The local analytic solution to some nonlinear diffusion-reaction problems

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Abstract: -The positive radially symmetric solutions to the nonlinear problem $div\left(|\nabla u|^{p-2}\nabla u\right)+f(u)=0$ in $B_R=\left\{x\in\mathbf{R}^n\ :\ |x|<R\right\}$, $1<p<n$, with $f(u)=u^{\gamma}+u^{\delta}$ for $u\geq 0$ are considered. We examine the existence of local solutions and give a method for the determination of power series solutions. The comparison of the local analytic and entire solutions is given for some special values of parameters p, n, γ and δ.

Key-Words: - Nonlinear partial differential equations, ***p***-Laplacian, non-Newtonian fluid, polytrophic gas, local analytic solutions

1 Introduction

In a model for diffusion-reaction problem the concentration of the steady state satisfies

$$\Delta_p u+f(u)=0 \text{ in } \Omega\subset\mathbf{R}^n, \tag{1}$$

where $\Delta_p u=div\left(|\nabla u|^{p-2}\nabla u\right)$ is the p-Laplacian of u, f is an increasing function. The equation above appears in the generalized reaction-diffusion theory [12] and in non-Newtonian fluid theory [10].

In the case of compressible fluid flows in a homogeneous isotropic rigid porous medium the continuity equation is given by

$$\theta\frac{\partial\rho}{\partial t}+\mathrm{div}\left(\rho\vec{V}\right)=0, \tag{2}$$

where ρ denotes the density of the fluid, $\vec{V}$ the seepage velocity and θ the volumetric moisture content.

The linear Darcy's law is not valid here, since the molecular and ion effects have to take into account in case of a non-Newtonian fluid. Therefore the nonlinear relation

$$\rho\vec{V}=-C|\nabla P|^{\alpha-1}\nabla P \tag{3}$$

is valid, where P denotes pressure, C and α are positive physical constants.

If we consider the fluid as polytrophic gas then we have the relation between the thermal pressure P and the fluid density ρ as

$$P=k\rho^{\bar{\gamma}} \tag{4}$$

with positive constant k. Here $\bar{\gamma}$ is called polytrophic exponent. We note that for isothermal flows $\bar{\gamma}=1$ ($\bar{\gamma}=0$ corresponds to the isobaric flows). After changing variables and making substitutions in equations (2-4) we obtain

$$\frac{\partial u}{\partial t}=\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right).$$

The case $p>2$ is called slow diffusion and the case $1<p<2$, the fast diffusion (see e.g., [20]).

When reaction term is added to the diffusion then equation (1) appears in the steady-state case.

The asymptotic and numerical solution of problem (1) has been attracted considerable interest in the last decades (see [7], [18], [20]).

Nonlinear partial differential equation of type (1) was considered previously for different function f. In paper [4] we considered function

$f(u)=(-1)^i |u|^{q-2}u,$ i.e., the quasilinear differential equation

$$\Delta_p u + (-1)^i |u|^{q-2} u = 0,\ u = u(x),\ x \in \mathbf{R}^n,$$

where $n \geq 1,$ $p>1$ and $q>1,$ $i=0,1$ and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the so-called $p-$Laplacian ( ). If $n=1,$ then the equation is reduced to

$$\left(\Phi_p(y')\right)' + (-1)^i \Phi_q(y) = 0,$$

where for $r \in \{p,q\}$

$$\Phi_r(y) := \begin{cases} |y|^{r-2} y, & \text{for } y \in \mathbf{R} \setminus \{0\} \\ 0, & \text{for } y = 0. \end{cases}$$

Note that function Φ_r is an odd function. For $n>1$ we restrict our attention to radially symmetric solutions. The problem under consideration is reduced to

$$\left(t^{n-1}\Phi_p(y')\right)' + (-1)^i t^{n-1}\Phi_q(y) = 0, \text{ on } (0,\ a) \tag{5}$$

where $a>0.$ A solution of (5) means a function $y \in C^1(0,\ a)$ for which $t^{n-1}\Phi_p(y') \in C^1(0,\ a)$ and (5) is satisfied. We shall consider the initial values

$$\begin{aligned} y(0) &= A \neq 0, \\ y'(0) &= 0, \end{aligned} \tag{6}$$

for any $A \in \mathbf{R}$.

For the existence and uniqueness of radial solutions to (5) we refer to paper [17]. If $n=1$ and $i=0,$ then it was showed that the initial value problem (5)-(6) has a unique solution defined on the whole $\mathbf{R}$ (see [8], and [9]), moreover, its solution can be given in closed form in terms of incomplete gamma functions [9]. If $n=1,$ $i=0,$ Lindqvist gives some properties of the solutions [16]. If $n = 1$ and $p = q = 2,$ then (5) is a linear differential equation, and its solutions are well-known:

if $i=0,$ the solution (5)-(6) with $A=1$ is the cosine function,

if $i=1,$ the solution (5)-(6) with $A=1$ is the hyperbolic cosine function, and both the cosine and hyperbolic cosine functions can be expanded in power series.

In the linear case, when $n=2,$ $p=q=2,$ $i=0,$ the solution of (5)-(6) with $A=1$ is $J_0(t),$ the Bessel function of first kind with zero order, and for $n=3$, $p=q=2,$ $i=0$ then the solution of (5)-(6) with $A=1$ is $j_0(t)=\sin t/t,$ called the spherical Bessel function of first kind with zero order.

In the cases above, for special values of parameters $n,$ $p,$ $q,$ i we know the solution in the form of power series.

The type of singularities of (5)-(6) was classified in [5] in the case when $i=0,$ and $p=q$. If $n=1,$ then a solution of (5) is not singular.

In paper [4] a method was presented for the evaluation of local analytic solution for problem (5)-(6) in the neighborhood of zero. Here we intend to generalize that method for a more general class of equations.

In this paper we consider the radial symmetric solution $u=u(|x|)$ for the problem

$$\Delta_p u + f(u) = 0 \text{ and } u>0 \text{ in } B_R, \tag{7}$$

where $B_R = \{x \in \mathbf{R}^n : |x| < R\},$ $1<p<n$ and

$$f(u) = \begin{cases} u^\gamma + u^\delta & \text{for } u \geq 0, \\ 0 & \text{for } u < 0. \end{cases}$$

In the special case $p=2,$ problem (7), i.e., the semilinear problem with superlinear and supercritical exponents was considered by Lin and Ni in [15].

