Qualitative properties of the ice-thickness in a 3D model

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Abstract: In this work we consider a 3D isothermal mathematical model for ice sheets flows over a horizontal bedrock. The model is derived from the mechanics and dynamics of ice sheets and experimental results carried out in Glaciology. The final formulation of the model gives rise to a degenerate quasi-linear elliptic-parabolic equation for the ice-thickness function. Under appropriated initial and Dirichlet boundary conditions, we discuss the existence and uniqueness of weak solutions for this problem. Then, we prove that the local speed of propagations of disturbances from the initial ice-thickness is finite. We prove also that the solutions of this problem have the waiting-time local behavior. To establish these properties we use here a suitable local energy method.

Key–Words: ice sheet dynamics, existence, uniqueness, finite speed of propagations, waiting time.

1 Introduction

Mathematical models of ice sheets flows deal with the evolution equation for the ice-thickness function H:

$$\frac{\partial H}{\partial t}+\mathbf{u}_b\cdot\nabla H=\operatorname{div}\left(\int_b^h\mathcal{A}(\theta)(h-z)^{n+1}dz|\nabla h|^{n-1}\nabla h\right)+a. \tag{1}$$

In (1), $H=h-b$, where h and b are, respectively, the top surface and the bedrock of the ice sheet, $\mathbf{u}_b$ is the sliding velocity, $\mathcal{A}(\theta)=2(\rho\, g)^nA(\theta)$, where ρ is the constant density of the ice sheet, n is called Glen's exponent and $A(\theta)$ is a temperature-depending function (θ is the absolute temperature), and a is the accumulation/ablation rate function (see Sections 3 and 4).

In this work, we shall only consider isothermal motions, which causes, in (1), that $\mathcal{A}$ does not depend on θ. This can be a consequence of approximately zero changes of temperature in the ice sheet, or more generally if, in the Arrhenius relationship (12), $|-Q/(k\theta)|\ll 1$ and $A_0=1$. The exothermic model shall be considered by the authors in a future work. Another simplification of the model, results from an usual assumption in ice sheet modeling, the bedrock b is a horizontal surface, *i.e.* $b=constant$. Under these assumptions, and after an integration procedure (see Fowler [11]), (1) becomes

$$\frac{\partial H}{\partial t}+\mathbf{u}_b\cdot\nabla H=\operatorname{div}\left(\frac{H^{n+2}}{n+2}|\nabla H|^{n-1}\nabla H\right)+a\,. \tag{2}$$

On the other hand, it should be pointed out that, as any measure, the ice-thickness must be non-negative. A different mathematical model was considered by the authors in [3, 5]. There, it was used the arguing of Fowler [12] to justify the replacement of the sliding velocity $\mathbf{u}_b$ by $-\nabla H$.

When formulating mathematical models for the study of ice sheets, usually it is necessary to take into account that the flow domain is not prescribed and is itself part of the solution (see Calvo *et al.* [7] and Rodrigues and Santos [21]). However, in this work, we are mainly interested with the local properties of the ice-thickness function. Therefore we may assume the ice sheet based domain is known. We assume the ice sheet occupies a sufficiently large area where can possibly vanish the ice-thickness in some relatively small subareas. On the boundary of this large area, we assume the ice-thickness vanishes.

Most of the mathematical works in Theoretical Glaciology deal only with 2D models (see *e.g.* Calvo *et al.* [7], Díaz et al. [9] and the references cited therein). However, in this article, we shall consider the 3D model of (1), which causes to consider two spatial coordinates in the ice sheet based domain. Let us then consider the cylinder

$$Q_T:=\Omega\times(0,T)\subset\mathbb{R}^2\times\mathbb{R}^+\,,$$

whose boundary is defined by $\Gamma_T:=\partial\Omega\times(0,T)$ and Ω is assumed to be a large enough open bounded domain with a sufficiently smooth boundary $\partial\Omega$.

The (strong) formulation of our problem can be stated in the following terms. Given an accumulatoin/ablation rate function $a=a(x,y,t)$ and a sliding velocity $\mathbf{u}_b=\mathbf{u}_b(x,y)$ defined in Q_T, and an initial ice-thickness $H_0=H_0(x,y)\geq 0$, bounded and compactly supported in Ω, to find a sufficiently smooth function $H=H(x,y,t)$ defined in Q_T such that (2) is fulfilled in Q_T,

$$H=H_0 \quad\text{in}\quad \Omega \quad\text{for}\quad t=0\,, \tag{3}$$

$$H=0 \quad\text{on}\quad \Gamma_T\,. \tag{4}$$

The mathematical (strong) solutions of (2)-(4) must be physically admissible, *i.e.* they have to be nonnegative compactly supported solutions. A sketch of this work containing some of the ideas and formulated in a different manner was published in the conference proceedings [4]. Calvo *et al.* [7] have consider a similar problem. They have considered a 2D model and there the ice sheet domain is itself part of the solution.

2 Mathematical framework

Notation. The notation used throughout this text is largely standard in Mathematical Fluid Mechanics - see, *e.g.*, Antontsev *et al.* [2]. We distinguish vectors from scalars by using boldface letters. For functions and function spaces we will use this distinction as well. Symbols C and K will denote generic positive constants, whose value will not be specified. It can change from one inequality to another and particularly when we use different subscripts. The dependence of C and K on other parameters will always be clear from the exposition and the same subscripted symbols used in different sections account for different constant values. In this article, the notation Ω stands always for a bounded 2D domain, *i.e.*, a connected open bounded subset of $\mathbb{R}^2$, whose compact boundary is denoted by $\partial\Omega$.

Function spaces. Let $1 \leq p \leq \infty$. We shall use the classical Lebesgue spaces $\mathrm{L}^p(\Omega)$, whose norm is denoted by $\|\cdot\|_{\mathrm{L}^p(\Omega)}$. For any nonnegative integer k, $\mathrm{W}^{k,p}(\Omega)$ denotes the Sobolev space and its norm is denoted by $\|\cdot\|_{\mathrm{W}^{k,p}(\Omega)}$. By $\mathrm{W}^{k,p}_0(\Omega)$ we denote the closure of $\mathrm{C}^\infty_0(\Omega)$ in $\mathrm{W}^{k,p}(\Omega)$, where $\mathrm{C}^\infty_0(\Omega)$ denotes the set of all continuously differentiable functions with compact support in Ω. For $1 \leq p < \infty$ and any nonnegative integer k, the dual spaces of $\mathrm{L}^p(\Omega)$ and $\mathrm{W}^{k,p}_0(\Omega)$ are denoted, respectively, by $\mathrm{L}^{p'}(\Omega)$ and $\mathrm{W}^{-k,p'}(\Omega)$, where $1/p + 1/p' = 1$. Given $T > 0$ and a Banach space X, $\mathrm{L}^p(0,T;X)$ and $\mathrm{W}^{k,p}(0,T;X)$ denote the usual Lebesgue and Sobolev spaces used in evolutive problems, with norms denoted by $\|\cdot\|_{\mathrm{L}^p(0,T;X)}$ and $\|\cdot\|_{\mathrm{W}^{k,p}(0,T;X)}$. The corresponding spaces of vector-valued functions are denoted by boldface letters. All these spaces are Banach spaces and the Hilbert framework corresponds to $p = 2$. Finally, $\mathrm{C}^1(0,T;\mathbf{C}^\alpha(\Omega))$, with $0 < \alpha < 1$, denotes the space of continuous functions $\mathbf{u} : [0,T] \to \mathbf{C}^\alpha(\Omega)$, where $\mathbf{C}^\alpha(\Omega)$ is the space of Hölder continuous functions. For a detailed exposition of these spaces and its properties, we address the reader, for instance, to Antontsev *et al.* [2] and especially to the monograph by Adams cited therein.

