Identification of a Heat Transfer Coefficient when it is a Function Depending on Temperature

ANDRÉS FRAGUELA
Benemérita Universidad Autónoma de Puebla
Facultad de Ciencias Físico–Matemáticas
Av. San Claudio s/n, Col. San Manuel, Puebla
MEXICO
fraguela@fcfm.buap.mx

JUAN–ANTONIO INFANTE
Universidad Complutense de Madrid
Departamento de Matemática Aplicada
Plaza de Ciencias 3, 28040–Madrid
SPAIN
ja_infante@mat.ucm.es

ÁNGEL MANUEL RAMOS
Universidad Complutense de Madrid
Departamento de Matemática Aplicada
Plaza de Ciencias 3, 28040–Madrid
SPAIN
angel@mat.ucm.es

JOSÉ MARÍA REY
Universidad Complutense de Madrid
Departamento de Matemática Aplicada
Plaza de Ciencias 3, 28040–Madrid
SPAIN
jose_rey@mat.ucm.es

Abstract: This paper deals with an inverse problem concerning the identification of the heat exchange coefficient H (assumed depending on the temperature) between a certain material with the external environment (see, e.g., [12], [20] for real applications modelled with equations involving this coefficient). Only experimental measurements of the temperature are supposed to be known. The goal is to identify H in order to get a solution for the corresponding model, approximating some given temperature measurements. The main difficulty is that we consider the case of functions H depending on the solution of the state equation. We begin by setting several scenarios for the inverse problem. For each scenario, we know the initial and ambient temperatures, we identify function H through different methods and we obtain error bounds in adequate norms (uniform and square integrable). Finally, we study the inverse problem in the framework of the classical theory for Hilbert spaces. Several methods are used (Tikhonov, Morozov, Landweber,. . .) and the approximations obtained, as well as the one provided by our method, are shown.

Key–Words: Function identification, Inverse Problems, Heat exchange, Regularization strategies.

1 Description of the inverse problem and their physical motivation.

Let us suppose we have a homogeneous sample of a material that is getting warm (respectively, cool) due to heat exchange with the external environment. Complex models based on partial differential equations are needed to describe the distribution of the temperature inside the sample. These equations (direct equations) involve functions and parameters that need to be known before we can compute solutions. Typically, these functions and parameters are computed either by experimental methods or, as in this work, by solving inverse problems in suitable mathematical frameworks (see, for instance, [5], [6], [7], [10]).

An experimental procedure was proposed in [24] based on a genetic algorithm for determining a heat transfer coefficient. In [1] and [18] some methods based on inverse analysis are developed in order to identify the heat transfer on a machine tool surface. A method for the determination of the heat transfer coefficient was proposed in [21] for the first falling drying period of potato cubes where heat and mass transfer were considered as coupled phenomena. In [8] an identification problem for the heat transfer coefficient in foods during freezing using cooling curves obtained from an industrial survey is solved. The diffusion coefficient is supposed to be constant in all the works cited in this paragraph, which do not use regularizing algorithms able to compensate the sensitivity of the identification process to experimental measurement errors, as it is done in this paper. Other approaches in inverse problem theory can be seen in [2], [11] and [13]

For parameter identification, the least squares method may provide a good tool for solve inverse problems (see, for instance, [12]). When the goal is to identify a function, the problem becomes more complicated, especially if the function depends on the solution of the state and experimental data can be given with measurement errors.

For simplicity, let us suppose that the sample is small enough to be able to assume that the temperature gradient inside it is negligible. The Newton Cooling Law provides a simple mathematical model describing this phenomenon through the following initial value problem (*direct problem*):

$$\begin{cases} T'(t) = H(T(t))(T^e - T(t)),\ t \geq t_0 \\ T(t_0) = T_0, \end{cases} \tag{1}$$

where $T(t)$ is the temperature of the sample at time t, T^e is the external environment temperature, T_0 is the temperature at the initial time t_0 and H is the temperature dependent heat exchange coefficient. To solve problem (1) we need to know the model data: constants $T_0, T^e \in \mathbb{R}$ and function $H(\cdot) : (T_a, T_b) \to \mathbb{R}$, where (T_a, T_b) is a range of temperatures suitable for the problem we are considering.

In real cases, the values of T_0 and T^e can be obtained through simple devices measuring temperature. However, obtaining function $H(\cdot)$ is not so easy by experimental methods.

The goal of this work is to solve the *inverse problem* of identify $H(\cdot)$, knowing just certain experimental measurements of temperature. The main difficulties are the following:

- The function $H(\cdot)$ to be identified depends on the temperature T, which is the solution of the state equation.
- Temperature data may be given with a certain error due to measurement equipment accuracy limitations.

In this paper we develop, in a rigorous mathematical way, suitable strategies for identifying the heat transfer coefficient when it is a function with such a kind of dependency. A numerical algorithm is also developed in a framework different from that of the Classical Theory. Other works regarding numerical approaches for inverse problems can be seen in [9], [14], [15], [22].

In some contexts, and under certain conditions, it can be assumed that H has a known expression (e.g., H constant or a function with a few real parameters to identify). The challenge that we face in this work is to identify function H when continuity and positivity are the only information available about H.

2 Scenarios of the inverse problem.

The model is not very sensitive to changes in $H(s)$ for s close to T^e in the following sense: if for some t_μ, $T(t_\mu) = T^e - \mu$ then, monotonicity of T implies that T remains in the interval $[T^e - \mu, T^e]$ for every $t \geq t_\mu$ and arbitrary values of H. For this reason, it is unrealistic (and unnecessary) pretend to identify H near T^e. These considerations lead us to pose the problem of identifying function H as follows:

i) A *threshold* $\mu > 0$, depending on the admissible error in the approximation of the temperature, is fixed so that the identification of H in the interval $[T^e - \mu, T^e]$ is not part of our goal. From this threshold, a time $t_f = t_f(\mu, T_0, T^e, H)$ is determined (by arguments explained later) such that

$$|T^e - T(t)| < \mu,\ t \geq t_f. \tag{2}$$

Thus, the error in the temperature will be smaller than μ for $t \geq t_f$.

ii) We use model (1) in $[t_0, t_f]$ and identify H in $[T_0, T(t_f)] \supset [T_0, T^e - \mu]$.

According to the available information about $T(t)$ in $[t_0, t_f]$ we set the inverse problem in several scenarios:

- The trivial (and unrealistic) case is to suppose that functions $T(t)$ and $T'(t)$ are known in $[t_0, t_f]$. Then, assuming $H \in \mathcal{C}([T_0, T(t_f)])$ and positive, we can identify H in a direct way from

$$H(s) = \frac{T'(T^{-1}(s))}{T^e - s}. \tag{3}$$

- If function T can be **evaluated without error** in a **finite number** of **arbitrary instants** $t \in [t_0, t_f]$, the identification of H in $[T_0, T(t_f)]$ becomes a standard problem of numerical differentiation (in order to approximate $T'(t)$ from data).
- Next scenario arises when a function $\widetilde{T}$, representing an **approximate value** of the temperature **in every instant**, is supposed to be known.
- However, in a realistic context, only **discrete values $\widehat{T}_k$ approximating the temperature** at some instants are available.

