The stability of collocation methods for approximate solution of singular integro- differential equations.

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Abstract: In this article we obtained that the collocation methods are stable in according with the small perturbations of coefficients, kernels and right part of studied equations. We proved that the condition number of the approximate operator exists and bounded. The condition number of collocation methods is appropriated with condition number for exact singular integro- differential equations.

key–words: condition number, stability, collocation methods, singular integro- differential equations.

1 Introduction

Singular integral equations (SIE) and singular integro-differential equations (SIDE) have been used to model many physical problems, for example, elasticity theory, aerodynamics etc. [1]-[4].

It is known that the exact solution for SIDE is possible in some particular cases. That is why we are looking for an approximate solution using the direct methods[1] with corresponding theoretical background.

It should be mentioned that using a conformal mapping we can transform an arbitrary smooth closed contour to the unit circle. However, this approach may not simplify the problem due to the following:

- The coefficients, the kernels and right part of the transformed equation may be more complicated.
- The convergence analysis may be more difficult due to the transformation of contour.

The problem for approximate solution of SIDE by collocation methods and mechanical quadrature methods was studied in [5]-[9] . The equations were defined on the unit circle.

The convergence for collocation methods was proved in [22], [17]. The equations were defined on an arbitrary smooth closed contour.

The main results about of the stability of projection methods were obtained in S. G. Mihlin [10], [11], G.M. Vainikko [12] for Hilbert spaces and B.G. Gabdulhaev [13] for Banach spaces.

[1]The collocation methods, mechanical quadrature- methods are direct methods.

The definition of condition number for system of linear algebraic equations was introduced for example in [14],[15], [16] and generalized for operators and operator equations in [13].

The classical continuous function space can not be used because the singular operator of integration is unbounded. That is why we studied the Hölder spaces.

2 Definitions and Notations

In this section we introduce the main definitions from [13],[16], [15]. Let

$$Ax = y,\ (x \in X, y \in Y), \tag{1}$$

be an exact equation and

$$A_n x_n = y_n, \quad (x_n \in X_n, y_n \in Y_n), \tag{2}$$

be an approximate equation.

Let A and A_n be a linear operators which acting from Banach space X to the Banach space Y and from subspace $X_n \subset X$ to the subspace $Y_n \subset Y$.

In practice the approximate solution of equation (2) is solved approximate because of the elements of these equations are not defined exactly. It means that the equation (2) is changed by new one

$$B_n x_n = z_n \quad (x_n \in X_n, z_n \in Y_n), \tag{3}$$

where B_n is linear operator acting from X_n to Y_n and so A_n and B_n, as y_n and z_n are appropriated.

So we study the error

$$\delta_n = ||x_n^{(*)} - x_n^{(1)}||, \tag{4}$$

where $x_n^{(*)}$ and $x_n^{(1)}$ are solutions (if the solutions exist) of equations (2) and (3), respectively.

We introduce the definition of condition number defined by operator.

The value $\eta = \eta(A) = ||A||||A^{-1}||$ is named the condition number of operator A and equation (1).

The operator A and the equation (1) are named well-conditioned if η is small and ill-conditioned in another case.

The following equality was obtained in [16] :

$$\eta(A) = \sup_{x^*}\left\{\sup_{y}\frac{||x^*-x_n^*||}{||x^*||} : \frac{||A(x^*-x_n^*)||}{||y||}\right\}, \quad (5)$$

where x^*, x_n^* are solutions of equations (1) and (2) and y is right part in (1).

3 Preliminaries

We study the stability of collocation methods. We suppose that the operator A in (1) are invertible.

The following theorems holds: [17]

Theorem 1. *Let the following conditions be satisfied*

1) $dimX_n = dimY_n\ (= n < \infty)$ *and* $Y_n = Q_nY$, *where* Q_n *is bounded projector for all* n;

2) *the operators* $A_n : X_n \to Y_n$ *are invertible and* $||A_n^{-1}||_{Y_n\to X_n} \le c_1\ (< \infty)^2$.

3) $||A_n - B_n||_{X_n\to Y_n} = O(\varepsilon_n^{(1)})$;

4) $||y_n - z_n|| = O(\varepsilon_n^{(2)})$; $y_n, z_n \in Y_n$;

5) $\lim\limits_{n\to\infty}\varepsilon_n^{(1)} = \lim\limits_{n\to\infty}||Q_n||\varepsilon_n^{(1)} = \lim\limits_{n\to\infty}\varepsilon_n^{(2)} = 0.$

Then for numbers n *large enough* $(n \ge N_0)$ *the operators* $B_n : X_n \to Y_n$ *are invertible and*

a) $||B_n^{-1}||_{Y_n\to X_n} \le c_2\ (< \infty)$;

b) $\lim\limits_{n\to\infty}\delta_n = 0$, $\delta_n = ||x_n^* - x_n^{(1)}||$, *and* $\delta_n \le c_3||Q_n||_Y\varepsilon_n^{(1)} + c_4\varepsilon_m^{(2)}$;

c) $||x^* - x_n^{(1)}||_X \le ||x^* - x_n^*|| + ||Q_n||_Y O(\varepsilon_n^{(1)}) + O(\varepsilon_n^{(2)})$.

