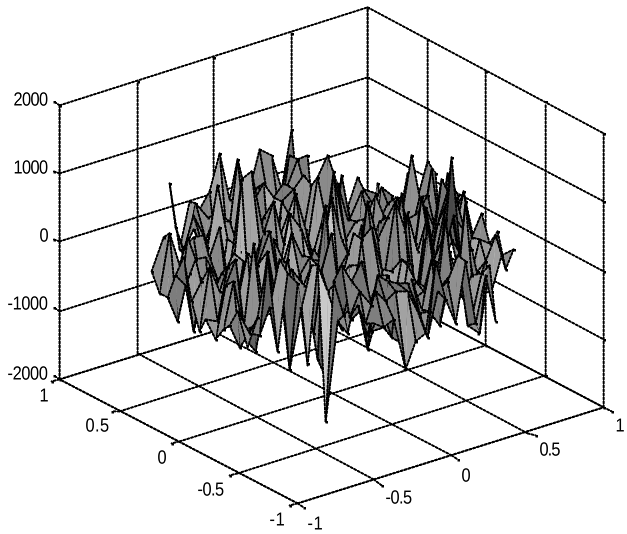


Fig. 13. Function $\tilde{y}_0$

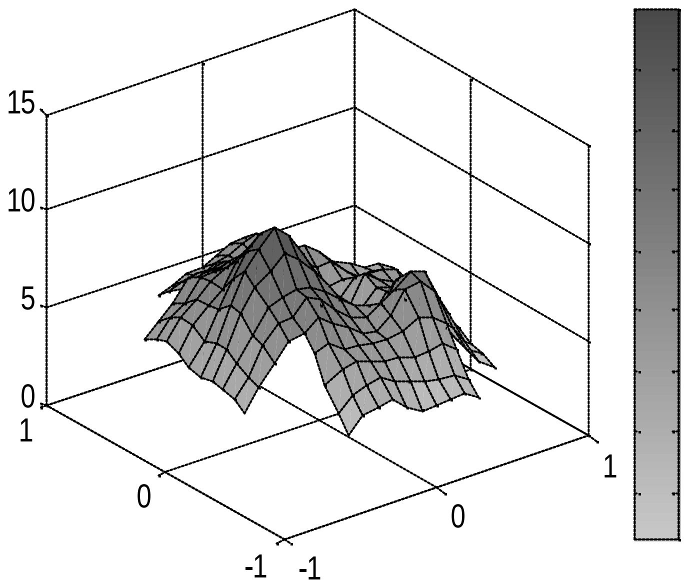


Fig. 14.
Approximate solution with noised data without regularization

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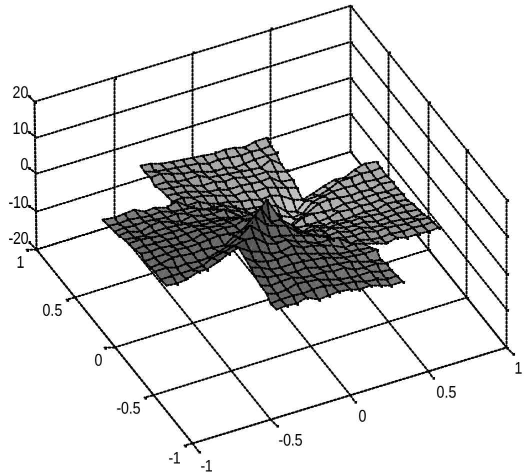