For the last two scenarios we use a "stable" method to approach $T'(t)$ from data. Then, formula (3) provides discrete values approximating H in points of interval $[T_0, T(t_f)]$.

Let us see how to determine t_f satisfying (2) in the non trivial situations described before:

a) In the second scenario, given $p+1$ exact values $\{T_0, T_1, \ldots, T_p\}$ of the temperature at instants $\{\tau_0 = t_0 < \tau_1 < \cdots < \tau_p\}$, we consider $\mu_k = T^e - T_k$. Then μ is chosen as one of the values μ_k or any number smaller than all of them. We take $t_f = \tau_m$, where

$$m = \begin{cases} p, \text{ if } \mu < \mu_k \text{ for all } k. \\ \min\limits_k\{\mu = \mu_k\}, \text{ otherwise.} \end{cases} \tag{4}$$

b) The assumptions in the third scenario are that function $\widetilde{T}$ is known in some interval $[t_0, t^*]$ and

$$\left|\left|T - \widetilde{T}\right|\right|_{\mathcal{C}([t_0,t^*])} < \delta,$$

where $\delta < \mu$ (if $\mu \le \delta$ we would need to increase the value of μ). Then, we consider t_f as

$$t_f = \begin{cases} t^*, \text{ if } \widetilde{T}(t) < T^e - \mu + \delta \text{ for all } t \le t^* \\ \min\limits_t\{\widetilde{T}(t) = T^e - \mu + \delta\}, \text{ otherwise.} \end{cases} \tag{5}$$

c) Finally, in the fourth scenario, t_f is defined in a more sophisticated way. Measurements $\{\widehat{T}_k\}_{k=0}^p$ such that $|T(\tau_k) - \widehat{T}_k| < \widehat{\delta}$, with $\widehat{\delta} > 0$, are available. Let $\widetilde{T}$ be an interpolation function of values $\{\widehat{T}_0, \widehat{T}_1, \ldots, \widehat{T}_p\}$ in $\{\tau_0, \tau_1, \ldots, \tau_p\}$ such that

$$\left|\left|T - \widetilde{T}\right|\right|_{\mathcal{C}([\tau_0,\tau_p])} < \delta$$

for some $\delta > 0$, and take $\mu_k = T^e - \widehat{T}_k + \delta$ for $k = 1, 2, \ldots, p$. Now, we assume that $\mu > 3\delta$ (otherwise, the value of μ will be increased) and that μ is lower or equal than all previous values μ_k. Then, taking m as in (4), we may define

$$t_f = \tau_m. \tag{6}$$

3 A first approach to the inverse problem.

3.1 Identifying from a finite amount of exact values of temperature.

Given $n \in \mathbb{N}$, the values of the temperature T at $t_k = t_0 + kh$ for $k = 0, 1, \ldots, n$, are supposed to be known, where $h = \dfrac{t_f - t_0}{n}$. Lets denote $T_k = T(t_k)$, $k = 0, 1, \ldots, n$. The differential equation of problem (1) can be rewritten as

$$\frac{T'(t)}{T^e - T(t)} = H(T(t)),\ t_0 < t < t_f. \tag{7}$$

Therefore, our goal is to find, for $k = 0, 1, \ldots, n$, an approximation $\widetilde{H}_k$ of

$$\frac{T'(t_k)}{T^e - T(t_k)},$$

which is also an approximation of $H(T_k)$. Considering the first order approximate differentiation operator $R_h : \mathcal{C}([t_0, t_f]) \to \mathcal{C}([t_0, t_f])$ given by

$$R_h(v)(t) = \begin{cases} \Phi_h(v)(t), & t \in [t_0, \hat{t}] \\ \Psi_h(v) + \Phi_h(v)(t-h), & t \in [\hat{t}, t_f] \end{cases}$$

where $\hat{t} = t_f - h$,

$$\Phi_h(v)(t) = \frac{v(t+h) - v(t)}{h}$$

and

$$\Psi_h(v) = \frac{v(t_f) - 2v(t_f - h) + v(t_f - 2h)}{h}.$$

Let us denote by $||\cdot||$ the norm in $\mathcal{C}([t_0, t_f])$. The following result holds:

Lemma 1 *If $v \in \mathcal{C}^2([t_0, t_f])$ then*

$$\left|\left|v' - R_h(v)\right|\right| \le \frac{7h}{2}\left|\left|v''\right|\right|.$$

PROOF. Taylor expansion provides

$$v(t+h) = v(t) + hv'(t) + \frac{h^2}{2}v''(\xi),\ t \in [t_0, t_f - h],$$

where $\xi \in (t, t+h)$. Then

$$v'(t) - R_h(v)(t) = -\frac{h}{2}v''(\xi) \le \frac{h}{2}\left|\left|v''\right|\right|.$$

On the other hand, for $t \in [t_f - h, t_f]$ it follows

$$v(t-h) = v(t) - hv'(t) + \frac{h^2}{2}v''(\eta)$$

and

$$\begin{aligned} &-2v(t_f - h) + v(t_f - 2h) \\ &= -v(t_f) + h^2\Big(2v''(\zeta_2) - v''(\zeta_1)\Big) \end{aligned}$$

for some $\eta \in (t-h, t)$, $\zeta_1 \in (t_f - h, t_f)$ and $\zeta_2 \in (t_f - 2h, t_f)$. Therefore,

$$\begin{aligned} v'(t) - R_h(v)(t) &= h\left(\frac{1}{2}v''(\eta) - 2v''(\zeta_2) + v''(\zeta_1)\right) \\ &\le \frac{7h}{2}\left|\left|v''\right|\right|. \square \end{aligned}$$

In order to approach $H(T_k)$ we take

$$\widetilde{H}_k = \frac{R_h(T)(t_k)}{T^e - T_k},$$

for $k = 0, 1, \dots, n$. Thus, the following bound for the error is obtained:

Proposition 2 *If* $T \in \mathcal{C}^2([t_0, t_f])$ *then*

$$\max_{k=0,1,\dots,n} \left| H(T_k) - \widetilde{H}_k \right| \leq \frac{7M_2}{2\mu} h, \qquad (8)$$

where $M_2 = ||T''||$.