[2] By $c_1, c_2, \dots$ we denote the constants ;

Proof As A_n and $B_n : X_n \to Y_n$ and the condition 2) holds then we have

$$B_n = A_n[I - A_n^{-1}(A_n - B_n)]. \quad (6)$$

Using the conditions 2),3) and 5) we have

$$||A_n^{-1}(A_n - B_n)||_{X_n} \le c_1O(\varepsilon_n^{(1)}) \le q_1 < 1, \quad (7)$$

therefore the operator $I - A_n^{-1}(A_n - B_n)$ is invertible in X_n for $n(\ge N_1)$ and we have that the operator $B_n : X_n \to Y_n$ is invertible:

$$B_n^{-1} = \sum_{j=0}^{\infty}[A_n^{-1}(A_n - B_n)]^jA_n^{-1}.$$

From the last relation, from (6) and from the condition 2) we obtain

$$||B_n^{-1}||_{Y_n\to X_n} \le \frac{c_1}{1-q_1}(= c_2)$$

The condition a) from theorem was received .

We will verify condition b).

We have that

$$B_n^{-1} - A_n^{-1} = \sum_{j=1}^{\infty}[A_n^{-1}(A_n - B_n)]^jA_n^{-1},$$

from (7)

$$||B_n^{-1} - A_n^{-1}|| \le \frac{c_1^2}{1-q_1}O(\varepsilon_n^{(1)}) = O(\varepsilon_n^{(1)}).$$

Then

$$\begin{gathered}||x_n^* - x_n^{(1)}||_{X_n} \le ||A_n^{-1}y_n - B_n^{-1}z_n||_X \\ \le ||(A_n^{-1} - B_n^{-1})y_n||_X + \\ ||B_n^{-1}(y_n - z_n||_X \le \\ O(\varepsilon_n^{(1)})||y_n||_{Y_n} + c_2O(\varepsilon_n^{(2)}).\end{gathered}$$

We know that $y_n = Q_ny$. So we obtain $||y_n||_{Y_n} \le ||Q_n||||y||_Y$. Using (4) we obtain

$$\delta_n = ||x_n^* - x_n^{(1)}||_{X_n} \le O(\varepsilon_n^{(1)})||Q_n||||y||_Y + O(\varepsilon_n^{(2)}).$$

Using the last relation and the theorem condition 5) we obtain condition b).

The condition c) followed by from b) and triangle rule. Theorem 1 was proved.

Theorem 2. *Let the operators* A *and* A_n *be linear and invertible as operators mapping from* X *to* Y *and from* X_n *to* Y_n *respectively where* $dimX_n = dimY_n\ (< \infty)$ *and*

$$||A - A_n||_{X_n\to Y} = O(\varepsilon_n); \quad \lim_{n\to\infty}\varepsilon_n = 0. \quad (8)$$

Then the condition numbers $\eta(A)$ and $\eta(A_n)$ of operators A and A_n exist. The following relations hold:

$$\eta(A_n) \leq c\eta_A, \quad 1 \leq c \leq \frac{1+\varepsilon}{1-\varepsilon}$$

$$\textit{for}\, n \geq N_3(\varepsilon), \tag{9}$$

where ε is an arbitrary positive less then 1 and

$$\lim_{n\to\infty} \eta(A_n) = \eta(A). \tag{10}$$

Proof

We will prove the relations (9) and (10).

$$\eta(A_n) = ||A_n|| ||A_n^{-1}|| = ||A[I - A^{-1}(A - A_n)]||$$

$$||[I - A^{-1}(A - A_n)]^{-1} A_n^{-1}||$$

$$\leq \eta(A)||I - A^{-1}(A - A_n|| || \sum_{j=0}^{\infty} [A^{-1}(A - A_n)]^j ||$$

$$\leq \eta(A) \frac{1 + O(\varepsilon_n)}{1 - O(\varepsilon_n)},$$

We note the function $(1 + \theta)/(1 - \theta)$, $\theta \in (0; 1)$ is increase monotonically and from the condition (8) we have (9).

We will prove that the relation (10).