Auxiliary results. Throughout this text we will make reference, at least once, to the following inequalities (see Antontsev *et al.* [2]):

(1) Algebraic inequalities - whenever it make sense, for every α, β, A, $B \geq 0$,

$$A^\alpha B^\beta \leq (A+B)^{\alpha+\beta}, \quad (A+B)^\alpha \leq 2^\alpha (A^\alpha + B^\alpha)\,; \tag{5}$$

(2) Algebraic equality - whenever it make sense, for every A, B, C, α, β, $\xi \in \mathbb{R}$ and for every $\gamma \in [0,1]$,

$$A^\alpha B^\beta + CB^\xi = A^\alpha B^{\gamma\beta} B^{(1-\gamma)\beta} + CB^{\alpha+\gamma\beta} B^{\xi-(\alpha+\gamma\beta)}\,; \tag{6}$$

(3) Young's inequality - for every $a,\ b \geq 0$, $\varepsilon > 0$ and $1 < p,\ q < \infty$ such that $1/p + 1/q = 1$,

$$ab \leq \varepsilon a^p + C(\varepsilon) b^q\,;$$

(4) Hölder's inequality - for every $u \in \mathrm{L}^p(\Omega)$, $v \in \mathrm{L}^q(\Omega)$, with $1 \leq p,\ q \leq \infty$ such that $1/p + 1/q = 1$,

$$\int_\Omega u\, v\, d\mathbf{x} \leq \|u\|_{\mathrm{L}^p(\Omega)} \|v\|_{\mathrm{L}^q(\Omega)}\,.$$

For the main properties we shall establish in this article, play an important role the following result.

Lemma 1 *Assume that $B_\rho \subset \mathbb{R}^2$ is an open ball with radius ρ and $0 \leq \sigma \leq q - 1 < \infty$. Then there exists a constant C depending on σ, q and B_ρ such that, for any $u \in \mathrm{W}^{1,q}(B_\rho)$, we have*

$$\|u\|_{\mathrm{L}^q(\partial B_\rho)} \leq C\left(\|\nabla u\|_{\mathrm{L}^q(B_\rho)} + \rho^{-\delta}\|u\|_{\mathrm{L}^{\sigma+1}(B_\rho)}\right)^\theta \times \|u\|^{1-\theta}_{\mathrm{L}^{\sigma+1}(B_\rho)}, \tag{7}$$

where

$$\theta = 1 - \frac{(q-1)(\sigma+1)}{2q + (q-2)(\sigma+1)}, \quad \delta = 1 + 2\frac{q-(\sigma+1)}{q(\sigma+1)}\,. \tag{8}$$

Lemma 1 is derived from the well-known trace-interpolation inequality (see Díaz and Veron [10]).

3 Mechanics of ice sheets

The common Fluid Mechanics model adopted for cold ice is a non-Newtonian, viscous, heat-conducting, incompressible fluid. But, it should be pointed out that, strictly speaking, it is not possible to assume ice to be incompressible and yet still presume density variations under phase changes. It is, however, justified to ignore density variations since associated changes in bulk density are very small. On the other hand, it is worth to know that ice sheets are assumed to be isotropic materials, but they can develop an induced anisotropy when stressed over sufficiently long time scales.

Governing equations. The model adopted for ice sheet flows results from the principles of conservation of mass and momentum:

$$\mathrm{div}\mathbf{u} = 0\,; \quad \mathbf{0} = \rho\,\mathbf{g} + \mathrm{div}\mathbf{T}. \tag{9}$$

Notice that in $(9)_2$ we have neglected the inertial terms because we are in the presence of very slow flows. The notation used in (9) is well known: $\mathbf{u}$ is the velocity field, p is the pressure, ρ is the constant density, $\mathbf{g}$ is the gravitational force and $\mathbf{T}$ is the Cauchy stress tensor:

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}\,; \tag{10}$$

$\mathbf{I}$ is the unit tensor and $\mathbf{S}$ is the deviatoric part of $\mathbf{T}$. Notice that from $(9)_1$, $\mathrm{tr}(\mathbf{S}) = 0$.

Glen's law. Extra stress tensor $\mathbf{S}$ and strain rate tensor $\mathbf{D}$ are related by a rheological flow law. According to the common usage in Glaciology to write stretching as a function of stress, this law states that the strain rate $\mathbf{D}$, at a given strain, is proportional to the stress $\mathbf{S}$ raised to the power n:

$$\mathbf{D} = A(\theta)\mathrm{sgn}(\mathbf{S})|\mathbf{S}|^n, \quad \mathrm{sgn}(\mathbf{S}) = |\mathbf{S}|^{-1}\mathbf{S}\,. \tag{11}$$

This law was suggested by J.W. Glen and, for this reason, is called Glen's law in Glaciology. The basic postulate is that ice is an incompressible nonlinear viscous fluid. Here n is a positive constant and the function A may depend on the temperature and usually is postulated an Arrhenius-type relationship

$$A(\theta) = A_0 \exp\left(-(k\theta)^{-1}Q\right)\,, \tag{12}$$

where Q is the so-called activation energy, k the Boltzman constant, θ the absolute temperature and A_0 a constant.

4 Dynamics of ice sheets

A thorough analysis of ice sheets dynamics is made in many monographs, for instance, Hutter [14] and Paterson [19]. However, many authors deal only with 2D mathematical models, see *e.g.* Fowler [11]. Present-day 3D mathematical models including full thermo-mechanical coupling are those developed by Huybrechts [15], Greve [13] and Patyn [20], to name a few. The mathematical model approach is based on the continuum mechanics equations (9)-(11). We consider a Cartesian coordinate system (x, y, z) with the z-axis vertically pointing upward and being $z = 0$ at the mean sea level.