PROOF. Monotonicity of T implies

$$T^e - T_k \geq T^e - T(t_f) = \mu.$$

Now, it suffices to apply Lemma 1. □

Remark 3 Note that this estimate for the error in H has the same order as the approximate differentiation method used. Thus, if an upper order method is chosen, the estimate (8) will be better. □

Remark 4 As noted at the beginning of Section 2, t_f is fixed, *a priori*, from the value of μ. Then, the bound in estimate (8) does not blow up. □

3.2 Identifying from a function that approximates the temperature.

In this context, we suppose to know a function $\widetilde{T} \in \mathcal{C}([t_0, t_f])$, where t_f is chosen according to (5) and

$$\left|\left| T - \widetilde{T} \right|\right| < \delta \qquad (9)$$

for some $\delta \in (0, \mu)$. For the sake of simplicity and consistency with the properties of T, we assume that $\widetilde{T}(t) \geq T_0,\ t \in [t_0, t_f]$. From (7), we define

$$u(t) = \frac{T'(t)}{T^e - T(t)},\ t_0 < t < t_f$$

and the approximation

$$\widetilde{u}_h(t) = \frac{R_h(\widetilde{T})(t)}{T^e - \widetilde{T}(t)},\ t_0 < t < t_f.$$

Next, an error estimate is obtained:

Proposition 5 *If* $T \in \mathcal{C}^2([t_0, t_f])$ *and* $\widetilde{T} \in \mathcal{C}([t_0, t_f])$ *satisfies* (9) *with* $0 < \delta < \dfrac{\mu}{3}$, *then*

$$||u - \widetilde{u}_h|| \leq \frac{1}{\mu - 2\delta} \left(\frac{7M_2}{2} h + \frac{3\delta}{h} \frac{T^e - T_0 + \mu - 2\delta}{\mu - 3\delta} \right). \qquad (10)$$

PROOF. Note that

$$u(t) - \widetilde{u}_h(t) = \frac{T'(t) - R_h(T)(t)}{T^e - T(t)} + R_h(T)(t) \frac{T(t) - \widetilde{T}(t)}{(T^e - T(t))(T^e - \widetilde{T}(t))} + \frac{R_h(T)(t) - R_h(\widetilde{T})(t)}{T^e - \widetilde{T}(t)}.$$

Monotonicity of T and (5) give

$$\begin{cases} T^e - T(t) \geq T^e - \widetilde{T}(t_f) - \delta = \mu - 2\delta \\ T^e - \widetilde{T}(t) \geq T^e - \widetilde{T}(t_f) - 2\delta = \mu - 3\delta. \end{cases} \qquad (11)$$

¿From Lemma 1, the first part of the right hand term can be bounded by

$$\frac{7h}{2(T^e - T(t))} \left|\left|T''\right|\right| \leq \frac{7M_2}{2(\mu - 2\delta)} h.$$

In order to estimate the second and third parts, we consider the following two cases:

a) Let $t \in [t_0, t_f - h]$. Since

$$|R_h(T)(t)| \leq \frac{T(t_f) - T_0}{h} \leq \frac{\widetilde{T}(t_f) + \delta - T_0}{h} = \frac{T^e - T_0 - \mu + 2\delta}{h},$$

then

$$\left| R_h(T)(t) \frac{T(t) - \widetilde{T}(t)}{(T^e - T(t))(T^e - \widetilde{T}(t))} \right| \leq \frac{T^e - T_0 - \mu + 2\delta}{h} \frac{\delta}{(\mu - 2\delta)(\mu - 3\delta)} = \frac{\delta}{(\mu - 3\delta)h} \left(\frac{T^e - T_0}{\mu - 2\delta} - 1 \right).$$

For the third part, since

$$\left| R_h(T)(t) - R_h(\widetilde{T})(t) \right| \leq \frac{1}{h} \left(\left| T(t+h) - \widetilde{T}(t+h) \right| + \left| T(t) - \widetilde{T}(t) \right| \right) \leq \frac{2\delta}{h},$$

then

$$\left| \frac{R_h(T)(t) - R_h(\widetilde{T})(t)}{T^e - \widetilde{T}(t)} \right| \leq \frac{1}{h} \frac{2\delta}{\mu - 3\delta}.$$

b) Let $t \in [t_f - h, t_f]$. Since

$$|R_h(T)(t)| \le \frac{2(T(t_f)-T_0)}{h} + \frac{T(t_f)-T_0}{h} \le \frac{3(\widetilde{T}(t_f)+\delta-T_0)}{h} = \frac{3(T^e-T_0-\mu+2\delta)}{h}$$

then

$$\left| R_h(T)(t)\, \frac{T(t)-\widetilde{T}(t)}{(T^e-T(t))(T^e-\widetilde{T}(t))} \right| \le \frac{3(T^e-T_0-\mu+2\delta)}{h}\, \frac{\delta}{(\mu-2\delta)(\mu-3\delta)} = \frac{3\delta}{(\mu-3\delta)h}\left(\frac{T^e-T_0}{\mu-2\delta}-1\right).$$

Finally, since

$$\left|R_h(T)(t) - R_h(\widetilde{T})(t)\right| \le \frac{1}{h}\left(\left|T(t_f)-\widetilde{T}(t_f)\right| + 2\left|T(t_f-h)-\widetilde{T}(t_f-h)\right| + \left|T(t_f-2h)-\widetilde{T}(t_f-2h)\right| + \left|T(t)-\widetilde{T}(t)\right| + \left|T(t-h)-\widetilde{T}(t-h)\right|\right) \le \frac{6\delta}{h},$$

we obtain

$$\left|\frac{R_h(T)(t)-R_h(\widetilde{T})(t)}{T^e-\widetilde{T}(t)}\right| \le \frac{1}{h}\,\frac{6\delta}{\mu-3\delta}.$$

All this leads to

$$||u-\widetilde{u}_h|| \le \frac{7M_2}{2(\mu-2\delta)}\,h + \frac{3\delta}{(\mu-3\delta)h}\left(1+\frac{T^e-T_0}{\mu-2\delta}\right) = \frac{1}{\mu-2\delta}\left(\frac{7M_2}{2}\,h + \frac{3\delta}{h}\,\frac{T^e-T_0+\mu-2\delta}{\mu-3\delta}\right). \square$$

The following result determines how to optimize the above estimate by choosing a suitable step time h:

Proposition 6 *Under the assumptions of Proposition 5, the minimum value for the right hand side in* (10) *is obtained for*

$$h^* = \sqrt{\frac{6(T^e-T_0+\mu-2\delta)}{7(\mu-3\delta)M_2}\,\delta}. \tag{12}$$

In this case, estimate (10) *becomes*

$$||u-\widetilde{u}_{h^*}|| \le \frac{1}{\mu-2\delta}\sqrt{\frac{42M_2(T^e-T_0+\mu-2\delta)}{\mu-3\delta}\,\delta}.$$

PROOF. It suffices to note that function

$$g(x) = c\left(ax+\frac{b}{x}\right),\ x>0$$

with $a, b, c > 0$, attains his minimum value at point

$$x_{\min} = \sqrt{\frac{b}{a}}$$

and $g(x_{\min}) = 2c\sqrt{ab}$, taking

$$a = \frac{7M_2}{2},\ b = 3\delta\left(\frac{T^e-T_0+\mu-2\delta}{\mu-3\delta}\right)$$

and

$$c = \frac{1}{\mu-2\delta}. \square$$

¿From Proposition 6, choosing h^* as in (12), taking n as the entire part of $\frac{t_f-t_0}{h^*}$, denoting $t_k = t_0 + kh^*$, $\widetilde{T}_k = \widetilde{T}(t_k)$ and

$$\widetilde{H}_k = \widetilde{u}_{h^*}(t_k) = \frac{R_{h^*}(\widetilde{T})(t_k)}{T^e-\widetilde{T}_k}$$

for $k = 0, 1, \dots, n$, we obtain the main result of this section:

Theorem 7 *If* $H \in \mathcal{C}^1([T_0, T^e])$ *and* $\widetilde{T} \in \mathcal{C}([t_0, t_f])$ *satisfies* (9) *with* $0 < \delta < \frac{\mu}{3}$, *then*

$$\max_{k=0,1,\dots,n}\left|H(\widetilde{T}_k)-\widetilde{H}_k\right| \le \delta\,||H'||_{\mathcal{C}([T_0,T^e])} + \frac{1}{\mu-2\delta}\sqrt{\frac{42M_2(T^e-T_0+\mu-2\delta)}{\mu-3\delta}\,\delta} = O(\sqrt{\delta}).$$

PROOF. Triangular inequality provides

$$\left|H(\widetilde{T}_k)-\widetilde{H}_k\right| \le \left|H(\widetilde{T}_k)-H(T_k)\right| + \left|H(T_k)-\widetilde{H}_k\right| \le ||H'||_{\mathcal{C}([T_0,T^e])}\left|\left|T-\widetilde{T}\right|\right| + ||u-\widetilde{u}_{h^*}||.$$

Now, the result follows from Proposition 6. $\square$

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Remark 8 We point out that, by using (11), we can change $||H'||_{\mathcal{C}([T_0,T^e])}$ by $||H'||_{\mathcal{C}([T_0,T^e-\mu+3\delta])}$ in Theorem 7, which provides a slightly better estimation. □

3.3 Identifying from a finite number of approximated values of the temperature.

We assume that the interpolation method used is such that the error δ between T and $\widetilde{T}$, and the measurement error $\widehat{\delta}$, are of the same order, i.e., $\delta = C\widehat{\delta}$.

For example, if $\widetilde{T}$ is the piecewise linear interpolation of measurements $\{\widehat{T}_0, \widehat{T}_1, \ldots, \widehat{T}_p\}$ and we denote T_{int} the piecewise linear interpolation of values of T at points τ_k, the monotonicity of T provides

$$\begin{aligned}\left|\left|T-\widetilde{T}\right|\right| &\le ||T-T_{\text{int}}|| + \left|\left|T_{\text{int}}-\widetilde{T}\right|\right| \\ &\le \max_{1\le k\le p} |T(\tau_k)-T(\tau_{k-1})| + \widehat{\delta} \\ &\le \max_{1\le k\le p}\left(|\widetilde{T}(\tau_k)-\widetilde{T}(\tau_{k-1})| + 2\widehat{\delta}\right) + \widehat{\delta} \\ &= \max_{1\le k\le p} |\widehat{T}_k-\widehat{T}_{k-1}| + 3\widehat{\delta}.\end{aligned}$$

Therefore, when the interpolation considered is the piecewise linear interpolation, if the difference between consecutive measurements is of order $\widehat{\delta}$, then δ and $\widehat{\delta}$ are of the same order. The number of measurements will be increased if needed.

3.3.1 Algorithm for determining H

The input data are: $\{\widehat{T}_k\}_{k=0}^p$, $\widehat{\delta} > 0$ and the admissible threshold $\mu > 0$. First of all, we construct a function $\widetilde{T}(t)$ interpolating $\{\widehat{T}_k\}_{k=0}^p$. Then, we estimate the error $\delta > 0$ due to the interpolation. Next, t_f is fixed by using (6).

The algorithm is based on an iterative process beginning from an initial guest Λ_2 for M_2. From this value, the time step is calculated by

$$h = \sqrt{\frac{6(T^e - T_0 + \mu - 2\delta)}{7(\mu - 3\delta)\Lambda_2}\,\delta}, \tag{13}$$

according to (12). With this election of h, the corresponding values

$$\widetilde{H}_k = \widetilde{u}_h(t_k) = \frac{R_h(\widetilde{T})(t_k)}{T^e - \widetilde{T}_k} \tag{14}$$

are obtained. Approximating T'' by (15) in nodes t_k and taking the absolute maximum, a new Λ_2 (and a new h) is obtained, and so on. This iterative process finishes when h stabilizes. Since

$$T'' = \left(H'(T)(T^e - T) - H(T)\right)H(T)(T^e - T),$$

we approximate $T''(t_k)$ as:

$$\begin{cases}\left(\dfrac{\widetilde{H}_{k+1}-\widetilde{H}_k}{\widetilde{T}_{k+1}-\widetilde{T}_k}(T^e-\widetilde{T}_k)-\widetilde{H}_k\right)\widetilde{H}_k(T^e-\widetilde{T}_k), \\ \qquad\qquad\qquad k=0,1,\ldots,n-1 \\ \left(\dfrac{\widetilde{H}_n-\widetilde{H}_{n-1}}{\widetilde{T}_n-\widetilde{T}_{n-1}}(T^e-\widetilde{T}_n)-\widetilde{H}_n\right)\widetilde{H}_n(T^e-\widetilde{T}_n).\end{cases} \tag{15}$$

Algorithm

DATA	$\{\widehat{T}_k\}_{k=0}^p$: measurements of $T(t_k)$. $\widehat{\delta} > 0$: bound for measurement errors. $\mu > 0$: threshold. ε: stopping test precision. Λ_2: initial guest for M_2.
Step 1:	Determine $\widetilde{T}$ and δ according to $\widehat{\delta}$.
Step 2:	Fix t_f from (6) adapting μ if needed.
Step 3:	Initialize h using (13).
Step 4:	While the relative error in h is bigger than ε: a) Compute $\widetilde{T}_k$. b) Calculate $\widetilde{H}_k$ from (14). c) Set Λ_2 as the maximum of the absolute value of (15). d) Set h using (13).

4 Functional framework of the inverse problem. Classical theory.

Let us suppose the fourth scenario (the more general one) exposed in Section 2. Once t_f is determined, we consider the initial value problem (1) over the interval $[t_0, t_f]$. By denoting $u(t) = H(T(t))$, $t \in [t_0, t_f]$, we have that

$$\int_{t_0}^t u(s)\,ds = \int_{t_0}^t \frac{T'(s)}{T^e - T(s)}\,ds = -\ln\left(\frac{T^e - T(t)}{T^e - T_0}\right).$$

Thus, for suitable functional spaces X and Y, by defining the operator $K : X \to Y$ as

$$Kx(t) = \int_{t_0}^t x(s)\,ds,$$

our problem can be written as $Ku = y$, where

$$y(t) = -\ln\left(\frac{T^e - T(t)}{T^e - T_0}\right),\ t \in [t_0, t_f]. \tag{16}$$

Note that function y is well defined and it is positive. In order to apply the Classical Regularization Theory in Hilbert spaces (see, e.g., [4], [16], [17]), we choose $X = Y = L^2(t_0, t_f)$. We remind that

$$L^2(t_0, t_f) = \left\{ f : (a, b) \to \mathbb{R} : \int_{t_0}^{t_f} (f(s))^2 ds < \infty \right\}.$$

We also consider

$$H^1(t_0, t_f) = \left\{ f \in L^2(t_0, t_f) : \ f' \in L^2(t_0, t_f) \right\}.$$

Next result shows some properties of operator K:

Proposition 9 *$K : L^2(t_0, t_f) \to L^2(t_0, t_f)$ is a linear and compact operator. Moreover:*

a) $Kx \in H^1(t_0, t_f)$ and $(Kx)' = x$ in $L^2(t_0, t_f)$ for every $x \in L^2(t_0, t_f)$.

b) K is an injective operator and has dense rank in $L^2(t_0, t_f)$.

c) The adjoint operator $K^* : L^2(t_0, t_f) \to L^2(t_0, t_f)$ is given by $K^*y(t) = \int_t^{t_f} y(s)\, ds$.