For $p \neq 2$ the existence of solutions of (1) has been examined in [1] and [19]. Moreover in [1], the multiplicity of radially symmetric solutions of the Dirichlet boundary value problem was proved; namely, that there exists $R^* = R^*(\gamma,\delta)$ such that the problem has at least two distinct radial solutions if $R>R^*$ and at least one radial solution provided $R=R^*.$

We shall study the positive radial solution of (7), i.e., the initial value problem (here $r=|x|$):

$$\begin{cases} r^{1-n}\left(r^{n-1}\left|u_r'\right|^{p-2}u_r'\right)' + f(u)=0 \\ \qquad \text{in} \quad (0,\infty), \\ u_r'(0)=0, \quad u(0)=\alpha \ge 0, \end{cases} \tag{8}$$

or equivalently,

$$\begin{cases} (p-1)\left|u_r'\right|^{p-2}u_{rr}'' + \frac{n-1}{r}\left|u_r'\right|^{p-2}u_r' + f(u)=0 \\ \qquad \text{in } (0,\infty), \\ u_r'(0)=0, \quad u(0)=\alpha \ge 0. \end{cases} \tag{9}$$

We say that $u=u(r)$ is a positive radial solution of (7), if it solves the initial value problem associated with (8) (or (9)) for some $\alpha \ge 0$, $u(r)>0$ for $r\in(0,\infty)$ and $\lim_{r\to\infty} u(r)=0.$

We remark that (8) has singularity at zero if $n\ge p+1$.

Our goal is to give the exact solution for the initial value problem of (8) (or (9)) for some special values of the parameters. Moreover, we examine the existence of local solution to the initial value problem of (9) and we give a method for the determination of the power series solution for given values of parameters p, n, α, γ and δ. We shall compare the exact and power series solutions and find the error for special values of parameters.

2 Positive entire solution

Positive entire radial solution of

$$\Delta_p u + f(u) = 0 \tag{10}$$

is a function $u=u(r)$ which solves (9) for some $\alpha>0$, $u(r)>0$ for $r\in(0,\infty)$ and $\lim_{r\to\infty} u(r)=0$. Moreover, we suppose that

$p-1<\gamma<p^*-1$ and $\delta>p^*-1$, $\quad p^*=\frac{np}{n-p}$,

the exponents γ and δ satisfy

$$\gamma=\frac{\beta+1}{\beta}(p-1) \quad \text{and} \quad \delta=\gamma+\frac{1}{\beta}, \tag{11}$$

where

$$\beta\in\left(\frac{(n-p)(p-1)}{p^2},\ \frac{n-p}{p}\right), \tag{12}$$

i.e., $\gamma=\frac{\delta+1}{p'}$, $p'=\frac{p}{p-1}$.

Proposition 1. *Let p, p', n, β, γ, δ, and f be as above. Set*

$$a=\frac{1}{\left(\frac{n}{(\beta+1)p}-1\right)^{\beta}}, \tag{13}$$

$$b=\frac{\left(n-(\beta+1)p\right)^{p'}}{(\beta+1)\ p}\left(\frac{\beta\ p}{p-1}\right)^{\frac{1}{p-1}}, \tag{14}$$

$$u(r)=a\left(\frac{b}{b+r^{p'}}\right)^{\beta}. \tag{15}$$

Then u is a positive entire solution of (10)

From straight forward computations we get

$$r^{1-n}\left(r^{n-1}|u'(r)|^{p-2}u'(r)\right)' =$$

$$-\frac{\left(\beta p' a b^{\beta}\right)^{p-1}}{\left(b+r^{p'}\right)^{(\beta+1)(p-1)+1}}\left(n(b+r^{p'})-(\beta+1)pr^{p'}\right),$$

moreover,

$$u^{\gamma}(r)+u^{\delta}(r)=$$

$$\frac{a^{\frac{(\beta+1)(p-1)}{\beta}}b^{(\beta+1)(p-1)+1}\left(1+a^{\frac{1}{\beta}}\right)}{\left(b+r^{p'}\right)^{(\beta+1)(p-1)+1}}$$

$$+\frac{a^{\frac{(\beta+1)(p-1)}{\beta}}b^{(\beta+1)(p-1)}r^{p'}}{\left(b+r^{p'}\right)^{(\beta+1)(p-1)+1}}.$$

To fulfill equation (8) we must have

$$a^{\frac{(\beta+1)(p-1)}{\beta}} b^{(\beta+1)(p-1)} \left(1+a^{\frac{1}{\beta}}\right)$$

$$= \left(\beta p' a b^{\beta}\right)^{p-1} n$$

and

$$(\beta+1)\, p \left(\beta p' a b^{\beta}\right)^{p-1} = \left(\beta p' a b^{\beta}\right)^{p-1} n$$

$$- a^{\frac{(\beta+1)(p-1)}{\beta}} b^{(\beta+1)(p-1)}.$$

From here we get for parameters a and b the same as in (13) and (14). Both constants fit with the case $p=2$ investigated in [15] for the choice $\delta = \frac{\beta+2}{\beta}$, $p=2$.

For the determination of the exact solution of (7) we refer to the paper by Bognár and Drábek [1].

3 The existence of local solution

We shall form problem (9) as the system of special Briot-Bouquet differential equations. For this type of differential equations we refer to the book by E. Hille [13] and E. L. Ince [14].

Theorem 2. *(Briot-Bouquet Theorem) Let us assume that for the system of equations*

$$\left.\begin{aligned} \xi \frac{dz_1}{d\xi} &= u_1(\xi, z_1(\xi), z_2(\xi)), \\ \xi \frac{dz_2}{d\xi} &= u_2(\xi, z_1(\xi), z_2(\xi)), \end{aligned}\right\} \qquad (16)$$

where functions u_1 *and* u_2 *are holomorphic functions of* ξ, $z_1(\xi)$, *and* $z_2(\xi)$ *near the origin, moreover* $u_1(0,0,0) = u_2(0,0,0) = 0$, *then a holomorphic solution of (16) satisfying the initial conditions* $z_1(0)=0$, $z_2(0)=0$ *exists if none of the eigenvalues of the matrix*

$$\begin{bmatrix} \left.\frac{\partial u_1}{\partial z_1}\right|_{(0,0,0)} & \left.\frac{\partial u_1}{\partial z_2}\right|_{(0,0,0)} \\ \left.\frac{\partial u_2}{\partial z_1}\right|_{(0,0,0)} & \left.\frac{\partial u_2}{\partial z_2}\right|_{(0,0,0)} \end{bmatrix}$$

is a positive integer.