So we evaluated $\eta(A_n)$ as we would evaluate $\eta(A)$.

$$\eta(A) = ||A|| ||A^{-1}|| = ||A_n[I - A_n^{-1}(A_n - A)]|| \times$$

$$|| \sum_{j=0}^{\infty} [A_n^{-1}(A_n - A)]^j A_n^{-1}|| \leq (\eta(A_n)) \frac{1 + O(\varepsilon_n)}{1 - O(\varepsilon_n)}.$$

From the obtained inequalities

$$\eta(A_n) \leq (\eta(A)) \frac{1 + O(\varepsilon_n)}{1 - O(\varepsilon_n)}.$$

$$\eta(A) \leq (\eta(A_n)) \frac{1 + O(\varepsilon_n)}{1 - O(\varepsilon_n)}.$$

follow

$$\eta(A_n) \leq \eta(A) \left(\frac{2O(\varepsilon_n)}{1 - O(\varepsilon_n)} + 1 \right);$$

$$\eta(A_n) \geq \eta(A) \frac{1 - O(\varepsilon_n)}{1 + O(\varepsilon_n)}$$

$$= \eta(A) \left(1 - \frac{2O(\varepsilon_n)}{1 + O(\varepsilon_n)} \right).$$

We obtain

$$-\eta(A) \frac{O(\varepsilon_n)}{1 + O(\varepsilon_n)} \leq \eta(A_n) - \eta(A) \leq$$

$$\eta(A) \frac{O(\varepsilon_n)}{1 - O(\varepsilon_n)}.$$

From the last inequality and (8) we have (10). Theorem 2 is proved.

So if the exact solutions of equation (1) are well conditioned then from the conditions of Theorem 2. the approximate solutions of (2) are also well conditioned.

4 Numerical schemes of the collocation methods

The numerical schemes of collocation methods for the approximate solution of SIDE are presented in this section. The theorems of the convergence of the approximate solutions to the exact solution are proved in [17], [22].

Let Γ be an arbitrary smooth closed contour bounding a simply-connected region F^+ of complex plane, let $t = 0 \in F^+$, $F^- = C \setminus \{F^+ \cup \Gamma\}$, C is the complex plane.

Let $z = \psi(w)$ be a Riemann function, mapping conformably and unambiguously the outside of unit circle $\Gamma_0 = \{|w| = 1\}$ on the domain F^-, so that $\psi(\infty) = \infty$, $\psi^{(\prime)}(\infty) = 1$. The class of these contours we denote by $\tilde{\Lambda}$.

Let U_n be the Lagrange interpolating polynomial operator constructed on the points $\{t_j\}_{j=0}^{2n}$ (n is a natural number) for any continuous function on Γ

$$(U_n g)(t) = \sum_{j=0}^{2n} g(t_j) \cdot l_j(t), \quad t \in \Gamma,$$

where

$$l_j(t) = \left(\frac{t_j}{t}\right)^n \prod_{(k=0, k\neq j)}^{2n} \frac{t - t_k}{t_j - t_k} \equiv$$

$$\equiv \sum_{k=-n}^{n} \Lambda_k^{(j)} t^k, \quad t \in \Gamma. \tag{11}$$

By $H_\beta(\Gamma)$ we denote Hölder space with the exponent β $(0 < \beta < 1)$ and with norm

$$\|g\|_\beta = \|g\|_C + H(g; \beta),$$

$$H(g, \beta) = \sup_{t' \neq t''} \frac{\left| g(t'') - g(t') \right|}{|t' - t''|^\beta}, t', t'' \in \Gamma.$$

By $H_\beta^{(q)}(\Gamma)$ $q = 0, 1, \ldots,$ we denote the space of r times continuously- differentiable functions. The derivatives of the q-th order for these functions are elements of $H_\beta(\Gamma)$ ($g^{(q)} \in H_\beta(\Gamma)$.)

The norm on $H_\beta^{(q)}(\Gamma)$ is given by formula

$$||g||_{\beta,q} = \sum_{k=0}^{q} ||g^{(k)}||_c + H(g^{(q)}; \beta). \qquad (12)$$

In the complex space $H_\beta(\Gamma)$ we consider the SIDE

$$(Mx \equiv) \sum_{r=0}^{\nu} [\tilde{A}_r(t)x^{(r)}(t) + \tilde{B}_r(t)\frac{1}{\pi i}\int_\Gamma \frac{x^{(r)}(\tau)}{\tau - t}d\tau +$$

$$+\frac{1}{2\pi i}\int_\Gamma K_r(t,\tau)x^{(r)}(\tau)d\tau]$$

$$= f(t), t \in \Gamma, \qquad (13)$$

where $\tilde{A}_r(t), \tilde{B}_r(t)$, $K_r(t,\tau)(r = \overline{0,\nu})$ and $f(t)$ are known functions which belong to $H_\beta(\Gamma)$, $x^{(0)}(t) = x(t)$ is the unknown function from $H_\beta(\Gamma)$, and $x^{(r)}(t) = \dfrac{d^r x}{dt^r}$, $r = \overline{1,\nu}$, ν is a positive integer.