Field Equations. Denoting the velocity components in the correspondingly directions as (u, v, w), $(9)_1$ can be rewritten as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\,. \tag{13}$$

Once the gravitational force is only important in the vertical direction, *i.e.* considering $\mathbf{g} = (0, 0, -g)$, $(9)_2$ becomes

$$\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z} = 0, \tag{14}$$

$$\frac{\partial T_{yx}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{yz}}{\partial z} = 0, \tag{15}$$

$$\frac{\partial T_{zx}}{\partial x} + \frac{\partial T_{zy}}{\partial y} + \frac{\partial T_{zz}}{\partial z} = \rho g, \tag{16}$$

where T_{ij} means stress in the i-plane ($i = constant$) along j-direction.

Dynamic Boundary Condition. At the free surface, say $z = h(x, y, t)$, the model assumes that there is no applied traction, *i.e.*

$$\mathbf{T}\cdot\mathbf{n} = \mathbf{0} \quad \text{on} \quad z = h(x, y, t)\,, \tag{17}$$

where $\mathbf{n}$ is the exterior unit normal to the ice sheet top surface $z = h(x, y, t)$. If we write the free surface in the implicit form $z - h(x, y, t) = 0$, then $\mathbf{n} = |\nabla s|^{-1}\nabla s$, where $s(x, y, t) = z - h(x, y, t)$. It is a matter of practical evidence that everywhere in an ice sheet the slopes of the free surface $z = h(x, y, t)$ are small, except in a small neighborhood of ice domes and ice margins. Thus the normal unit vector of the free surface $z = h(x, y, t)$ is approximately vertical and (17) reduces to

$$T_{xz} = 0\,,\ T_{yz} = 0\,,\ T_{zz} = 0 \ \text{ on } \ z = h(x, y, t)\,, \tag{18}$$

Hydrostatic Approximation. Applying the hydrostatic approximation in the vertical direction, *i.e.* $p_z = -\rho g$, then (16) reduces to

$$\frac{\partial T_{zz}}{\partial z} = \rho g\,. \tag{19}$$

This means that, in all parts of an ice sheet, the shear stresses T_{xz} and T_{yz} are small compared to the vertical normal stress T_{zz}. Therefore the variational stress in the z-plane can be neglected. On the other hand, if we neglect atmospheric pressure, an integration of (19) from the surface $h(x, y, t)$ to a height z in the ice body and using (18), gives us an expression for the vertical normal stress

$$T_{zz} = \rho g(z - h)\,. \tag{20}$$

From (10) and (20), the pressure p reads

$$p = \rho g\,(h - z) - S_{xx} - S_{yy} \tag{21}$$

and the horizontal normal stresses can be expressed as

$$T_{xx} = 2S_{xx} + S_{yy} - \rho g\,(h - z)\,, \tag{22}$$

$$T_{yy} = S_{xx} + 2S_{yy} - \rho g\,(h - z)\,. \tag{23}$$

Inserting (22) and (23) in the horizontal components (x, y) of (14) and (15), we achieve to

$$\frac{\partial}{\partial x}\left(2S_{xx} + S_{yy}\right) + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z} = \rho g\frac{\partial h}{\partial x}\,, \tag{24}$$

$$\frac{\partial}{\partial y}\left(S_{xx} + 2S_{yy}\right) + \frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yz}}{\partial z} = \rho g\frac{\partial h}{\partial y}\,. \tag{25}$$

Shallow-Ice Approximation. The major simplification of the model ensues by considering the shallow-ice approximation. This is justified, since we assume a physical process in which important length scales in the longitudinal directions are much larger, compared to those in the transverse directions. Consistently, $x\,,\ y \gg z$, and also $u\,,\ v \gg w$, and thus the dominant stresses are the shear stresses in the horizontal plane, S_{xz} and S_{yz}, which are supported by the basal drag. Moreover, normal stresses S_{xx}, S_{yy}, S_{zz} are negligible , as well the shear stress S_{xy} in the vertical planes. In consequence,

$$T_{xx} = T_{yy} = T_{zz} = -p\,. \tag{26}$$

and, from (17)-(20), the pressure is close to hydrostatic

$$p = \rho\, g(h - z)\,. \tag{27}$$

Then the horizontal components of (24)-(25) simplify to

$$\frac{\partial T_{xz}}{\partial z} = \rho\, g \frac{\partial h}{\partial x}\,, \quad \frac{\partial T_{yz}}{\partial z} = \rho\, g \frac{\partial h}{\partial y}\,. \tag{28}$$

On the free surface $z = h(x, y, t)$ we obtain, after using (18) and (26),

$$T_{xz} = 0\,,\ T_{yz} = 0\,,\ p = 0 \ \text{ on } z = h(x, y, t)\,. \tag{29}$$

Then a vertical integration of (28) from $h(x, y, t)$ to a height z in the ice body and using (29), lead us to

$$T_{xz} = -\rho\, g(h - z)\frac{\partial h}{\partial x}\,, \quad T_{yz} = -\rho\, g(h - z)\frac{\partial h}{\partial y}\,. \tag{30}$$

From (11), strain rates are related with deviatoric stresses by

$$\mathbf{D} = A(\theta)\tau^{n-1}\mathbf{S}\,, \quad \tau = \sqrt{II_{\mathbf{S}}}\,, \tag{31}$$

where $II_{\mathbf{S}}$ denotes the second invariant of $\mathbf{S}$. Notice that $(9)_1$ implies $\tau = \sqrt{\mathrm{tr}(\mathbf{S}^2)/2}$ and from the simplifications of the shallow ice approximation, especially (30),

$$\tau = \sqrt{T_{xz}^2 + T_{yz}^2} = \rho\, g(h - z)|\nabla h|\,. \tag{32}$$

A common assumption in ice sheet modeling, and which is valid for most of the ice sheet domain, is that horizontal gradients of the vertical velocity are small compared to the vertical gradient of the horizontal velocity, *i.e.* $w_x \ll u_z$ and $w_y \ll v_z$. Using this assumption, (30) and (32), we obtain from (31)

$$\frac{\partial u}{\partial z} = -2A(\theta)\,[\rho\, g(h - z)]^n\, |\nabla h|^{n-1}\frac{\partial h}{\partial x}\,, \tag{33}$$

$$\frac{\partial v}{\partial z} = -2A(\theta)\,[\rho\, g(h - z)]^n\, |\nabla h|^{n-1}\frac{\partial h}{\partial y}\,. \tag{34}$$

Integrating (33) and (34) from the ice base, say $z = b(x, y, t)$, to an arbitrary point z in the ice sheet, we obtain

$$u = u_b - 2(\rho\, g)^n|\nabla h|^{n-1}\frac{\partial h}{\partial x}\int_b^z A(\theta)(h - s)^n ds\,, \tag{35}$$

$$v = v_b - 2(\rho\, g)^n|\nabla h|^{n-1}\frac{\partial h}{\partial y}\int_b^z A(\theta)(h - s)^n ds\,, \tag{36}$$

where $\mathbf{u}_b = (u_b, v_b)$ is the ice velocity at the ice base.