For a proof of Theorem 2 we refer to [6].

The differential equation (9) has singularity at $r=0$ for the case $n>1$. Theorem 2 ensures the existence of formal solutions

$$z_1 = \sum_{k=1}^{\infty} a_k \xi^k \text{ and } z_2 = \sum_{k=1}^{\infty} b_k \xi^k$$

for system (16) and it provides the convergence of formal solutions.

Theorem 3. *For any* p, γ, δ, n *as above, the initial value problem (9)* $u(0)=\alpha$, $u'(0)=0$ *has an unique analytic solution of the form* $u(r) = Q\left(r^{p/(p-1)}\right)$ *in* $(0, A)$ *for small real value of* A, *where* Q *is a holomorphic solution to*

$$Q'' = \frac{-1}{p\,(1+1/p)^{p+1}} r^{-\frac{p+1}{p}} \frac{Q^{\gamma}+Q^{\delta}}{|Q'|^{p-1}}$$

$$-\frac{n}{p\,\alpha} r^{-(1+1/p)} Q'$$

near zero satisfying

$$Q(0)=\alpha,\ \ Q'(0) = \frac{1-p}{p}\left(\frac{\alpha^{\gamma}+\alpha^{\delta}}{n}\right)^{\frac{1}{p-1}}.$$

Proof. We shall now present a formulation of (9) as a system of Briot-Bouquet type differential equations (16). Let us take solution of (9) in the form

$$u(r) = Q\left(r^{\sigma}\right),\ r \in (0, A),$$

where $Q \in C^2(0,a)$ and $\sigma > 0$. Let us take $u(r) = Q\left(r^{\sigma}\right)$ into (9) we obtain

$$\begin{aligned} Q''\left(r^{\sigma}\right) &= -\frac{Q^{\gamma}+Q^{\delta}}{|Q'|^{p-2}} \frac{r^{-p\,(\sigma-1)}}{(p-1)\,\sigma^{p}} \\ &\quad - \frac{n-1+(p-1)(\sigma-1)}{(p-1)\,\sigma} r^{-\sigma} Q' \end{aligned} \qquad (17)$$

and substituting $\xi = r^{\sigma}$ we have

$$Q''(\xi) = -\frac{Q^{\gamma} + Q^{\delta}}{|Q'|^{p-2}} \frac{\xi^{-p\frac{\sigma-1}{\sigma}}}{(p-1)\,\sigma^{p}} \tag{18}$$

$$-\frac{n-1+(p-1)(\sigma-1)}{(p-1)\,\sigma}\xi^{-1}Q'.$$

Here, we introduce function Q as follows

$$Q(\xi) = g_0 + g_1\xi + F(\xi), \tag{19}$$

where $F \in C^2(0,a)$,

$$F(0) = 0,\ F'(0) = 0.$$

Therefore we have

$$Q(0) = g_0,\ Q'(0) = g_1,$$

$$Q'(\xi) = g_1 + F'(\xi),\ Q''(\xi) = F''(\xi).$$

From initial condition $u(0) = \alpha$ we have that

$$g_0 = \alpha.$$

We reformulate (18) as a system of equations:

$$z_1(\xi) = F(\xi),$$
$$z_2(\xi) = F'(\xi),$$

with initial conditions

$$z_1(0) = 0,$$
$$z_2(0) = 0.$$

According to (17) we get that

$$F''(\xi) = -\frac{\xi^{-\frac{\sigma-1}{\sigma}p}}{(p-1)\,\sigma^{p}}\, G(\xi,\ z_1,\ z_2)$$

$$-\frac{n-1+(p-1)(\sigma-1)}{(p-1)\,\sigma}\xi^{-1}(g_1 + F'(\xi)),$$

$$G(.,.,.) = \frac{[g_0 + g_1\xi + F(\xi)]^{\gamma} + [g_0 + g_1\xi + F(\xi)]^{\delta}}{|g_1 + F'(\xi)|^{p-2}}$$

We generate the system of equations

$$\left.\begin{aligned} u_1(\xi, z_1(\xi), z_2(\xi)) &= \xi\, z_1'(\xi) \\ u_2(\xi, z_1(\xi), z_2(\xi)) &= \xi\, z_2'(\xi) \end{aligned}\right\}$$

as follows

$$\left.\begin{aligned} u_1(\xi, z_1(\xi), z_2(\xi)) &= \xi\, z_2 \\ u_2(\xi, z_1(\xi), z_2(\xi)) &= -\frac{\xi^{1-\frac{\sigma-1}{\sigma}p}}{(p-1)\,\sigma^{p}} G(\xi,\ z_1,\ z_2) \\ &\quad -\frac{n-1+(p-1)(\sigma-1)}{(p-1)\,\sigma}(g_1 + z_2(\xi)) \end{aligned}\right\}.$$

In order to satisfy conditions $u_1(0,0,0) = 0$ and $u_2(0,0,0) = 0$ we must get zero for the power of ξ in the right-hand side of the second equation:

$$1 - \frac{p(\sigma - 1)}{\sigma} = 0,$$

i.e.,

$$\sigma = \frac{p}{p-1}.$$

To ensure $u_2(0,0,0) = 0$ we have the connection

$$n\, g_1 |g_1|^{p-2} + \left(\frac{p-1}{p}\right)^{p-1}\left(g_0^{\gamma} + g_0^{\delta}\right) = 0,$$

i.e.,

$$g_1 = \frac{1-p}{p}\left(\frac{g_0^{\gamma} + g_0^{\delta}}{n}\right)^{\frac{1}{p-1}}. \tag{20}$$

Therefore, we obtain

$$g_1 = \frac{1-p}{p}\left(\frac{\alpha^{\gamma} + \alpha^{\delta}}{n}\right)^{\frac{1}{p-1}}. \tag{21}$$

From initial conditions $u(0) = \alpha \neq 0,\ u'(0) = 0,$ and (19) it follows that $g_0 = \alpha.$