We assume that the function $x^{(\nu)}(t)$ belongs to $H_\beta(\Gamma)$, then

$$x^{(k)}(t) \in H_\beta(\Gamma), \quad k = \overline{0, \nu - 1}.$$

We search for a solution of equation (13) in the class of functions, satisfying the conditions

$$\int_\Gamma x(\tau)\tau^{-k-1}d\tau = 0, \quad k = \overline{0,\nu - 1}. \qquad (14)$$

Equation (13) with conditions (14) will be denoted as "problem (13), (14) "

Using the Riesz operators $P = \frac{1}{2}(I + S)$; $Q = \frac{1}{2}(I - S)$; (I an identity operator and S is a singular operator)

$$S\varphi(t) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)d\tau}{\tau - t}.$$

We rewrite the SIDE (13) in the form:

$$(Mx \equiv) \sum_{s=0}^{\nu} A_s(t)(Px^{(s)})(t) + B_s(t)(Qx^{(s)})(t) +$$

$$+\frac{1}{2\pi i}\int_\Gamma K_s(t,\tau)x^{(s)}(\tau)d\tau = f(t), \quad t \in \Gamma, \quad (15)$$

We search for the approximate solutions of problem (13), (14) in the polynomial form

$$x_n(t) = \sum_{k=0}^{n} \alpha_k^{(n)} t^{k+\nu} + \sum_{k=-n}^{-1} \alpha_k^{(n)} t^k, \quad t \in \Gamma, \ (16)$$

where $\alpha_k^{(n)} = \alpha_k$ ($k = \overline{-n,n}$) are unknowns; we note that the function $x_n(t)$, constructed by formula (16), obviously, satisfies the conditions (14).

Let $R_n(t) = Mx_n(t) - f(t)$ be the residual of SIDE. The collocation methods consist in setting it equal to zero at chosen points t_j, $j = 0, \ldots, 2n$ on Γ and thus obtaining system linear algebraic equations for the unknown coefficients α_k, which will be determined by solving it.

$$R_n(t_j) = 0, j = 0, \ldots, 2n. \qquad (17)$$

Using formulae [17] we have the following formulae:

$$(Px)^{(r)}(t) = (Px^{(r)})(t),$$
$$(Qx)^{(r)}(t) = (Qx^{(r)})(t) \qquad (18)$$

and the relations

$$(t^{k+q})^{(r)} = \frac{(k+q)!}{(k+q-r)!}t^{k+q-r}, \ k = 0, \ldots, n;$$

$$(t^{-k})^{(r)} = (-1)^r \frac{(k+r-1)!}{(k-1)!}t^{-k-r},$$
$$k = 1, \ldots, n; \qquad (19)$$

from (17), we obtain the following system of linear algebraical equations (SLAE) for collocation methods:

$$\sum_{k=-n}^{n}\sum_{r=0}^{\nu}\Big\{\frac{(k+\nu)!}{(k+\nu-r)!} \cdot sign(k)[A_r(t_j)t_j^{k+\nu-r}$$

$$+\frac{1}{2\pi i}\int_\Gamma K_r(t_j,\tau)\cdot\tau^{k+\nu-r}d\tau] +$$

$$+\frac{(k+r-1)!}{(k-1)!}sign(-k)\cdot[(-1)^r B_r(t_j)t_j^{-k-r} +$$

$$+\frac{1}{2\pi i}\int_\Gamma K_r(t_j,\tau)\tau^{-k-r}d\tau]\Big\}\,\alpha_k$$

$$= f(t_j), \quad j = \overline{0,2n}, \qquad (20)$$

where $A_r(t) = \tilde{A}_r(t) + \tilde{B}_r(t)$, $B_r(t) = \tilde{A}_r(t) - \tilde{B}_r(t)$,$r = \overline{0,\nu}$, sign$(k) = 1$,$k \geq 0$, sign$(k) = -1$,$k < 0$.

Let $\overset{o}{H}{}_\beta^{(\nu)}(\Gamma)$ is a subspace of $H_\beta^{(\nu)}(\Gamma)$ space. The elements of $\overset{o}{H}{}_\beta^{(\nu)}(\Gamma)$ are satisfied the condition (14) with the norm as in $H_\beta^{(\nu)}(\Gamma)$.