Kinematic Boundary Conditions. For this model, the possible presence of attached ice shelves will be ignored. If we write the free surface in the implicit form $s(x, y, t) = z - h(x, y, t) = 0$, then its exterior unit normal is given by $\mathbf{n} = |\nabla s|^{-1}\nabla s$. Let $\mathbf{u}$ and $\mathbf{w}$ denote, respectively, the ice surface velocity and the velocity at which the free surface points move. Then $\mathbf{w} \cdot \mathbf{n}$ represents the normal speed of propagation of the free surface and

$$a_h = (\mathbf{w} - \mathbf{u}) \cdot \mathbf{n} \tag{37}$$

is the ice volume flux through the free surface, also known as the accumulation/ablation function. The sign is chosen such that a supply (accumulation) is counted as positive and a loss (ablation) as negative. Then the time derivative of $s(x, y, t)$ following the motion of the free surface with velocity $\mathbf{w}$ must vanish and, by using (37), we obtain

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} - w = a_h N_h,\ N_h = \sqrt{h_x^2 + h_y^2 + 1}. \tag{38}$$

A similar boundary condition can be derived for the ice base. We proceed as above, considering the implicit form of the ice base $r(x, y, t) = b(x, y, t) - z = 0$, its exterior unit normal given by $\mathbf{n} = |\nabla r|^{-1}\nabla r$ and the ice volume flux through the ice base is given by

$$a_b = (\mathbf{w} - \mathbf{u}) \cdot \mathbf{n}\,. \tag{39}$$

Now $\mathbf{w}$ is the velocity at which the ice base points move and $\mathbf{w} \cdot \mathbf{n}$ represents the normal speed of propagation of the ice base. Arguing as before, we obtain

$$\frac{\partial b}{\partial t} + u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y} - w = a_b N_b\,, \quad N_b = \sqrt{b_x^2 + b_y^2 + 1}\,. \tag{40}$$

In both cases, free surface and ice base, their interior sides are identified with the ice and therefore the exterior sides are identified with the atmosphere and the lithosphere, respectively. Provided that accumulation/ablation functions (37) and (39) are given, equations (38) and (40) govern the evolution of the free surface and ice base, respectively.

Ice-Thickness Equation. We integrate (13) along the vertical from the ice base $z = b(x, y, t)$ to the free surface $z = h(x, y, t)$ and we use (38) and (40) to obtain

$$\frac{\partial}{\partial x}\int_b^h u\, dz + \frac{\partial}{\partial y}\int_b^h v\, dz + \frac{\partial h}{\partial t} - N_h a_h - \frac{\partial b}{\partial t} + N_b a_b = 0\,. \tag{41}$$

Replacing, in (41), u and v by its expressions (35) and (36), we obtain, after an integration by parts, the evolution equation for the ice sheet thickness (see (1)), where $\mathbf{u}_b = (u_b, v_b)$ is the sliding velocity, $\mathcal{A}(\theta) = 2(\rho\, g)^n A(\theta)$ and $a = a_h - a_b$ is the accumulation/ablation rate. We already have seen that everywhere in an ice sheet the slopes of the free surface $z = h(x, y, t)$ are small. The same happens with the slopes of the ice base $z = b(x, y, t)$. Then the exterior normal vectors to $z = h(x, y, t)$ and to $z = b(x, y, t)$ are approximately vertical and this justifies why we have taken $N_h = N_b = 1$ in (1).

5 Existence and uniqueness

General formulation. In order to obtain a more general framework than (2)-(4), let us introduce the new functions $\nu = \nu(x, y, t)$ and $b = b(s)$ defined by

$$\nu := H^m = \psi(H) \Longrightarrow \psi^{-1}(\nu) = \nu^{\frac{1}{m}} := b(\nu)\,, \qquad (42)$$

where $m = 2(n+1)/n$. Notice that the new variable $\nu := H^m$ is motivated by the relation

$$\frac{H^{n+2}}{n+2}|\nabla H|^{n-1}\nabla H = \frac{m^{1-p}}{n+2}\,|\nabla H^m|^{p-2}\,\nabla H^m\,,$$

with $p = n+1$. Let us assume that:

$$a \in \mathrm{L}^\infty(\Omega)\,; \qquad (43)$$

$$\operatorname{div}\mathbf{u}_b = 0 \quad \text{in} \quad Q_T\,; \qquad \mathbf{u}_b \in \mathbf{L}^\infty(Q_T)\,; \qquad (44)$$

$$\nu_0 \in \mathrm{L}^\infty(\Omega). \qquad (45)$$

Notice that, according to (42), condition (45) is equivalent to assume that $H_0 \in \mathrm{L}^\infty(\Omega)$. Then the general formulation of (2)-(4) can be stated in terms of ν and b as follows. Given Ω, a constant $k = m^{1-p}/(n+2)$ and a, $\mathbf{u}_b$ and H_0 satisfying (43)-(45), to find a function ν defined by (42) and solution of

$$\frac{\partial\, b(\nu)}{\partial\, t} = \operatorname{div}\,\left(k|\nabla\nu|^{p-2}\nabla\nu - \mathbf{u}_b b(\nu)\right) + a\,, \qquad (46)$$

$$b(\nu) = b(\nu_0) \quad \text{in} \quad \Omega \quad \text{for} \quad t = 0\,, \qquad (47)$$

$$\nu = 0 \quad \text{on} \quad \Gamma_T\,. \qquad (48)$$

It is worth to notice that, according to (42), ν and H have the same support and have the same value on the boundary Γ_T. Moreover, if H is a solution of (2)-(4) then ν is a solution of (46)-(48) and reciprocally. The general formulation (46)-(48) is the one used to establish existence and uniqueness of solutions (see Calvo *et al.* [7]) and goes back to mathematical works on quasi-linear elliptic-parabolic differential equations (see Alt and Luckhaus [1], Otto [18], Benilan and Wittbold [6], Carrillo and Wittbold [8], Ivanov and Rodrigues [16]), being our problem a particular case.