PROOF. Obviously, K is a linear operator. Compactness follows from Theorem A.33 (pag. 230) of [17] (see also [3]) taking the function $k(t, s)$ appearing in that Theorem as, for every $t \in (t_0, t_f)$, the characteristic function of interval (t_0, t), *i.e.*,

$$k(t, s) = \begin{cases} 1, & t_0 < s < t \\ 0, & \text{otherwise.} \end{cases}$$

Let us prove the rest of properties:

a) For all test function $\varphi \in \mathcal{C}_c^\infty(t_0, t_f)$, the space of infinitely many differentiable functions with compact support, we have

$$\begin{aligned} \langle (Kx)', \varphi \rangle &= -\langle Kx, \varphi' \rangle \\ &= -\int_{t_0}^{t_f} \left(\int_{t_0}^{t} x(s)\, ds \right) \varphi'(t)\, dt \\ &= \int_{t_0}^{t_f} x(t)\varphi(t)\, dt = \langle x, \varphi \rangle. \end{aligned}$$

Now, if $\varphi \in L^2(t_0, t_f)$, since $\mathcal{C}_c^\infty(t_0, t_f)$ is dense in $L^2(t_0, t_f)$ (see, for instance, [3]), there exists a sequence $\{\varphi_n\}_{n=1}^\infty \subset \mathcal{C}_c^\infty(t_0, t_f)$ such that

$$\varphi_n \to \varphi \ \text{ in } \ L^2(t_0, t_f).$$

Then,

$$\langle (Kx)', \varphi_n \rangle = \langle x, \varphi_n \rangle$$

and, passing to the limit ($n \to \infty$) we have that

$$\langle (Kx)', \varphi \rangle = \langle x, \varphi \rangle$$

and, therefore, $(Kx)' = x$ in $L^2(t_0, t_f)$.

b) Injectivity of K follows from

$$Kx = 0 \ \Rightarrow \ (Kx)' = 0 \ \Rightarrow \ x = 0.$$

On the other hand, note that

$$R(K) = \{v \in H^1(t_0, t_f) : \ v(t_0) = 0\}.$$

Since $R(K) \supset \mathcal{C}_c^\infty(t_0, t_f)$ and $\mathcal{C}_c^\infty(t_0, t_f)$ is dense in $L^2(t_0, t_f)$, the operator K has dense rank in $L^2(t_0, t_f)$.

c) For $u, y \in L^2(t_0, t_f)$ given and denoting

$$Y(t) = \int_{t_f}^{t} y(s)\, ds,$$

we obtain

$$\begin{aligned} \langle Ku, y \rangle &= \int_{t_0}^{t_f} \left(\int_{t_0}^{t} u(s) ds \right) y(t) dt \\ &= \left(\int_{t_0}^{t} u(s) ds \right) Y(t) \Big|_{t_0}^{t_f} - \int_{t_0}^{t} u(t) Y(t) dt \\ &= \int_{t_0}^{t} u(t)(-Y(t)) dt = \langle u, -Y \rangle. \end{aligned}$$

That is,

$$K^*y(t) = -Y(t) = \int_t^{t_f} y(s)\, ds. \ \square$$

In our problem we have measurements $\widehat{T}_k$ verifying $|T(\tau_k) - \widehat{T}_k| < \widehat{\delta}$, and an interpolation function $\widetilde{T}$ such that $\left\| T - \widetilde{T} \right\|_{\mathcal{C}([\tau_0, \tau_p])} < \delta$. This provides a right hand term

$$y_\delta(t) = -\ln \left(\frac{T^e - \widetilde{T}(t)}{T^e - T_0} \right) \tag{17}$$

and the approximate problem $Ku_\delta = y_\delta$. Next proposition estimates the error between y_δ and y in terms of error between $\widetilde{T}$ and T (given by δ).

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Proposition 10 *Let $y(t)$ and $y_\delta(t)$ given by* (16) *and* (17)*, respectively. By denoting*

$$e(\delta) = \frac{\sqrt{t_f - t_0}}{\mu - 3\delta}\,\delta, \tag{18}$$

the estimate

$$||y - y_\delta||_{L^2(t_0,t_f)} \le e(\delta)$$

holds.

PROOF. A first order Taylor expansion of function $s \mapsto \ln(T^e - s)$ about $s = T(t)$, provides

$$|y(t) - y_\delta(t)| = \left|\ln(T^e - \widetilde{T}(t)) - \ln(T^e - T(t))\right| = \left|\frac{T(t) - \widetilde{T}(t)}{T^e - T_\theta}\right|,$$

where T_θ is a value between $T(t)$ and $\widetilde{T}(t)$ which can be written as

$$T_\theta = \theta T(t) + (1-\theta)\widetilde{T}(t)$$

for some $0 < \theta < 1$. Estimates (11) imply

$$\begin{aligned} T^e - T_\theta &= \theta(T^e - T(t)) + (1-\theta)(T^e - \widetilde{T}(t)) \\ &\ge \theta(\mu - 2\delta) + (1-\theta)(\mu - 3\delta) \\ &= \mu - (3-\theta)\delta \\ &\ge \mu - 3\delta. \end{aligned}$$

Thus,

$$|y(t) - y_\delta(t)| \le \frac{|T(t) - \widetilde{T}(t)|}{\mu - 3\delta} \le \frac{\delta}{\mu - 3\delta},$$

which allows to conclude the result easily. □

4.1 Tikhonov's method

The *Tikhonov strategy* to solve $Ku_\delta = y_\delta$, (see, for instance, [22], [23]) consists of minimizing the Tikhonov functional

$$J_\alpha(x) = ||Kx - y_\delta||^2_{L^2(t_0,t_f)} + \alpha\,||x||^2_{L^2(t_0,t_f)}, \tag{19}$$

where $\alpha = \alpha(\delta) > 0$. Theorem 2.11 of [17], guarantees uniqueness of the minimum $u_{\alpha,\delta}$ of (19), which is also the unique solution of the *normal equation*

$$(\alpha + K^*K)x = K^*y_\delta. \tag{20}$$

The regularization strategy is given for the linear operators $R_\alpha : L^2(t_0, t_f) \to L^2(t_0, t_f)$ defined by

$$R_\alpha y = (\alpha + K^*K)^{-1}K^*y.$$

For $\alpha = 0$ this becomes the normal equation associated to operator K. Since minimizing operator J_0 is an ill–posed problem (see [17], Lemma 2.1), a penalty term is added.