Theorem 3. *Let $\Gamma \in \tilde{\Lambda}$ and the following conditions be satisfied:*

1. *the functions $\tilde{A}_r(t), \tilde{B}_r(t), K_r(t,\tau)(r = \overline{0,\nu})$ and $f(t)$ belong to the space $H_\alpha^{(r)}(\Gamma)$; $0 < \alpha < 1, r \geq 0$;*
2. *$A_\nu(t) \cdot B_\nu(t) \neq 0, t \in \Gamma$;*
3. *the index of function $t^\nu B_\nu^{-1}(t) A_\nu(t)$ is equal to zero;*
4. *the operator $M : \overset{o}{H}{}_\beta^{(\nu)}(\Gamma) \to H_\beta(\Gamma)$ is linear and invertible;*
5. *$t_j (j = \overline{0,2n})$ form a system of Fejér points [20], [21] on Γ:*
$$t_j = \psi\left[exp\left(\frac{2\pi i}{2n+1}(j-n)\right)\right], \; j = \overline{0,2n};$$
6. *$0 < \beta < \alpha < 1$.*

Then, beginning with $n \geq n_1$, SLAE (20) has the unique solution α_k, $k = \overline{-n,n}$. The approximate solution $x_n(t)$, constructed by formula (16,) converges when $n \to \infty$ in according to the norm of space $H_\beta^{(\nu)}(\Gamma)$ to the exact solution $x(t)$ of the problem (13), (14). The following estimation of convergence speed holds:

$$||x - x_n||_{\beta,\nu} = \frac{d_1 + d_2 \ln n}{n^{r+\alpha-\beta}} H(x^{(r)}, \alpha).$$

The proof of this theorem can be found in [17], [22].

5 Stability of collocation methods. Condition numbers

Theorem 4. *In conditions of Theorem 3 the collocation methods for the approximate solution of SIDE (13) is stable in Hölder spaces from different of small variations in approximate equations.*

Proof of theorem. From the proof of Theorem 3 we obtained that approximate collocation operator A_n starting from the numbers $n \geq n_1$, is invertible as operator mapping from $\overset{o}{X}_n$ to X_n, where X_n and $\overset{o}{X}_n$ are defined in [22], [17]

$$||A_n^{-1}|| = O(1), \quad A_n : \overset{o}{X}_n \to X_n.$$

From proof of Theorem 3 we have that the operators U_n is bounded in H_β and $X_n = U_n H_\beta$. Using the theorem 1. in conditions $A = M$, $X_n = \overset{o}{X}_n$, $Y_n = U_n H_\beta; Q_n = U_n, \varepsilon_n^{(1)} = \varepsilon_n^{(2)} = \dfrac{\ln n}{n^{\alpha-\beta}}$, we have the collocation operator A_n. Theorem 4. is proved

Theorem 5. *Let the conditions of Theorem 3. be satisfied. Then beginning with the number $n \geq N_1$ exist a condition numbers $\eta(A_n)$ for approximate equations of collocation methods and $\eta(A_n) \leq c \cdot \eta(M), 1 \leq c \leq \dfrac{1+\varepsilon}{1-\varepsilon}$, $\varepsilon(>0)$ is an arbitrary small number $n \geq N_1(\varepsilon)$:*

$$\lim_{n\to\infty}(A_n) = \eta(M).$$

From Theorem 3 we have,

$$||A_n - M||_{X_n} = const\frac{\ln n}{n^{\alpha-\beta}}.$$

We obtained the conditions (8) of the Theorem 2. Now Theorem 5 followed from the relations (9) and (10).

6 Stability of exact SIDE

In this section we study the stability of SIDE in Hölder spaces $H_\beta(\Gamma), \Gamma \in \tilde{\Lambda}$.

We consider the SIDE (15) as exact equation.

We suppose that equation (15) has an unique solution. The coefficients, nuclei and right part have small perturbations.

$$||A_s - \hat{A}_s||_c < \varepsilon, \; ||B_s - \hat{B}_s||_c < \varepsilon,$$
$$||f - \hat{f}||_c < \varepsilon, \; ||K_s(t,\tau) - \hat{K}_s(t,\varepsilon)||_c < \varepsilon,$$
$$(t, \tau \in \Gamma, \varepsilon < 1), \quad s = 0, \ldots \nu. \tag{21}$$

The following question appears: if the unique solution $x_\varepsilon(t)$ exists for equation

$$(M_1 x \equiv) \sum_{s=0}^{\nu} \hat{A}_s(t)(Px^{(s)})(t) + \hat{B}_s(t)(Qx^{(s)})(t) +$$
$$+\frac{1}{2\pi i}\int_\Gamma \hat{K}_s(t,\tau)x^{(s)}(\tau)d\tau = \hat{f}(t), \quad t \in \Gamma, \tag{22}$$

if yes we should study the error $\delta_n^{(1)} = ||x^*(t) - x_\varepsilon(t)||$, where $x^*(t)$ is an unique solution for equation (15) and $x_\varepsilon(t)$ is an unique solution for (22)?