Weak formulation We start this section by introducing the notion of solutions to the problem (46)-(48) we shall work with in the sequel. We multiply (46) by a test function ζ and integrate by parts over Q_T to obtain

$$\begin{aligned}\int_{Q_T}\left(b(\nu)\frac{\partial\zeta}{\partial t} + a\zeta\right)d\mathbf{z} \;+ \int_\Omega b(\nu_0)\zeta_0\,d\mathbf{x} = \\ \int_{Q_T}(k|\nabla\nu|^{p-2}\nabla\nu \;- b(\nu)\mathbf{u}_b)\cdot\nabla\zeta\,d\mathbf{z}\,,\end{aligned} \qquad (49)$$

where $\zeta_0 = \zeta(\cdot, 0)$ and where we have set $\mathbf{x} = (x, y)$ and $\mathbf{z} = (\mathbf{x}, t)$. Then the definition of weak solution follows as usual (see Alt and Luckhaus [1]).

Definition 2 . *Let (43)-(45) be fulfilled. A function ν is a weak solution of the problem (46)-(48), if:*
1. $\nu \geq 0$ a.e. in Q_T and $\nu \in \mathrm{L}^p(0, T; \mathrm{W}^{1,p}_0(\Omega))$;
2. $b(\nu) \in \mathrm{L}^\infty(0, T; \mathrm{L}^1(\Omega))$, $b(\nu)_t \in \mathrm{L}^{p'}(0, T; \mathrm{W}^{1,-p}(\Omega))$;
3. The relation (49) holds for every $\zeta \in \mathrm{L}^p(0, T; \mathrm{W}^{1,p}_0(\Omega)) \cap \mathrm{W}^{1,1}(0, T; \mathrm{L}^\infty(\Omega))$, such that $\zeta(\cdot, T) = 0$.

There are now many existence and uniqueness results which can be applied directly to the problem (46)-(48) (Alt and Luckhaus [1], Otto [18], Benilan and Wittbold [6], Ivanov and Rodrigues [16], Carrillo and Wittbold [8], to name a few). One of the first references to appear was the paper by Alt and Luckhaus [1], where is proved (Theorem 1.7) the existence of a weak solution to a general problem which includes the case of Definition 2. The existence result there is proved for any

$$u_0 = b(\nu_0) \quad \text{with} \quad B(\nu_0) \in \mathrm{L}^1(\Omega) \quad \text{(see (53 below))}\,,$$

$$a \in \mathrm{L}^{p'}(0, T; \mathrm{W}^{-1,p'}(\Omega)).$$

In order to apply Alt and Luckhaus [1, Theorem 1.7] to the problem (46)-(48), let us define the following functions

$$\mathcal{B} : \mathbb{R} \to \mathbb{R}\,, \quad \mathcal{B}(u) = u^{\frac{1}{m}}\,, \qquad (50)$$

$$\mathcal{A} : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2\,,\; \mathcal{A}(\mathbf{v}, u) = k|\mathbf{v}|^{p-2}\mathbf{v} - u\mathbf{u}_b\,, \qquad (51)$$

where m, k and $p = n+1$ are constants and $\mathbf{u}_b$ is a given vector - the sliding velocity at the ice base. One can easily sees that (50) is a nondecreasing continuous function in $\mathbb{R}$ such that $\mathcal{B}(0) = 0$ and (51) is a vector-valued continuous function in $\mathbb{R}^2 \times \mathbb{R}$ such that the growth condition

$$|\mathcal{A}(\nabla\nu, \mathcal{B}(\nu))|^{p'} \leq C_1\,(1 + |\nabla\nu|^p + B(\mathcal{B}(\nu)))\,, \qquad (52)$$

hold. In (52), $C_1 = const. \geq 0$ and $B(\mathcal{B}(\nu))$ is the Legendre transform of the primitive of $\mathcal{B}(\nu)$

$$B(\mathcal{B}(\nu)) := \int_0^\nu s\,d\mathcal{B}(s)\,. \qquad (53)$$

It should be noticed that B is super-linear in the sense that for any $\delta > 0$, there exists a $C(\delta) < \infty$, such that for all $u \in \mathbb{R}$, $|u| \leq \delta\,B(u) + C(\delta)$. From this property of B it is a easy task to prove (52). The proof of the strict monotonicity condition

$$(\mathcal{A}(\mathbf{v}, u) - \mathcal{A}(\mathbf{w}, u))\cdot(\mathbf{v}, \mathbf{w}) \geq C_2\,|\mathbf{v} - \mathbf{w}|^p\,, \qquad (54)$$

$C_2 = const. > 0$, is more involved. In fact, after some algebraic manipulations, we can prove successively

$$\begin{gathered}k^{-1}(\mathcal{A}(\mathbf{v}, u) - \mathcal{A}(\mathbf{w}, u))\cdot(\mathbf{v}, \mathbf{w}) = \\ |\mathbf{v}|^p + |\mathbf{w}|^p - \left(|\mathbf{v}|^{p-2} - |\mathbf{w}|^{p-2}\right)\mathbf{v}\cdot\mathbf{w} = \\ \left(|\mathbf{v}|^{p-2} + |\mathbf{w}|^{p-2}\right)|\mathbf{v} - \mathbf{w}|^2 + |\mathbf{v}|^{p-1}|\mathbf{w}| + |\mathbf{v}||\mathbf{w}|^{p-1} \geq \\ C|\mathbf{v} - \mathbf{w}|^p\,, \quad C = C(p)\,, \quad p \geq 2\,.\end{gathered}$$

For our purposes, it is enough to consider $p \geq 2$, because $p = n+1$ and, as we shall see in Section 7,

$n \geq 1, 9$. Anyway, an extension of the result presented in Alt and Luckhaus [1] to the case $1 < p < 2$ is given by Ivanov and Rodrigues [16]. Moreover, Alt and Luckhaus [1] have shown that the natural energy associated to a weak solution ν of the problem (46)-(48) is given by the finite sum

$$\sup_{t\in(0,T)} \int_\Omega B(\mathcal{B}(\nu(\cdot,t)))\,d\mathbf{x} + \int_{Q_T} |\nabla\nu|^p d\mathbf{z} < \infty\,, \tag{55}$$

where $B(\mathcal{B}(\nu(\cdot,t)))$ is defined in (53). Benilan and Wittbold [6] under rather general assumptions than Alt and Luckhaus [1], and using the nonlinear semigroup theory, have proved the existence of mild solutions, which under certain conditions were shown to be weak solutions. Uniqueness of weak solutions of (46)-(48) is a much more difficult task because of the nonlinear term $b(\nu)$. The usual approach consists in to prove the L^1-contraction principle

$$\int_\Omega |\mathcal{B}(\nu_1(\cdot,t) - \mathcal{B}(\nu_2(\cdot,t)|\,d\mathbf{x} \leq e^{Lt} \int_\Omega \mathcal{B}(\nu_1(\cdot,0)) - \mathcal{B}(\nu_2(\cdot,0))|\,d\mathbf{x} \tag{56}$$

for any two weak solutions ν_1 and ν_2 satisfying (55) - L is the assumed Lipschitz constant of a. Under the additional continuity property

$$|\mathcal{A}(\mathbf{v},u) - \mathcal{A}(\mathbf{v},z)|^{p'} \leq C(1 + B(u) + B(z) + |\mathbf{v}|^p)|u - z|\,, \tag{57}$$