Fig. 15. Smoothed function $\tilde{y}_0$

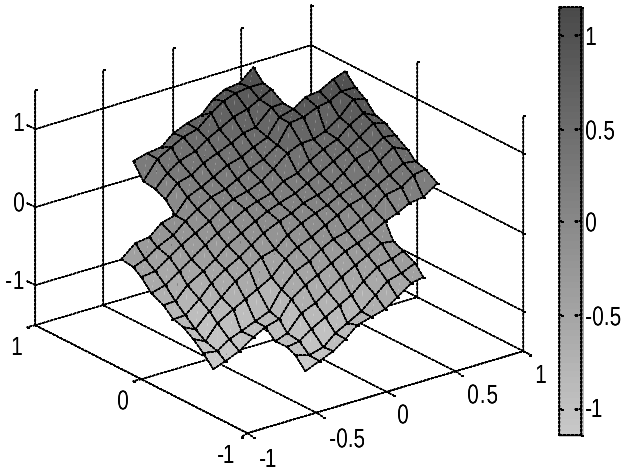

Fig. 16.
Regularized solution

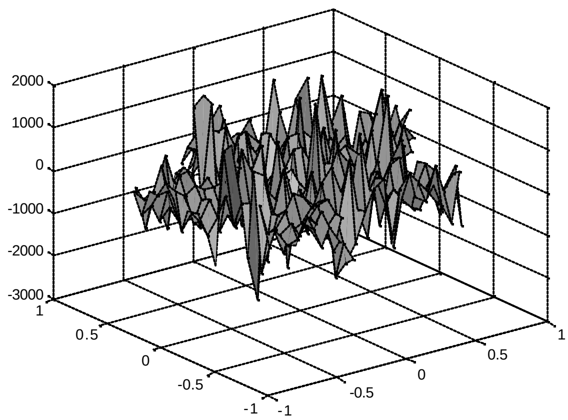

Fig. 17. Function $\tilde{y}_0$

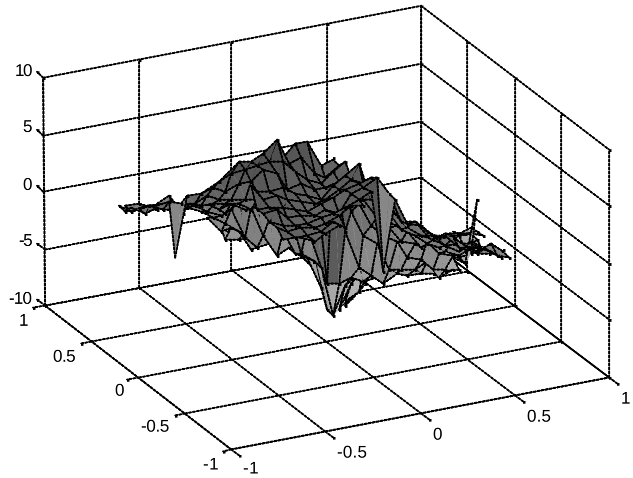

Fig. 18. Approximate solution with noised data without regularization

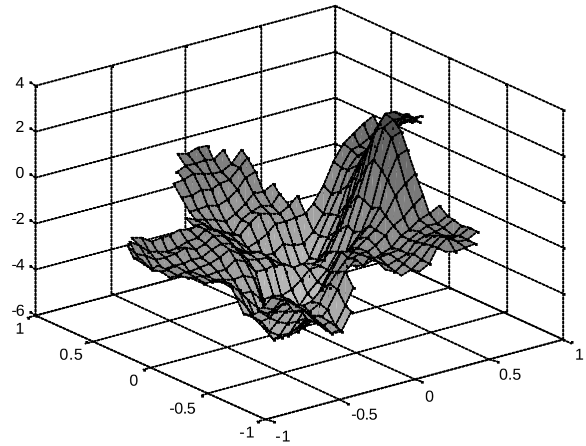

Fig. 19. Smoothed function $\tilde{y}_0$

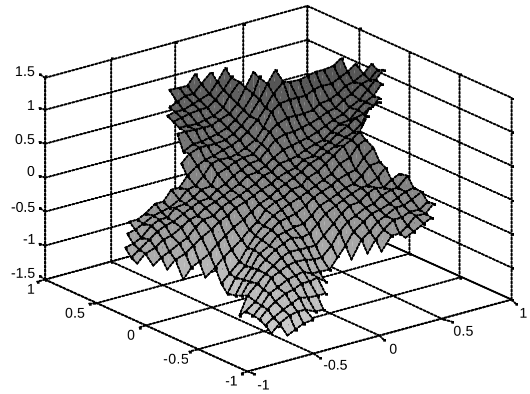

Fig. 20.
Regularized solution

8 Conclusion

New version of *GR*-method is constructed. It is based on application of the Radon transform directly to the partial differential equation. This version of *GR*-method for arbitrary simple connected star domains is justified theoretically, realized as algorithms and program package in MATLAB system, illustrated by numerical experiments.

Proposed version can be applied for solution of boundary value problems for another PDE with constant coefficients. In perspective it seams interesting to develop this approach for solution of the direct and inverse problems for the equations of mathematical physics with variable coefficients.

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