For u_1 and u_2 we find the following partial derivatives at $(0,0,0)$

$$\left.\frac{\partial u_1}{\partial z_1}\right|_{(0,0,0)} = 0,$$

$$\left.\frac{\partial u_1}{\partial z_2}\right|_{(0,0,0)} = 0,$$

$$\left.\frac{\partial u_2}{\partial z_1}\right|_{(0,0,0)} = -\frac{\gamma g_0^{\gamma-1} + \delta g_0^{\delta-1}}{(p-1)\sigma g_1^{p-2}},$$

$$\left.\frac{\partial u_2}{\partial z_2}\right|_{(0,0,0)} = -\frac{n\,(p-1)}{p}.$$

Therefore the eigenvalues of matrix

$$\begin{bmatrix} \partial u_1/\partial z_1 & \partial u_1/\partial z_2 \\ \partial u_2/\partial z_1 & \partial u_2/\partial z_2 \end{bmatrix}$$

at $(0,0,0)$ are 0 and $-n(p-1)/p$. Since both eigenvalues are non-positive, applying Theorem 2 we get the existence of unique analytic solutions z_1 and z_2 at zero. Thus we get the analytic solution

$$Q(\xi) = g_0 + g_1\xi + F(\xi)$$

satisfying (17) with

$$Q(0) = g_0,$$

$$Q'(0) = g_1,$$

where $g_0 = \alpha$ and g_1 is determined in (21).

Corollary 4. *From Theorem 3 it follows that solution $u(r)$ for (9) has an expansion near zero of the form*

$$u(r) = \sum_{k=0}^{\infty} g_k\, r^{\frac{kp}{p-1}}$$

satisfying

$$u(0) = \alpha,$$

and

$$u'(0) = 0.$$

4 Determination of local solution near zero

This section is devoted to the construction of the power series solution for (9) with initial conditions

$$u(0) = \alpha > 0, \quad u'(0) = 0. \tag{22}$$

We seek a solution of the form

$$u(r) = g_0 + g_1\, r^{\frac{p}{p-1}} + g_2\, r^{2\left(\frac{p}{p-1}\right)} + \ldots, \quad r > 0, \tag{23}$$

with coefficients $g_k \in \mathbf{R}$, $k = 0,1,\ldots$.
From Section 3 we get the first two coefficients:

$$g_0 = \alpha > 0,$$

and

$$g_1 = \frac{1-p}{p}\left(\frac{\alpha^\gamma + \alpha^\delta}{n}\right)^{\frac{1}{p-1}}.$$

We assume that

$$u(r) > 0 \tag{24}$$

and

$$u'(r) < 0 \tag{25}$$

in the neighborhood of zero. Since

$$u'(r) = r^{\frac{1}{p-1}}\left[g_1\frac{p}{p-1} + g_2\frac{2p}{p-1}\, r^{\frac{p}{p-1}} + \ldots\right],$$

and function $f(s) = s|s|^{p-2}$ is an odd function $(s \in \mathbf{R})$, then we can write

$$|u'(r)|^{p-2}u'(r) = -[-u'(r)]^{p-1}, \tag{26}$$

$$|u'(r)|^{p-2}u'(r) =$$

$$-r\left[-g_1\frac{p}{p-1} - g_2\frac{2p}{p-1}\, r^{\frac{p}{p-1}} - \ldots\right]^{p-1} =$$

$$-r\left[P_0 + P_1\, r^{\frac{p}{p-1}} + P_2\, r^{2\left(\frac{p}{p-1}\right)} + \ldots\right],$$

moreover,

$$r^{1-n}\left(r^{n-1}|u'(r)|^{p-2}u'(r)\right)' =$$

$$= \left[-P_0 n - P_1 \left(n + \frac{p}{p-1} \right) r^{\frac{p}{p-1}} - \dots \right.$$

$$\left. - P_k \left(n + \frac{kp}{p-1} \right) r^{k\left(\frac{p}{p-1}\right)} + \dots \right],$$

where coefficients P_k will be expressed in terms of g_k $(k = 0,1,\dots)$. Now, $u^{\gamma}(t)$ and $u^{\delta}(t)$ can be written in the form

$$u^{\gamma}(t) = \left[g_0 + g_1 r^{\frac{p}{p-1}} + g_2 r^{2\left(\frac{p}{p-1}\right)} + \dots \right]^{\gamma}$$

$$= G_0 + G_1 r^{\frac{p}{p-1}} + G_2 r^{2\left(\frac{p}{p-1}\right)} + \dots$$

$$u^{\delta}(t) = \left[g_0 + g_1 r^{\frac{p}{p-1}} + g_2 r^{2\left(\frac{p}{p-1}\right)} + \dots \right]^{\delta}$$

$$= D_0 + D_1 r^{\frac{p}{p-1}} + D_2 r^{2\left(\frac{p}{p-1}\right)} + \dots,$$

where coefficients G_k and D_k can be expressed in terms of g_k $(k = 0,1,\dots)$ and γ, δ, respectively

Substituting them into the equation (9) we compare the coefficients of the proper powers of r and we find that

$$-P_k \left(n + \frac{kp}{p-1} \right) + G_k + D_k = 0 \qquad (27)$$

for $k \ge 0$.

Applying the J. C. P. Miller formula (see [11]) we derive P_k, G_k and D_k $(k = 0,1,\dots)$ in the forms

$$G_k = \frac{1}{k\alpha} \sum_{j=0}^{k-1} [(k-j)\gamma - j] G_j g_{k-j}, \qquad (28)$$

$$D_k = \frac{1}{k\alpha} \sum_{j=0}^{k-1} [(k-j)\delta - j] D_j g_{k-j}, \qquad (29)$$

$$P_k = \sum_{j=0}^{k-1} [(k-j)(p-1) - j] P_j g_{k+1-j} \frac{(k+1-j)}{g_1 k} \qquad (30)$$

for $k \ge 1$, and

$$G_0 = g_0^{\gamma},$$

$$D_0 = g_0^{\delta},$$

$$P_0 = \left(-g_1 \frac{p}{p-1} \right)^{p-1}.$$

From (27) we obtain coefficients g_k for $k \ge 2$:

$$g_0 = \alpha,$$

$$g_1 = -\frac{p-1}{p} \left(\frac{\alpha^{\gamma} + \alpha^{\delta}}{n} \right)^{\frac{1}{p-1}},$$

$$g_2 = \left(\frac{p-1}{p} \right)^2 \left(\frac{\alpha^{\gamma} + \alpha^{\delta}}{n} \right)^{\frac{3-p}{p-1}} \frac{\gamma\alpha^{\gamma} + \delta\alpha^{\delta}}{2\alpha(np - n + p)}$$

$$\vdots$$

Example 1. Let us consider the solution of (9) with parameters

$$p = 2,$$

$$n = 5,$$

$$\gamma = 1.8,$$

$$\delta = 3,$$

$$\alpha = 2.3233.$$

Using software MAPLE we obtain the coefficients of the power series solution

$$u(r) = \sum_{k=0}^{\infty} g_k r^{2k}$$

from recursive formulas (27)-(30) as follows

$$g_0 = 2.3233,$$

$$g_1 = -1.7101,$$

$$g_2 = 1.2047,$$

$$g_3 = -0.8505,$$

$$g_4 = 0.6013,$$

$$g_5 = -0.4254,$$

$$g_6 = 0.3011$$

$$\vdots$$

Therefore, we have the local analytic solution of the form

$$\begin{aligned} u(r) &= 2.3233 - 1.7101 r^2 + 1.2047 r^4 \\ &- 0.8505 r^6 + 0.6013 r^8 - 0.4254 r^{10} \qquad (31) \\ &+ 0.3011 r^{12} - 0.21318 r^{14} + \ldots \end{aligned}$$

We keep parameters n, γ and δ fixed and evaluate the power series solution for different values of p. The value of the radial solution of (9) at 0 denoted by g_0 will be considered

$$g_0 = \left(\frac{n}{(\beta+1)p} - 1 \right)^{-\beta}$$

in each cases (as it was done in Example 1 as well). This value is the same as parameter value a in the formula (15) of the exact solution (see Section 2).

Example 2. Evaluate the power series solution of (9)-(22) with parameters

$$\begin{aligned} p &= 1.8, \\ n &= 5, \\ \gamma &= 1.8, \\ \delta &= 3, \\ \alpha &= 1.7380. \end{aligned}$$

After making calculations from the recursive formulas we have the solution in the form

$$\begin{aligned} u(r) &= 1.7380 - 0.7941 r^{1.44} + 0.4053 r^{2.88} \\ &- 0.2137 r^{4.32} + 0.1145 r^{5.76} - 0.0619 r^{7.2} \qquad (32) \\ &+ 0.0337 r^{8.64} - 0.0184 r^{10.08} + \ldots . \end{aligned}$$

Example 3. Evaluate the power series solution of (9)-(22) with parameters

$$\begin{aligned} p &= 2.2, \\ n &= 5, \\ \gamma &= 1.8, \\ \delta &= 3, \\ \alpha &= 3.2884. \end{aligned}$$

From formulas (27)-(30) we shall obtain the solution in the form

$$\begin{aligned} u(r) &= 3.2884 - 3.3458 r^{2.64} + 2.8728 r^{5.28} \\ &- 2.3882 r^{7.92} + 1.9539 r^{10.56} - 1.5827 r^{13.20} \qquad (33) \\ &+ 1.2732 r^{15.84} - 1.0191 r^{18.48} + \ldots . \end{aligned}$$

We note that in this section we have assumed that $u = u(r)$ satisfies (24) and (25). If we assume that

$$u(r) > 0$$

and

$$u'(r) > 0$$

in the neighborhood of zero, then we have the following modified version of (26)

$$|u'(r)|^{p-2} u'(r) = [u'(r)]^{p-1},$$

and

$$[u'(r)]^{p-1} =$$

$$r \left[g_1 \frac{p}{p-1} + g_2 \frac{2p}{p-1} r^{\frac{p}{p-1}} + \ldots \right]^{p-1} =$$

$$r \left[P_0 + P_1 r^{\frac{p}{p-1}} + P_2 r^{2\left(\frac{p}{p-1}\right)} + \ldots \right].$$

In this case the connection between P_k and G_k, D_k simplifies to the form

$$P_k\left(n+\frac{kp}{p-1}\right)+G_k+D_k=0 \qquad (34)$$

for $k \geq 0$

and moreover,

$$G_0 = g_0^{\gamma},$$
$$D_0 = g_0^{\delta},$$
$$P_0 = \left(g_1 \frac{p}{p-1}\right)^{p-1}.$$

With these changes we can determine the coefficients g_0, g_1, ... g_k, ... of the power series solution.

5 The comparison of exact and local analytic solutions

In the previous three examples we gave approximate solutions to the problem (9) in the neighborhood of zero. The parameters p, n, γ and δ were chosen such a way to fulfill conditions (11) in Proposition 1. It means that for such specific values entire solution of the problem (9)-(22) exists. Now we consider problem (9) with the same values of n, γ and δ, (i.e., with the same parameter value $\beta = \frac{5}{6}$) and with different values of p (namely $p = 1.8, \quad 2, \quad 2.2$).

Using Proposition 1 the entire solutions for all three cases can be determined with the calculated parameter values by using (13)-(15).

In Example 1 the parameters are the following:

$$p = 2$$
$$a = 2.3233,$$
$$b = 0.8081,$$
$$\beta = 5/6,$$
$$p' = 2$$

and the exact solution can be given as follows

$$\bar{u}(r) = 2.3233\left(\frac{0.8081}{0.8081+r^2}\right)^{\frac{5}{6}} \qquad (35)$$

(see Fig. 1).

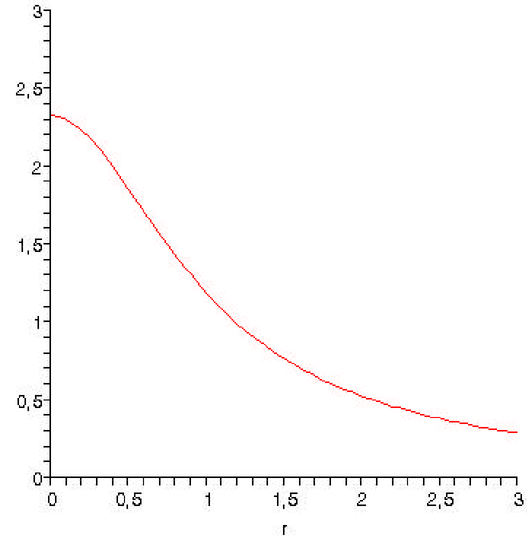

Fig.1. Entire solution for p=2

Hence, we have the possibility to compare the two solutions, i.e., (31) with (35). On Fig. 2. the difference between the exact (31) and local analytic solutions (35) are represented.