Alt and Luckhaus [1, Theorem 2.3] also have proved the uniqueness of a weak solution ν provided

$$\frac{\partial\nu}{\partial t} \in \mathrm{L}^1(Q_T)\,. \tag{58}$$

It is a easy task to prove that (51) satisfies (57). Latter, Otto [18], by using Kruzhkov method of doubling variables both in space and time, have proved (56), and consequently the uniqueness result, for ν_i, $i = 1,\ 2$, satisfying (55) without assuming (58). Carrillo and Wittbold [8] have generalized the uniqueness result of Otto [18] and have proved a comparison result by using also Kruzhkov method.

6 Qualitative properties

Existence and uniqueness of a weak solution $\nu = H^m$ to the equivalent problem (46)-(48) have been established in the previous section. According to (42), that results allow us to state the existence and uniqueness of a weak solution H for (2)-(4) and such that:

1. $H \geq 0$ a.e. in Q_T and $H^m \in \mathrm{L}^p(0,T;\mathrm{W}_0^{1,p}(\Omega))$;
2. $H \in \mathrm{L}^\infty(0,T;\mathrm{L}^1(\Omega))$ and $H_t \in \mathrm{L}^{p'}(0,T;\mathrm{W}^{1,-p}(\Omega))$;
3. for every $\zeta \in \mathrm{L}^{n+1}(0,T;\mathrm{W}_0^{1,n+1}(\Omega)) \cap \mathrm{W}^{1,1}(0,T;\mathrm{L}^\infty(\Omega))$, with $\zeta(\cdot,T) = 0$, the equivalent of (49) holds:

$$\int_{Q_T} \left(H\frac{\partial\zeta}{\partial t} + a\zeta\right) d\mathbf{z} + \int_\Omega H_0\zeta_0\,d\mathbf{x} = \int_{Q_T} \left(k|\nabla H^m|^{p-2}\nabla H^m - H\mathbf{u}_b\right)\cdot\nabla\zeta\,d\mathbf{z}\,.$$

Moreover, from (55) and by using (53), one can easily proves that the energy associated with the problem (2)-(4) is finite

$$E(Q_T) := \sup_{t\in[0,T]} \int_\Omega |H(\cdot,t)|^{m+1}d\mathbf{x} + \int_{Q_T} |\nabla H^m|^p d\mathbf{z} < \infty\,, \tag{59}$$

where we already have seen that

$$p = n + 1\,, \quad m = \frac{2(n+1)}{n}\,. \tag{60}$$

In order to establish the qualitative properties, let us fix $\mathbf{x}_0$ in Ω, $\rho_0 \in (0, \mathrm{dist}(\mathbf{x}_0, \partial\Omega))$ and assume that

$$H_0(\mathbf{x}) = 0 \text{ in } B_{\rho_0}(\mathbf{x}_0) = \{\mathbf{x}\in\Omega : |\mathbf{x}-\mathbf{x}_0| < \rho_0\} \subset \Omega\,. \tag{61}$$

In this section, we shall assume that

$$\mathrm{div}\,\mathbf{u}_b = 0 \text{ in } Q_T\,,\ \ \mathbf{u}_b \in \mathrm{C}^1(0,T;\mathbf{C}^\alpha(\Omega))\,,\ 0 < \alpha < 1\,. \tag{62}$$

To proceed our study, let us consider the Lagrange variables $\mathbf{X}$ defined as usual in Continuum Mechanics (see, *e.g.*, Meirmanov *et al.* [17]):

$$\frac{d\mathbf{X}(\mathbf{x},t)}{dt} = \mathbf{u}_b(\mathbf{X},t),\ t\in(0,T);\ \mathbf{X}(\mathbf{x},0) = \mathbf{x},\ \mathbf{x}\in\Omega. \tag{63}$$

Under conditions expressed in (62), there exists a unique solution $\mathbf{X}(\mathbf{x},t)$ of the problem (63), which is a homeomorphism between Ω and $\Omega^t = \{\mathbf{y} : \mathbf{z} = \mathbf{X}(\mathbf{x},t),\ \mathbf{x}\in\Omega\}$ for any $t\in[0,T]$. This solution transforms the ball $B_\rho(\mathbf{x}_0)$ into

$$B^t_\rho(\mathbf{x}_0) = \{\mathbf{z} : \mathbf{y} = \mathbf{X}(\mathbf{x},t), \text{for some } \mathbf{x}\in B_\rho(\mathbf{x}_0)\}.$$

Moreover, the following formulas hold

$$\frac{d}{dt}\int_{B^t_\rho(\mathbf{x}_0)} \Phi\,d\mathbf{z} = \int_{B^t_\rho(\mathbf{x}_0)} \left(\frac{\partial\Phi}{\partial t} + \mathbf{u}_b\nabla\Phi\right) d\mathbf{z}\,, \tag{64}$$

$$\frac{dJ}{dt} = J\mathrm{div}\mathbf{u}_b,\ J = \det\left(\frac{\partial\mathbf{X}(\mathbf{x},t)}{\partial\mathbf{x}}\right),\ J(\mathbf{x},0) = 1. \tag{65}$$

In the considered case ($\mathrm{div}\,\mathbf{u}_b = 0$), we have that $J(\mathbf{x},0) = J(\mathbf{x},t) = 1$. In order to simplify the expressions, let us introduce the energy functions

$$E(\rho,s) := \int_0^s \int_{B^t_\rho(\mathbf{x}_0)} |\nabla H^m|^p d\mathbf{z}dt \tag{66}$$

and

$$B(\rho,s) := \int_{B^t_\rho(\mathbf{x}_0)} H^{m+1}d\mathbf{z}\,. \tag{67}$$

6.1 Finite speed of propagations

Theorem 3 *Let H be a weak solution to the problem (2)-(4) with $a = 0$. Assume $\mathbf{u}_b$ satisfies (62) and (59) is finite. If (61) is verified, then there exists t^*, $0 < t^* < T$, such that*

$$H(\mathbf{x},t) = 0 \quad \text{a.e. in } B_{\rho(t)}(\mathbf{x}_0), \quad \forall\, t\in[0,t^*],$$

with $\rho(t)$ given by

$$\rho^\nu(t) = \rho_0^\nu - \frac{\nu}{C\,\tau}t^\lambda E(\rho_0,t)^\tau,$$

for some positive constants α, λ, ν and C, provided $n > (1+\sqrt{17})/4$.