Suppose $A_s(t), B_s(t), f(t)$ and $K_s(t,\tau) \in H_\alpha^r(\Gamma)$, $r = 0, 1, 2, \ldots$, $s = 0, \ldots \nu$ (by both variables).

As It was proved in [18], for small ε the coefficients $\hat{A}_s(t), \hat{B}_s(t)$ and $\hat{K}_s(t,\tau)$, $s = 0, \dots \nu$, belong to the $H^r_\alpha(\Gamma)$, $r = 0, 1, 2, \dots$, $s = 0, \dots \nu$.

We estimate the function norm $\Delta M x$,

$$\Delta M x \stackrel{df}{=} (M - M_1)x,$$

in $H_\beta(\Gamma)$ $(0 < \beta < \alpha)$:

$$\Delta M x = \sum_{s=0}^{\nu}\{[A_s(t) - \hat{A}_s(t)](Px^{(s)})(t) + [B_s(t) - \hat{B}_s(t)](Qx^{(s)})(t) + \frac{1}{2\pi i}\int_\Gamma [K_s(t,\tau) - \hat{K}_s(t,\tau)]x^{(s)}(\tau)d\tau\}, t \in \Gamma. \quad (23)$$

It is enough to estimate $||\Delta M x||_c$ and $H(\Delta M x; \beta)$.

a) $|\Delta M x|(t)|$:

$$|\Delta M x)(t)| \le \sum_{s=0}^{\nu}|[A_s(t) - \hat{A}_s(t)](Px^{(s)})(t)| + \sum_{s=0}^{\nu}|[B_s(t) - \hat{B}_s(t)](Px^{(s)})(t)| + \frac{1}{2\pi i}\sum_{s=0}^{\nu}\{\int_\Gamma |K_s(t,\tau) - \hat{K}_s(t,\tau)| |x(\tau)||d\tau|\} = M_1 + M_2 + M_3.$$

Taking into consideration that the operators P, Q is bounded in Hölder spaces, (21) and evident equality $||\cdot||_c \le ||\cdot||_\beta$ for M_1 and M_2, we obtain.

$$M_1 \le \sum_{s=0}^{\nu}\{|A_s(t) - \hat{A}_s(t)||(Px^{(s)})(t)|\} \le \varepsilon\sum_{s=0}^{\nu}||Px^{(s)}||_\beta \le \varepsilon||P||_\beta||x||_{\beta,\nu};$$

$$M_2 \le \sum_{s=0}^{\nu}\{|B_s(t) - \hat{B}_s(t)||(Qx^{(s)})(t)|\} \le \varepsilon\sum_{s=0}^{\nu}||Qx^{(s)}||_\beta \le \varepsilon||Q||_\beta||x||_{\beta,\nu}.$$

Analog, using (21), we obtain $M_3 \le \frac{l}{2\pi}\varepsilon||x||_{c,\nu} \le \frac{l}{2\pi}\varepsilon||x||_{\beta,\nu}$ (where l is length of contour Γ).

So,

$$|(\Delta M x)(t)| \le \varepsilon(||P||_\beta + ||Q||_\beta + \frac{l}{2\pi})||x||_{\beta,\nu}. \quad (24)$$

b) $H(\Delta M x; \beta)$. Let t' and $t'' \in \Gamma$. Then

$$|(\Delta M x)(t') - (\Delta M x)(t'')| \le \sum_{s=0}^{\nu}|[A_s(t') - \hat{A}_s(t')](Px^{(s)})(t') - [A_s(t'') - \hat{A}_s(t'')](Px^{(s)})(t'')| + \sum_{s=0}^{\nu}|[B_s(t') - \hat{B}_s(t')](Qx^{(s)})(t') - [B_s(t'') - \hat{B}_s(t'')](Qx^{(s)})(t'')| + \sum_{s=0}^{\nu}\frac{1}{2\pi i}\int_\Gamma |[K_s(t',\tau) - \hat{K}_s(t',\tau)] - [K_s(t'',\tau) - \hat{K}_s(t'',\tau)]|x^{(s)}(\tau)||d\tau| = M_4 + M_5 + M_6.$$

We estimate M_4 and M_5.