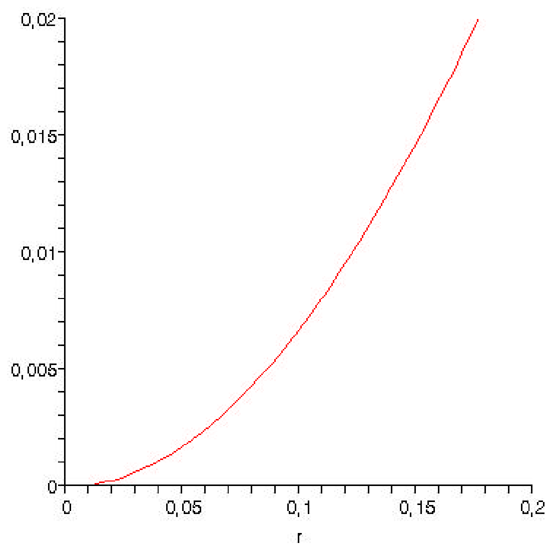

Figure 2.

In Example 2 the parameters are

$$p = 1.8$$
$$a = 1.7380,$$
$$b = 1.4275,$$
$$\beta = 5/6,$$
$$p' = 1.44$$

and the exact solution can be given as follows

$$\bar{u}(r) = 1.7380\left(\frac{1.4275}{1.4275 + r^{1.44}}\right)^{\frac{5}{6}} \qquad (36)$$

(see Fig. 3).

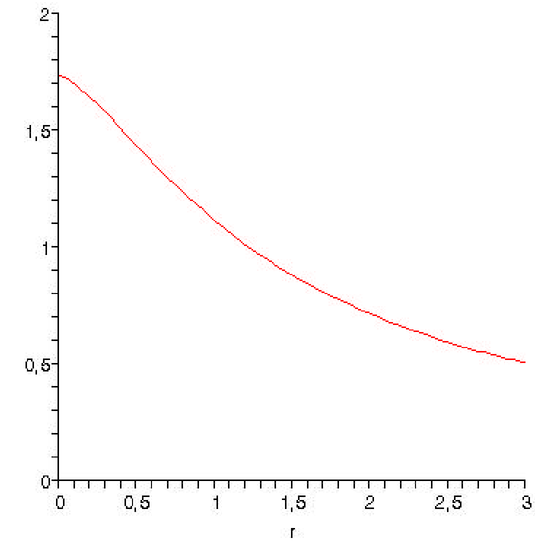

Fig.3. Entire solution for p=1.8

The difference between the entire and power series solution is exhibited on Fig. 4.

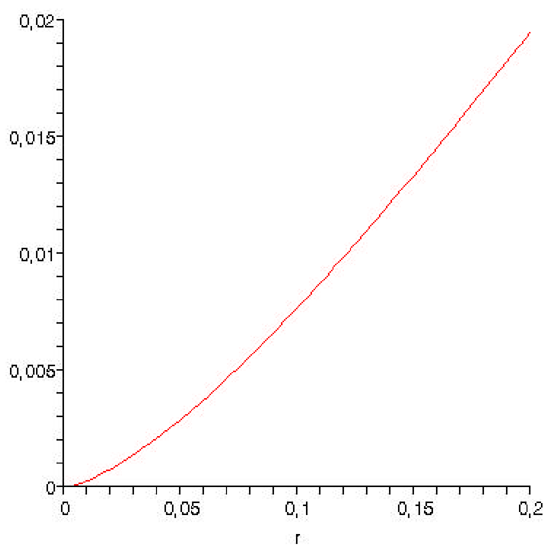

Figure 4.

In Example 3 the parameters are

$$p = 2.2$$
$$a = 3.2884,$$
$$b = 0.3227,$$
$$\beta = 5/6,$$
$$p' = 2.64,$$

and the exact solution can be given by

$$\bar{u}(r) = 3.2884\left(\frac{0.3227}{0.3227 + r^{2.64}}\right)^{\frac{5}{6}} \qquad (37)$$

(see Fig. 5).

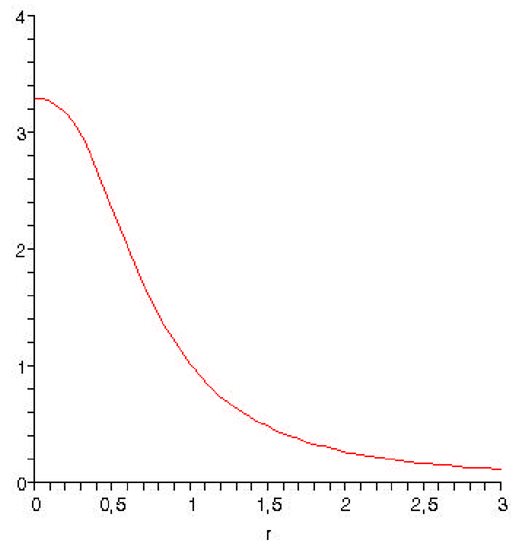

Fig. 5. Entire solution for p=2.2

The difference between solution (33) and (37) is illustrated on Fig.6.

On Fig.1, Fig.3 and Fig. 5 we see that the shapes of the solutions of the initial value problem (9) differ considerably if we keep the parameters the same and only value of p is varying.

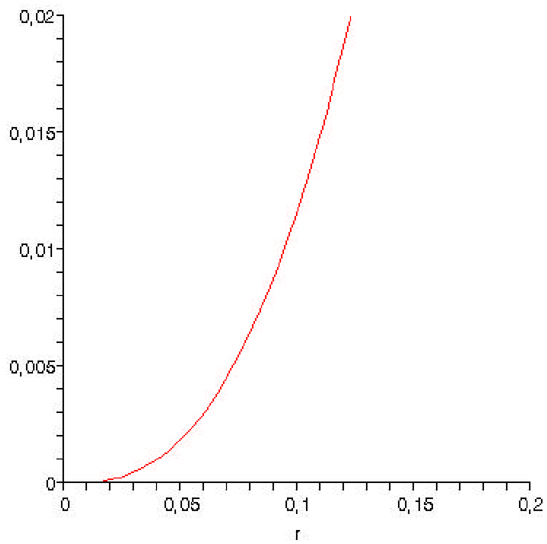

Figure 6.