PROOF. We formally multiply (2) (with $a = 0$) by H, a weak solution of (2)-(4) and integrate by parts over $B^t_\rho(\mathbf{x}_0) \times (0, s)$, with $s \le t \le T$. To be precise, we should multiply (2) by a regularized H function, with compact support in Ω, and then pass to the limit in the obtained integral inequality (see Díaz & Veron [10, Lemma 2.1]). Using $(62)_1$ and the notations introduced in (66)-(67), we obtain the following energy relation

$$\frac{1}{m+1}B(\rho,s) + kE(\rho,s) \le \frac{1}{m+1}B(\rho,0) + kI(\rho,s), \quad (68)$$

where $k = m^{1-p}/(n+2)$,

$$I(\rho,s) := \int_0^s \int_{S^t_\rho(\mathbf{x}_0)} |\nabla H^m|^{p-2}\nabla H^m \cdot \mathbf{n}\, H^m\, dS\,,$$

$S^t_\rho(\mathbf{x}_0)$ is the boundary of $B^t_\rho(\mathbf{x}_0)$, *i.e.* $S^t_\rho(\mathbf{x}_0) = \partial B^t_\rho(\mathbf{x}_0)$, and $\mathbf{n}$ is the unit exterior normal to $S^t_\rho(\mathbf{x}_0)$. Taking the supreme for $s \in [0, t]$ in (68), we achieve to

$$\sup_{0\le s\le t} B(\rho,s) + C_1E(\rho,t) \le B(\rho,0) + C_1I(\rho,t)\,, \quad (69)$$

where $C_1 = (m+1)k$. Noticing that

$$\frac{\partial E(\rho,t)}{\partial\rho} = \int_0^t \int_{S^t_\rho(\mathbf{x}_0)} |\nabla H^m|^p dS\,,$$

we obtain, by using Hölder's inequality,

$$|I(\rho,t)| \le (J(\rho,t))^{\frac{1}{p}} \left(\frac{\partial E(\rho,t)}{\partial\rho}\right)^{\frac{p-1}{p}}\,, \quad (70)$$

where

$$J(\rho,t) := \int_0^t J(\rho)dt, \quad J(\rho) := \int_{S^t_\rho} H^{mp}dS\,.$$

Now, we shall estimate $J(\rho,t)$ in terms of $E(\rho,t)$ and $B(\rho,t)$ by using Lemma 1. We apply (7) with $q = p$ and $\sigma = m$ to obtain

$$J(\rho) \le C^p \left(\|\nabla H^m\|_{\mathrm{L}^p(B_\rho)} + \rho^{-\delta}\|H\|^m_{\mathrm{L}^{m+1}(B_\rho)}\right)^{p\theta} \times \|H\|^{mp(1-\theta)}_{\mathrm{L}^{m+1}(B_\rho)}\,, \quad (71)$$

where, from (8) and (60),

$$\theta = 1 - \frac{n(3n+2)}{5n^2+n-2}\,, \qquad \delta = 1 + 2\frac{n^2-2n-2}{(3n+2)(n+1)}\,. \quad (72)$$

The analysis of (72) and knowing that Glen's exponent n is positive, we have

$$0 < \theta < 1 \quad \text{and} \quad \delta > 0 \quad \text{iff} \quad n > (1+\sqrt{17})/4\,. \quad (73)$$

Integrating (71) over the interval $(0, t)$ and then using Hölder's inequality and the algebraic inequality $(5)_2$

$$J(\rho,t) \le C_2 \left(E(\rho,t) + \rho^{-\delta\, p}\int_0^t B(\rho,s)^{\frac{pm}{m+1}}\, ds\right)^\theta \times \left(\int_0^t B(\rho,s)^{\frac{pm}{m+1}}\, ds\right)^{1-\theta}\,, \quad C_2 = (2^\theta C)^p\,. \quad (74)$$

We use

$$\int_0^t B(\rho,s)^{\frac{pm}{m+1}}\, ds \le t \sup_{0\le s\le t} B(\rho,s)^{\frac{pm}{m+1}}$$

and the algebraic inequality $(5)_2$ in (74), which yield

$$J(\rho,t)^{\frac{1}{p}} \le C_3 t^{(1-\theta)\frac{1}{p}} K(\rho,t)^\theta\,, \quad C_3 = (2C_2)^{1/p}\,, \quad (75)$$

$$K(\rho,t) := E(\rho,t)^{\frac{1}{p}} \left(\sup_{0\le s\le t} B(\rho,s)\right)^{\frac{m}{m+1}\frac{1-\theta}{\theta}} + \rho^{-\delta} t^{\frac{1}{p}} \left(\sup_{0\le s\le t} B(\rho,s)\right)^{\frac{m}{m+1}\frac{1}{\theta}}\,.$$

Then, we use the algebraic equality (6) on the term $K(\rho,t)$, and we obtain for any $\gamma \in (0,1)$

$$K(\rho,t) = E(\rho,t)^{\frac{1}{p}}\left(\sup_{0\le s\le t} B(\rho,s)\right)^{\gamma\, b}\left(\sup_{0\le s\le t} B(\rho,s)\right)^{(1-\gamma)\, b} + \rho^{-\delta} t^{\frac{1}{p}}\left(\sup_{0\le s\le t} B(\rho,s)\right)^{\frac{1}{p}+\gamma\, b}\left(\sup_{0\le s\le t} B(\rho,s)\right)^{\frac{m}{m+1}\frac{1}{\theta}-\left(\frac{1}{p}+\gamma\, b\right)}\,,$$

$$b = \frac{m}{m+1}\frac{1-\theta}{\theta}\,. \quad (76)$$

The following obvious relations $E(\rho,t) \ge 0$, $t \le T$, $\rho \le \rho_0$ and $(73)_2$ imply $B(\rho,t) \le B(\rho_0,t)$ and $\rho^\delta \le \rho_0^\delta$, and the algebraic inequality $(5)_1$, allow us to estimate (76) in the form

$$K(\rho,t) \le C_4\rho^{-\delta}\left(E(\rho,t) + \sup_{0\le s\le t} B(\rho,s)\right)^\alpha\,, \quad (77)$$

where

$$C_4 = \max\left\{1, \rho_0^\delta\right\} \max\left\{1, T^{\frac{1}{p}}\right\} \times \max\left\{\left(\sup_{0\le t\le T} B(\rho_0,t)\right)^\alpha, \left(\sup_{0\le t\le T} B(\rho_0,t)\right)^\beta\right\}\,,$$

$$\alpha = \frac{1}{p} + \gamma\frac{m}{m+1}\frac{1-\theta}{\theta}\,, \ \beta = \frac{m}{m+1}\frac{1}{\theta} - \left(\frac{1}{p} + \gamma\frac{m}{m+1}\frac{1-\theta}{\theta}\right)\,. \quad (78)$$