Proposition 11 *The solution $u_{\alpha,\delta}$ of* (20) *is the solution of the boundary problem*

$$\begin{cases} -\alpha x''(t) + x(t) = y'_\delta(t),\ t \in (t_0, t_f) \\ x'(t_0) = 0,\ x(t_f) = 0. \end{cases} \tag{21}$$

Moreover, denoting $\gamma(r) = \dfrac{t_f - r}{\sqrt{\alpha}}$, the solution is

$$u_{\alpha,\delta}(t) = \frac{1}{\sqrt{\alpha}}\left(\varphi_{\alpha,\delta}(t)\cosh\gamma(t) + \psi_{\alpha,\delta}(t)\sinh\gamma(t)\right),$$

where

$$\varphi_{\alpha,\delta}(t) = \int_t^{t_f} y'_\delta(s)\sinh\gamma(s)\,ds$$

and

$$\psi_{\alpha,\delta}(t) = \int_{t_0}^t y'_\delta(s)\cosh\gamma(s)\,ds - \tanh\gamma(t_0)\varphi_{\alpha,\delta}(t_0).$$

PROOF. Proposition 9 allows to write equation (20) as

$$\alpha x(t) + \int_t^{t_f}\left(\int_{t_0}^s x(\tau)\,d\tau\right)ds = \int_t^{t_f} y_\delta(s)ds.$$

Thus $x(t_f) = 0$. Further, since $y_\delta \in R(K)$, we have $y_\delta \in H^1(t_0, t_f)$ and $y_\delta(t_0) = 0$. Therefore, by differentiating the above expression, we obtain

$$\alpha x'(t) - \int_{t_0}^t x(s)\,ds = -y_\delta(t),$$

and, in particular, $x'(t_0) = -y_\delta(t_0) = 0$. By differentiating again, we get to

$$\alpha x''(t) - x(t) = -y'_\delta(t).$$

Finally, standard calculations for solving the boundary value problem (21) lead to the above expression for $u_{\alpha,\delta}$. □

Remark 12 Theorem 2.12 of [17] states that if one chooses $\alpha = \alpha(\delta)$ such that $\lim_{\delta\to 0}\alpha(\delta) = 0$ and $\lim_{\delta\to 0}\dfrac{\delta^2}{\alpha(\delta)} = 0$, then the Tikhonov regularization strategy is admissible, i.e.,

$$\lim_{\delta\to 0}\left|\left|u_{\alpha(\delta),\delta} - u\right|\right|_{L^2(t_0,t_f)} = 0,$$

since $||y - y_\delta||_{L^2(t_0,t_f)} \le e(\delta)$ (see Proposition 10). □

4.1.1 Morozov's discrepancy principle.

This principle (see [19]) provides a way to choose the parameter $\alpha = \alpha(\delta)$ for the Tikhonov regularization strategy: it is chosen so that the solution $u_{\alpha(\delta),\delta}$ of (20) satisfies

$$\left|\left|Ku_{\alpha(\delta),\delta} - y_\delta\right|\right|_{L^2(t_0,t_f)} = e(\delta), \qquad (22)$$

supposing that

$$||y - y_\delta||_{L^2(t_0,t_f)} \leq e(\delta) < ||y_\delta||_{L^2(t_0,t_f)} .$$

Theorem 2.17 of [17] assures that the regularization strategy associated to this choice of $\alpha(\delta)$ is admissible.

4.2 Landweber's iterative method.

Landweber's iterative method is defined as

$$\begin{cases} x_0 = 0 \\ x_m = (I - aK^*K)x_{m-1} + aK^*y, \; m = 1, 2, \ldots, \end{cases}$$

where $a > 0$. Using Theorem 2.19 of [17], we choose a such that $0 < a < \frac{1}{||K||^2}$ and we consider the stopping test

$$||Kx_m - y_\delta||^2_{L^2(t_0,t_f)} \leq r(e(\delta))^2$$

for some $r > 0$ satisfying

$$||y_\delta|| \geq re(\delta), \; \delta \in (0, \delta_0).$$

4.3 Comparison between the methods.

We present two test problems in order to compare between the methods considered above. For each example, we start from a known function H and we solve the corresponding direct problem in order to compute the temperature T. Then, we evaluate T in some time instants and, finally, we introduce some perturbations (measurement errors) of these values.

With these new temperature values we use linear piecewise interpolation to get a function $\widetilde{T}$ providing the temperature with simulated measurement errors. Every definite integral appearing in the computations is approximated by means of the trapezoidal rule by using only points at which measurements of the temperature are available. Thus, these calculations are "independent" of the interpolation method used to compute $\widetilde{T}$.

For the Tikhonov method we compute $u_{\alpha,\delta}$ as stated in Proposition 11 for $\alpha(\delta) = \delta^\gamma$, by using discrete values of $0 < \gamma < 2$ with step size 0.025.

For the Morozov discrepancy principle we approximate the solution of (22) by applying the secant method to the function

$$F(\alpha) = ||Ku_{\alpha,\delta} - y_\delta||^2_{L^2(t_0,t_f)} - (e(\delta))^2,$$

where $u_{\alpha,\delta}$ is stated in Proposition 11 and y_δ and $e(\delta)$ are given in (17) and (18), respectively.

Finally, for the Landweber iterative method we consider $a = 10$ and the stopping criterium of Section 4.2 for $r = 1$.

In the first example we consider a constant function H whereas the second one deals with a smooth but strongly oscillating function H.

4.3.1 Example 1

Consider the test problem

$$\begin{cases} T'(t) = 4\big(1 - T(t)\big), \; t \in (0, 0.23) \\ T(0) = 0. \end{cases}$$

We take $t_f = 0.23$ corresponding to the threshold $\mu = 0.4$. The goal is to identify $H(s) \equiv 4$ in $(0, T(t_f)) \simeq (0, 0.6)$. We consider a uniform partition of $(0, 0.23)$ with step $h = 0.01$. At these instants, approximate measurements of temperature with error $\delta = 0.001$ are supposed to be known.

First, we consider the algorithm described in Section 3.3 and obtain the results shown in Figure 1 (the error is computed in the L^∞–norm).

When applying the Tikhonov method as explained above, the exponent with lower error in the L^2–norm is attained when $\gamma = 1.25$, which corresponds to $\alpha(\delta) = 1.778 \times 10^{-4}$. For this value of α, Figure 2 shows the computed approximations for H and T.

Figure 3 shows the results obtained with the Morozov discrepancy principle by applying the secant method as explained above. The value that has been obtained is $\alpha = 0.5 \times 10^{-3}$.

Finally, the Landweber iterative method, after 2539 iterations, provides a residual L^2–norm of 1.181×10^{-3} and the results are shown in Figure 4.

4.3.2 Example 2

Consider now the test problem

$$\begin{cases} T'(t) = \big(2 + \sin(14T(t))\big)\big(1 - T(t)\big), \; t \in (0, 0.48) \\ T(0) = 0 \end{cases}$$

We approximate its solution trough an adaptive Runge–Kutta method.