In Fig. 7 the entire solutions of (9) are illustrated for the examined three cases.

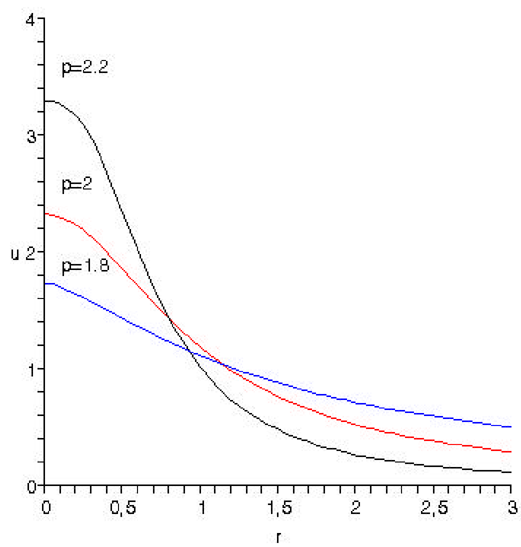

Figure 7.

6 Perturbation analysis

In this section we intend to discuss the influence of change of parameters for the analytical solution of initial value problem (8). We present the power series solutions for different values of parameters n, p, γ, and δ. From the four parameters we always fix three ones and allow to change only one. For all investigated cases we take sublinear and superlinear exponents in function f, i.e., $\gamma < 1$, and $\delta > 1$, and we fix the initial value of the solution at zero:

$$\alpha = 2.$$

6.1 The effect of change in p

Let us fix parameters n, γ, and δ:

$$n = 3,$$
$$\gamma = 0.8,$$
$$\delta = 2,$$

and evaluate the power series solution according to Section 4 for

$$p = 1.5,\ 2,\ 3.$$

The solutions u_p for different values of p have the form:

$p = 1.5$

$$u_{p=1.5}(r) = 2 - 1.2207r^3 + 0.6095r^6 - 0.3047r^9 + 0.1526r^{12} - 0.0765r^{15} + 0.0384r^{18} - 0.0193r^{21},$$

$p = 2$

$$u_{p=2}(r) = 2 - 0.9568r^2 + 0.2247r^4 - 0.4616r^6 + 0.0886r^8 - 0.0016r^{10} + 0.0003r^{12},$$

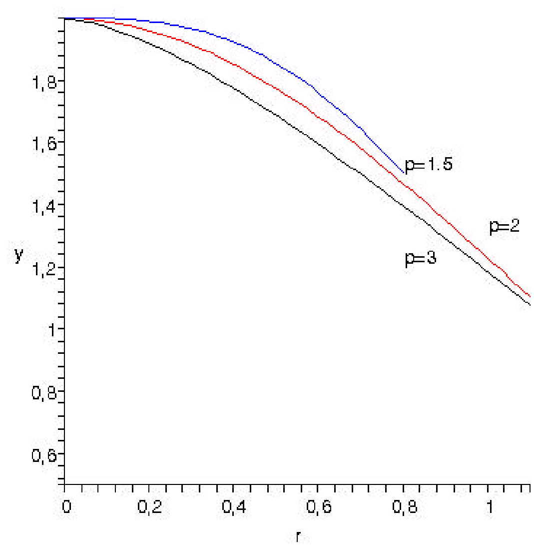

Figure 8.

$p = 3$

$$u_{p=3}(r) = 2 - 0.9222r^{1.5} + 0.1159r^3 - 0.0086r^{4.5} + 0.0004r^6 - 0.1094053423\,10^{-4}r^{7.5} + 0.878\,10^{-6}r^9.$$

On Figure 8 we demonstrate the figures of solutions $u_{p=1.5}$, $u_{p=2}$, $u_{p=3}$ and the influence of different values of p.

6.2 The effect of change in δ

Let us fix parameters n, γ, and p:

$$n = 3, \quad \gamma = 0.8, \quad p = 2,$$

and evaluate the local analytic solution in the neighborhood of zero for

$$\delta = 1,\ 2,\ 3.$$

The solutions u_δ for different values of δ have the form

$\delta = 1$

$$u_{\delta=1}(r) = 2 - 0.6235r^2 + 0.0529r^4 - 0.0018r^6 + 0.3429\,10^{-4}r^8 - 0.2010\,10^{-6}r^{10} + 0.1850\,10^{-7}r^{12},$$

$\delta = 2$

$$u_{\delta=2}(r) = 2 - 0.9568r^2 + 0.2247r^4 - 0.0462r^6 + 0.0088r^8 - 0.0016r^{10} + 0.0003r^{12} - 0.4990\,10^{-4}r^{14},$$

$\delta = 3$

$$u_{\delta=3}(r) = 2 - 1.6235r^2 + 1.0306r^4 - 0.6859r^6 + 0.4581r^8 - 0.3057r^{10} + 0.2040r^{12} - 0.1362r^{14}.$$

On Figure 9 the graphs of $u_{\delta=1}$, $u_{\delta=2}$, and $u_{\delta=3}$ are illustrated and we see the effect of δ for the shape of the solutions.

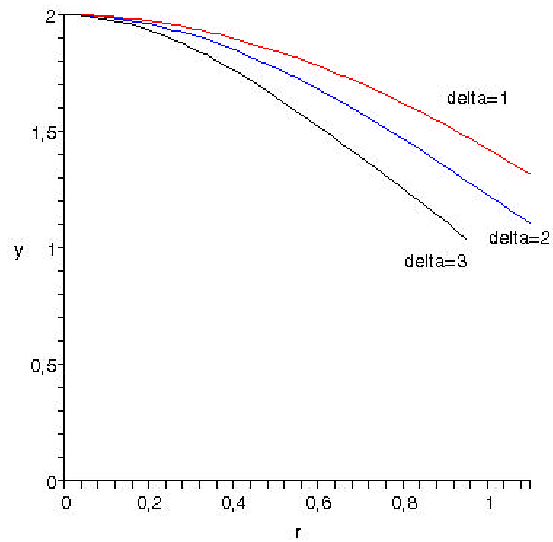

Figure 9.