Let $|t' - t''| \ge \varepsilon$. Then from (21) we have

$$M_4 \le \sum_{s=0}^{\nu}\{|[A_s(t') - \hat{A}_s(t')]|(Px^{(s)})(t')| + |A_s(t'') - \hat{A}_s(t'')||(Px^{(s)})(t'')|\} \le \le 2\varepsilon\sum_{s=0}^{\nu}||Px^{(s)}||_\beta \le 2\varepsilon^{1-\beta}\varepsilon^\beta||P||_\beta||x||_{\beta,\nu} \le 2\varepsilon^{1-\beta}||P||_\beta||x||_{\beta,\nu}|t' - t''|^\beta.$$

If $|t' - t''| < \varepsilon$, then

$$M_4 \le \sum_{s=0}^{\nu}|[A_s(t') - \hat{A}_s(t')][(Px^{(s)})(t') - (Px^{(s)})(t'')]| + \sum_{s=0}^{\nu}|(Px^{(s)})(t'')[A_s(t') - \hat{A}_s(t') - \hat{A}(t'') + \hat{A}_s(t'')]| \le \le \varepsilon\sum_{s=0}^{\nu}H(Px^{(s)}; \beta) + \sum_{s=0}^{\nu}||Px^{(s)}||_c[H(A_s; \alpha) + H(\hat{A}_s, \alpha)]|t' - t''|^\alpha \le \varepsilon||P||_\beta||x||_{\beta,\nu} +$$

$$||P||_\beta||x||_{\beta,\nu}[H(A_s;\alpha)+H(\hat{A}_s;\alpha)]|t^{'}-t^{''}|^\beta\varepsilon^{\alpha-\beta}.$$

The analog estimations are true for M_5 changing $||P||$ by $||Q||$ and functions $A_s(t)$, $\hat{A}_s(t)$ by $B_s(t)$ and $\hat{B}_s(t)$, $s=\overline{0,\nu}$. So in both cases

$$\sum_{s=0}^{\nu}\frac{|[A_s(t^{'})-\hat{A}_s(t^{'})](Px^{(s)})(t^{'})-}{|t^{'}-t^{''}|^\beta}$$

$$\frac{[A_s(t^{''})-\hat{A}_s(t^{''})](Px^{(s)})(t^{''})|}{|t^{'}-t^{''}|^\beta}\leq c_1\varepsilon^\delta||x||_{\beta,\nu},$$

$$\sum_{s=0}^{\nu}\frac{|[B_s(t^{'})-\hat{B}_s(t^{'})](Qx^{(s)})(t^{'})}{|t^{'}-t^{''}|^\beta}-$$

$$\frac{[B_s(t^{''})-\hat{B}_s(t^{''})](Qx^{(s)})(t^{''})|}{|t^{'}-t^{''}|^\beta}$$

$$\leq c_2\varepsilon^\delta||x||_{\beta,\nu}, \tag{25}$$

where

$$\delta=\min(\beta;\alpha-\beta). \tag{26}$$

For M_6, in similar way we will consider the case $|t^{'}-t^{''}|\geq\varepsilon$. Then

$$M_6\leq\sum_{s=0}^{\nu}\{\frac{1}{2\pi i}\int_\Gamma|K_s(t^{'},\tau)-$$

$$\hat{K}_s(t^{'},\tau)||x^{(s)}(\tau)|\}|d\tau|+$$

$$+\frac{1}{2\pi i}\sum_{s=0}^{\nu}\{\int_\Gamma|K_s(t^{''},\tau)-$$

$$-\hat{K}_s(t^{''},\tau)||x^{(s)}(\tau)||d\tau|\}\leq$$

$$\leq\frac{\varepsilon}{\pi}||x||_{c,\nu}l\leq\frac{\varepsilon^{1-\beta}}{\pi}l||x||_{\beta,\nu}|t^{'}-t^{''}|^\beta.$$

We used the fact that for functions $K_s(t,\tau)$ and $\hat{K}_s(t,\tau)$ the relation (21) holds.