We take $t_f = 0.48$ corresponding to the same threshold $\mu = 0.4$ as in the example above. In this case, the goal is to identify the function $H(s) = 2 + \sin(14s)$ in $(0, T(t_f)) \simeq (0, 0.6)$. We consider again a uniform partition of $(0, t_f)$ with step $h = 0.01$ and error $\delta = 0.001$.

Following the same schedule, we begin showing in Figure 5 the results obtained by applying the algorithm described in Section 3.3.

Figure 6 shows the computed approximations for H and T with the Tikhonov method when $\alpha(\delta) = 7.0795 \times 10^{-4}$. This value corresponds to exponent $\gamma = 1.05$, which minimizes the corresponding L^2–norm error for $\alpha(\delta) = \delta^\gamma$.

By applying the Morozov discrepancy principle we obtain $\alpha = 0.001$ and the corresponding approximations are shown in Figure 7.

Finally, Landweber's iterative method gives, after 620 iterations, the functions H and T in Figure 8. In this case, the residual L^2–norm is 1.725×10^{-3}.

5 Conclusions.

We have developed a numerical algorithm, well adapted to the problem considered, which improves those based on the Classical Theory, from a qualitative and quantitative point of view.

On the one hand, this algorithm is able to capture the qualitative properties of the solution, correcting (see Figures 1 and 5) the bad behavior obtained with classical methods for function H near final instant t_f (see Figures 2, 3, 4, 6, 7 and 8), which is due to the $x(t_f) = 0$ condition needed with the square integrable approximation used in Classical Theory.

On the other hand, from a quantitative point of view, the solution $\widetilde{H}$ provided by our algorithm approximates function H better than those corresponding to the classical approaches, when considering the L^∞–norm and L^2–norm of $H - \widetilde{H}$.

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References:

[1] P. Bárta. Real–time identification method of a heat transfer coefficient. *Proceedings of the 5th Australasian Congress on Applied Mechanics (ACAM 2007)*, ISBN 0 8582 5862 5, 259–264.

[2] A. Buikis and Š. Guseinovs. Inverse Initial Heat Flux and Relaxation Time Finding Problems for Hyperbolic Heat Equation. *WSEAS Transactions on Mathematics*, Issue 1, Vol. 6, 2007, 43–48.

[3] H. Brezis: *Analyse Fonctionelle. Théorie et applications*. Masson, Paris, 1983.

[4] A.M. Denisov: *Elements of the Theory of Inverse Problems*. VSP. Utrecht, The Netherlands, 1999.

[5] A. Fraguela, C. Gamio and D. Hinestroza: The inverse problem of Electrical capacitance Tomography and its application to Gas–Oil two phase flow imaging. *WSEAS Transactions on Systems.*, ISSUE2–V1, 2002, 130-137.

[6] A. Fraguela, J. Oliveros, L. Cervantes, M. Morin and S. Gómez: Un algoritmo no iterativo para la tomografía de capacitancia eléctrica. *Revista Mexicana de Física*, 51–N3, 2005, 236–242.

[7] A. Fraguela, J. Oliveros, M. Morin and L. Cervantes: Inverse Electroencephalography for cortical sources. *Applied Numerical Mathematics*, V55–I2, 2004, 191–203.

[8] B.A. Fricke and B.R. Becker. Calculation of heat transfer coefficients for foods. *Int. Comm. Heat Mass Transfer*, Vol. 29, No. 6, 2002, 731–740.

[9] S.G. Giliazov and N.L. Goldman: *Regularization of Ill Posed problems by iteration methods*. Kluwer Acad. Pub., 2000.

[10] S. Gómez, M. Ono, C. Gamio and A. Fraguela: Reconstruction of Capacitance Tomography Images of Simulated Two–Phase Flow Regimes. *Applied Numerical Mathematics*, 46, 2003, 197–208.

[11] A. Grebennikov. Mathematical and numerical modeling of scanning and penetrating Computer Tomography. *WSEAS Transactions on Computers*, Issue3, Vol. 2, July 2003, 548–553.

[12] B. Guignon, Á.M. Ramos, J.A. Infante, J.M. Díaz and P.D. Sanz: Determining thermal parameters in the cooling of a small–scale high pressure freezing vessel. *International Journal of Refrigeration*, Vol. 29, 2006, 1152–1159. doi:10.1016/j.ijrefrig.2006.01.007

[13] Š. Guseinovs. On one approach for reducing of 3–D inverse heat conductivity problem in a multilayered domain to 1–D inverse problem and the uniqueness theorem. *WSEAS Transactions on Mathematics*, Issue 5, Vol. 5, 2006, 292–499.

[14] M. Hanke and C. Hansen: Regularization method for large scale problems. *Surv. Math. Ind.*, 3, 1993, 253–315.

[15] C. Hansen: Analysis of discrete ill posed problems by means of the *L*–curve. *SIAM Rev.*, 34, 1992, 561–580.

[16] V. Isakov: *Inverse Problems for partial differential equations*. Springer Verlag, 1998.

[17] A. Kirsch: *An Introduction to the Mathematical Theory of Inverse Problems*. Applied Mathematical Sciences, 120. Springer. 1996.

[18] J. D. Lee, I. Tanabe and K. Takada. Identification of Heat Transfer Coefficient on Machine Tool Surface by Inverse Analysis. *JSME international journal. Series C, Mechanical systems, machine elements and manufacturing*, Vol.42, No. 4, 1999, 1056–1060.

[19] L. Morozov: *Regularization methods for Ill Posed Problems*. CRC Press, Boca Raton, 1993.

[20] L. Otero, Á.M. Ramos, C. de Elvira and P.D. Sanz: A Model to Design High-Pressure Processes Towards an Uniform Temperature Distribution. *Journal of Food Engineering*, Vol 78, 2007, 1463–1370. doi:10.1016/j.jfoodeng.2006.01.020

[21] S. Simal, Á. Berna, A. Mulet and C Rossell. A method for the calculation of the heat transfer coefficient in potato drying. *Journal of the Science of Food and Agriculture*, Volume 63, Issue 3, 1993, 365–367.

[22] A.N. Tikhonov and V.Y. Arsenin: *Solutions of ill posed problems*. V.H. Winston & Sons, Washington D.C., 1977.

[23] A.N. Tikhonov, A.V. Goncharsky, V.V. Stepanov and A.G. Yagola: *Numerical methods for the solution of Ill Posed Problems*. Kluwer, Dordrechdt, 1995.

[24] L. Zhang, N, Zhang, F. Zhao and Y. Chen. A genetic–algorithm–based experimental technique for determining heat transfer coefficient of exterior wall surface. *Applied Thermal Engineering*, Volume 24, Issues 2–3, February 2004, 339–349.

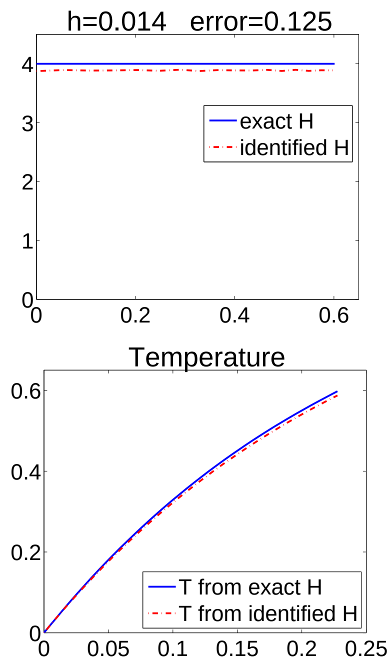

Figure 1: Results obtained with the algorithm developed in Section 3.3.1.