6.3 The effect of change in γ

Let us fix parameters n, p, and δ as follows

$$n = 3, \quad p = 3, \quad \delta = 1.5,$$

and evaluate the power series solution according to Section 4 for

$$\gamma = 0.1,\ 0.4,\ 0.9.$$

The solutions u_γ for different values of γ have the form

$\gamma = 0.1$

$$u_{\gamma=0.1}(r) = 2 - 0.7601\,r^{3/2} + 0.0537\,r^3 - 0.0017\,r^{9/2} - 0.2463\,10^{-4}r^6 - 0.2868\,10^{-5}r^{15/2} - 0.2102\,10^{-6}r^9 - 0.4263\,10^{-8}r^{21/2},$$

$\gamma = 0.4$

$$u_{\gamma=0.4}(r) = 2 - 0.7839\,r^{3/2} + 0.05889\,r^3 - 0.0014\,r^{9/2} + 0.1395\,10^{-4}r^6 + 0.1916\,10^{-5}r^{15/2} + 0.4301\,10^{-6}r^9 + 0.9059\,10^{-7}r^{21/2},$$

$\gamma = 0.9$

$$u_{\gamma=0.9}(r) = 2 - 0.8339\, r^{3/2} + 0.0731\, r^3 - 0.0014\, r^{9/2} + 0.1838\, 10^{-4} r^6 + 0.8063\, 10^{-6} r^{15/2} + 0.4372\, 10^{-7} r^9 - 0.1465\, 10^{-8} r^{21/2}.$$

On Figure 10 we exhibit the figures of $u_{\gamma=0.1}$, $u_{\gamma=0.4}$, and $u_{\gamma=0.9}$. We see that the influence of change of γ very small.

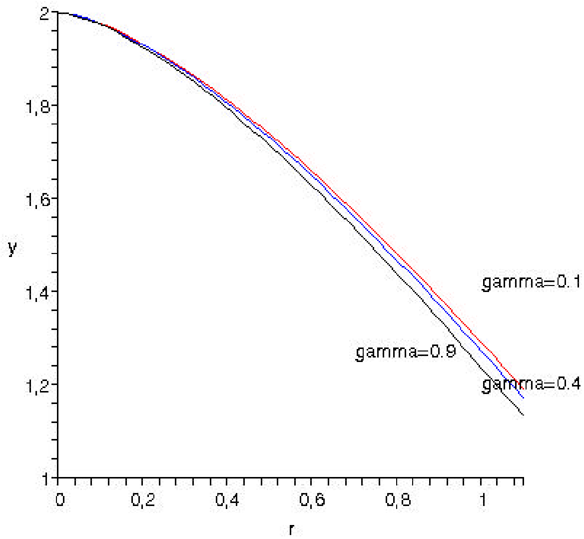

Figure 10.

6.4 The effect of change in n

Let us fix parameters n, γ, and δ such as

$$p = 2.5,\quad \gamma = 0.8,\quad \delta = 2,$$

and evaluate the local analytic solution in the neighborhood of zero for

$$n = 2,\ 4,\ 6.$$

The solutions u_n for different values of n have the form

$n = 2$

$$u_{n=2}(r) = 2 - 1.2119 r^{5/3} + 0.2184 r^{10/3} - 0.0298 r^5 + 0.3225\, 10^{-2} r^{20/3} - 0.2987\, 10^{-3} r^{25/3} + 0.2737\, 10^{-4} r^{10} - 0.1819\, 10^{-5} r^{35/3},$$

$n = 4$

$$u_{n=4}(r) = 2 - 0.7634 r^{5/3} + 0.1122 r^{10/3} - 0.0121 r^5 + 0.1089\, 10^{-2} r^{20/3} - 0.8425\, 10^{-4} r^{25/3} + 0.6128\, 10^{-5} r^{10} - 0.3999\, 10^{-6} r^{35/3},$$

$n = 6$

$$u_{n=6}(r) = 2 - 0.5826\, r^{5/3} + 0.0724\, r^{10/3} - 0.6679\, 10^{-2} r^5 + 0.5241\, 10^{-3} r^{20/3} - 0.3568\, 10^{-4} r^{25/3} + 0.2259\, 10^{-5} r^{10} - 0.1319\, 10^{-6} r^{35/3}.$$

On Figure 11 we present the figures of $u_{n=2}$, $u_{n=4}$, and $u_{n=6}$.

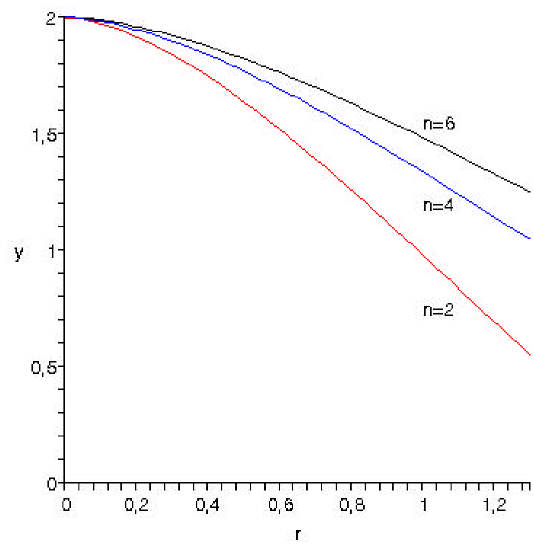

Figure 11.

7 Conclusion

For problem (9) the entire solutions can be evaluated under restrictions listed in (11).

In other cases (and in this case as well) we are able to show the existence of local analytic solution of initial value problem (9) and to give a method for the determination of the coefficients for convergent power series solution in the neighborhood of zero. We calculated the exact entire solutions and the power series solutions for different values when the parameters. The effect of change of parameter values was investigated.
We remark, that the local solution can be given in the neighborhood of any $r_0 \in R$ and for different parameter values of $p,\ n,\ \alpha,\ \gamma,\ \delta$.
Our future aim is to investigate the widest class of function f such that there exists locally analytic solution to problem (7).

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