If $|t^{'}-t^{''}|<\varepsilon$, then

$$M_6\leq\frac{1}{2\pi i}\sum_{s=0}^{\nu}\int_\Gamma|K_s(t^{'},\tau)-$$

$$K_s(t^{''},\tau)||x^{(s)}(\tau)||d\tau|+$$

$$+\frac{1}{2\pi i}\sum_{s=0}^{\nu}\int_\Gamma|\hat{K}_s(t^{'},\tau)-$$

$$\hat{K}_s(t^{''},\tau)||x^{(s)}(\tau)||d\tau|\leq$$

$$\leq\frac{1}{2\pi i}\sum_{s=0}^{\nu}\{||x^{(s)}||(H(K_s;\alpha)+$$

$$+H(\hat{K}_s;\alpha))\}|t^{'}-t^{''}|^\alpha\leq$$

$$\frac{1}{2\pi i}||x||_{c,\nu}\varepsilon^{\alpha-\beta}\sum_{s=0}^{\nu}(H(K_s;\alpha)+$$

$$+H(\hat{K}_s;\alpha))|t^{'}-t^{''}|^\beta.$$

From estimations of M_6, from (22) and (26) we obtain

$$||\Delta M||_{H_\beta^{(l)}(\Gamma)}\leq c\cdot\varepsilon^\delta;\quad \delta=\min(\beta;\alpha-\beta). \tag{27}$$

From relation (27) we have for ε enough small the equation (22) has unique solution $x_\varepsilon^*(t)$.

Using the theory of operator perturbation ([19]) and the relations (27) we can determine the relations between exact solutions $x^*(t)$ and $x_\varepsilon(t)$ of equations (15) and (22) in spaces $H_\beta(\Gamma)$.

Taking into consideration the definition of norm in Hölder spaces we obtain

$||x^*-x_\varepsilon^*||_\beta=O(\varepsilon^\delta)$;

Remark The same results we can obtain for Lebesgue and Generalized Hölder spaces.

7 Numerical Result

We present a test example in this section.

We take the exact solution as $x(t)=\dfrac{1}{t-1}$.

The coefficients are chosen as follows

$$\tilde{A}_0(t)=\tilde{A}_1(t)=\frac{1}{2}\left(t+\frac{1}{2}-\frac{1}{t}\right)\left(1+\frac{1}{t}\right)$$

$$\tilde{B}_0(t)=\tilde{B}_1(t)=\frac{1}{2}\left(t+\frac{1}{2}-\frac{1}{t}\right)\left(\frac{1}{t}-1\right)$$

$$K_r(t,\tau)=\frac{t+r+1}{\tau}, r=\overline{0,1}$$

The contour Γ is an ellipse $Rcos\phi+irsin\phi$. For this example, $R=3$ and $r=2$. The right part $f(t)$ of equation is determined automatically.

In table we show the results using the collocation scheme (20) . We approximate the integrals by quadrature formula [17]:

$$\frac{1}{2\pi i}\int_\Gamma g(\tau)\tau^{l+k}d\tau\cong$$

$$\frac{1}{2\pi i}\int_\Gamma U_n(\tau^{l+1}\cdot g(\tau))\tau^{k-1}d\tau, \tag{28}$$

(where $k = 0, \ldots, n$ for $l = 0, 1, \ldots$ and $k = -1, \ldots, -n$ for $l = -1, -2, \ldots$). Thus, we obtain the following SLAE:

$$\sum_{r=0}^{\nu}\{A_r(t_j)\sum_{k=0}^{n}\frac{(k+\nu)!}{(k+\nu-r)!}t_j^{k+\nu-r}\alpha_k+$$

$$B_r(t_j)\sum_{k=1}^{n}(-1)^r\frac{(k+r-1)!}{(k-1)!}t_j^{-k-r}\cdot\alpha_{-k}$$

$$+\sum_{k=0}^{n}\frac{(k+\nu)!}{(k+\nu-r)!}\sum_{s=0}^{2n}K_r(t_j,t_s)t_s^{1+\nu-r}\Lambda_{-k}^{(s)}\alpha_k$$

$$+\sum_{k=1}^{n}(-1)^r\frac{(k+r-1)!}{(k-1)!}$$

$$\sum_{s=0}^{2n}K_r(t_j,t_s)t_s^{-1-r}\Lambda_k^{(s)}\alpha_{-k}\} = f(t_j), \qquad (29)$$

for $j = 0, \ldots, 2n$.

We determine Λ_k, $k = -n, \ldots, n$ from relations (11).

$2n$	$Error$
8	0.0749
16	0.0215
24	0.0012
28	$2.8018e-04$
32	$6.4508e-05$

Table 1 In this table we presented the error between the exact and approximate solutions. The error is largest error in the magnitude of all selected points.

In our test, the non- collocation points have been obtained from formula

$$z(j) = R\cos\left(\frac{2\pi(j-1)}{k}+\frac{\pi}{16}\right)+$$

$$ir\sin\left(\frac{2\pi(j-1)}{k}+\frac{\pi}{16}\right)$$

where k is a natural integer and $j = 1, \ldots, k+1$. We observe that we should take enough collocation points to guarantee the convergence.

8 Conclusion

In this article we proved the stability of collocation methods. We demonstrated that condition numbers of approximate equations and exact equations existed and appropriated.

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