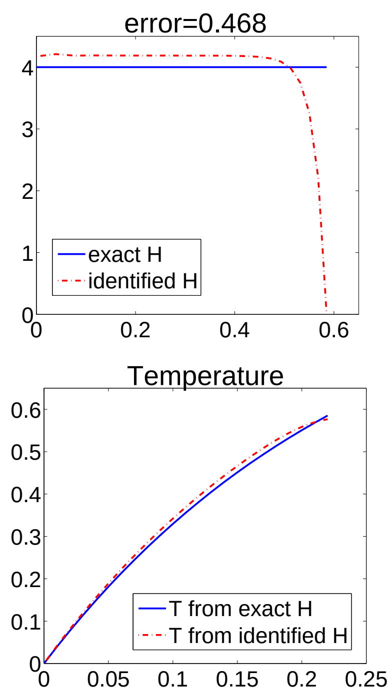

Figure 2: Tikhonov's method with the best exponent.

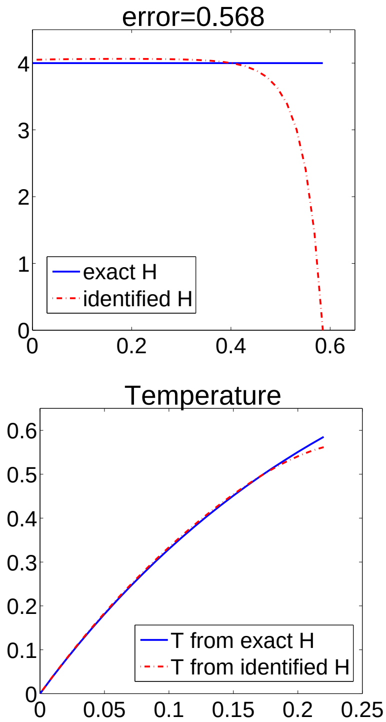

Figure 3: Morozov's discrepancy principle.

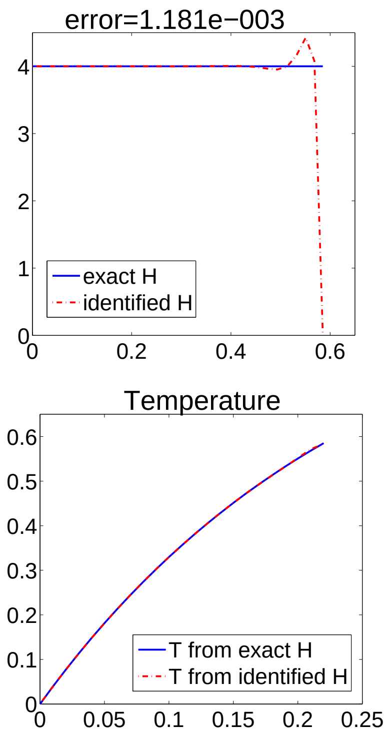

Figure 4: Landweber's iterative method.

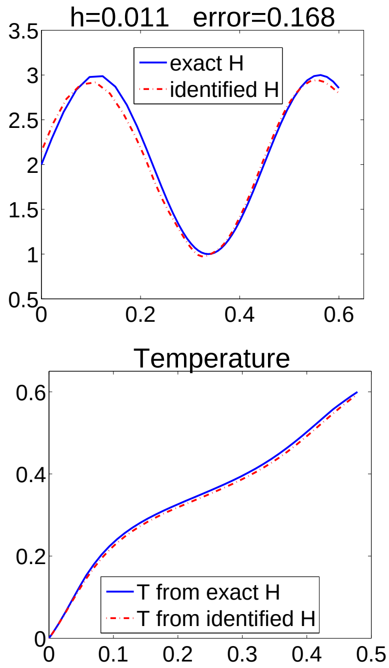

Figure 5: Results obtained with the algorithm developed in Section 3.3.1.

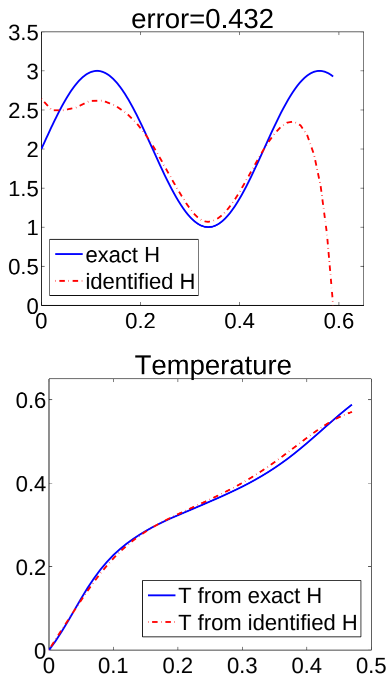

Figure 6: Tikhonov's method with the best exponent.

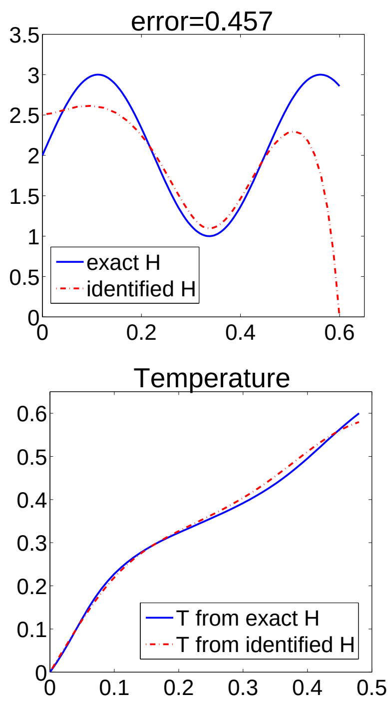

Figure 7: Morozov's discrepancy principle.

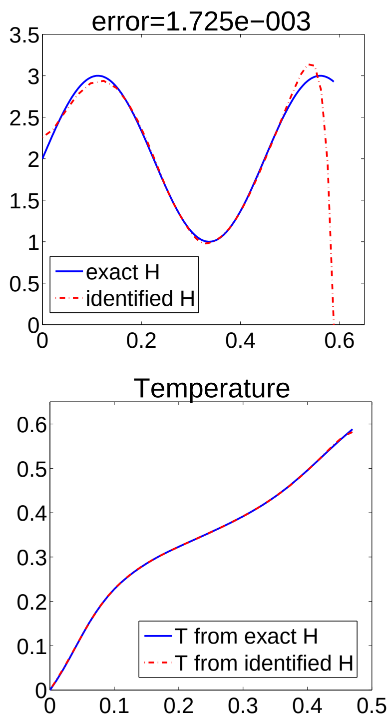

Figure 8: Landweber's iterative method.