Applying successively (77) in (75) and the resulting inequality in (70), we prove that

$$|I(\rho,t)| \le C_5\rho^{-\delta\theta} t^{\frac{1-\theta}{p}} \times \left(E(\rho,t) + \sup_{0\le s\le t} B(\rho,s)\right)^\alpha \left(\frac{\partial\, E(\rho,t)}{\partial\,\rho}\right)^{\frac{p-1}{p}}\,, \quad (79)$$

where $C_5 = C_3C_4^\theta$ and α is given by $(78)_2$. Now, we shall use (79) in (69), but first we notice that, if $\rho \le \rho_0$, we are inside the ball B_{ρ_0} and, from (61), $B(\rho, 0) = 0$. In consequence, we obtain

$$\sup_{0\le s\le t} B(\rho,s) + E(\rho,t) \le C_6\rho^{-\delta\theta} t^{\frac{1-\theta}{p}} \times \left(\sup_{0\le s\le t} B(\rho,s) + E(\rho,t)\right)^\eta \left(\frac{\partial\, E(\rho,t)}{\partial\,\rho}\right)^{\frac{p-1}{p}}\,, \quad (80)$$

where $C_6 = C_1C_5/\min\{1, C_1\}$ and

$$\eta = \alpha\theta = \frac{\theta}{p} + \gamma\frac{m}{m+1}(1-\theta)\,. \quad (81)$$

Notice that from $(73)_1$, $0 < \eta < 1$. Rising both sides of (80) to the power $p/(p-1)$, we achieve to

$$\left(\sup_{0\le s\le t} B(\rho,s) + E(\rho,t)\right)^{1-\tau} \le C_7 \rho^{1-\nu} t^{\lambda} \frac{\partial E(\rho,t)}{\partial \rho}, \quad (82)$$

where $C_7 = C_6^{\frac{p}{p-1}}$ and

$$\begin{aligned} \tau &= \frac{1}{p-1}\left(\gamma p \frac{m}{m+1} - 1\right)(1-\theta), \\ \nu &= 1 + \delta\theta \frac{p}{p-1}, \quad \lambda = \frac{1-\theta}{p-1}. \end{aligned} \quad (83)$$

Gathering $(73)_1$ and $(83)_1$, we have $\tau > 0$ if an only if $\gamma > (m+1)/(pm)$. Therefore, by virtue of (60) and once that Glen's exponent n is positive, γ must be chosen in (76) such that

$$\frac{2n+3}{2(n+1)^2} < \gamma < 1. \quad (84)$$

Once that $B(\rho,s) \ge 0$, we obtain from (82) the ordinary differential inequality for the variable ρ

$$E(\rho,t)^{\tau-1} \frac{\partial E(\rho,t)}{\partial \rho} \ge C_8 \rho^{\nu-1} t^{-\lambda}, \quad C_8 = C_7^{-1}. \quad (85)$$

Notice that in (85) the variable t is considered as a parameter. Integrating (85) between ρ and $\rho_0 \ge \rho$, we get

$$E(\rho,t)^{\tau} \le E(\rho_0,t)^{\tau} - C_8 \frac{\tau}{\nu} t^{-\lambda} (\rho_0^{\nu} - \rho^{\nu})$$

In consequence, from (83) and (84), $E(\rho,t) = 0$ if and only if

$$\rho^{\nu} \le \rho^{\nu}(t) := \rho_0^{\nu} - \frac{\nu}{C_8 \tau} t^{\lambda} E(\rho_0,t)^{\tau}. \blacksquare$$

6.2 Waiting time property

Theorem 4 *Let H be a weak solution to the problem (2)-(4) with $a = 0$. Assume $\mathbf{u}_b$ satisfies (62) and (59) is finite. If additionally to (61), the following condition holds*

$$\int_{B_\rho(\mathbf{x}_0)} |H_0|^{m+1} d\mathbf{x} \le \varepsilon (\rho - \rho_0)^{\mu}, \quad (86)$$

for some $\rho > \rho_0$, $\mu = \mu(n) > 0$, $\varepsilon > 0$. Then, there exist t^*, $0 < t^* < T$, and $\varepsilon^* > 0$, $0 < \varepsilon \le \varepsilon^*$, such that $H(\mathbf{x},t) = 0$ a.e. in $B_{\rho_0}(\mathbf{x}_0)$, for all $t \in [0,t^*]$, provided $n > (1+\sqrt{17})/4$.

PROOF. We proceed as we did in the proof of Theorem 3 (see (69)) and we obtain, after using the assumption (86),

$$B(\rho,s) + K_1 E(\rho,s) \le \varepsilon(\rho-\rho_0)^{\mu} + K_1 I(\rho,s) \quad (87)$$

where $K_1 = (m+1)k$ and ε is a positive constant to be defined later on. Using the same reasoning as we did to prove (79), we obtain

$$\begin{aligned} &|I(\rho,t)| \le K_2 t^{\frac{1-\theta}{p}} \times \\ &\left(E(\rho,t) + \sup_{0\le s\le t} B(\rho,s)\right)^{\eta} \left(\frac{\partial E(\rho,t)}{\partial \rho}\right)^{\frac{p-1}{p}}, \end{aligned} \quad (88)$$

where $K_2 = C_3 C_4^{\theta}$, η is given in (81), C_3 in (75) and C_4 has the form given in (77) but with the term $\max\{1,\rho_0^{\delta}\}$ replaced by $\max\left\{1,\rho_0^{-\delta}\right\}$ - notice that, in this case, $\rho_0 \le \rho$. Gathering (88) and (87), we obtain

$$\begin{aligned} &K_3\left(E(\rho,t) + \sup_{0\le s\le t} B(\rho,s)\right) \le \varepsilon(\rho-\rho_0)^{\mu} + \\ &K_2 t^{\frac{1-\theta}{p}} \left(E(\rho,t) + \sup_{0\le s\le t} B(\rho,s)\right)^{\eta} \left(\frac{\partial E(\rho,t)}{\partial \rho}\right)^{\frac{p-1}{p}}, \end{aligned} \quad (89)$$

where $K_3 = \min\{1, K_1\}$. Using Young's inequality with $\epsilon = K_3/(2K_2)$ in the second term of the right-hand side of (89) and knowing that $B(\rho,s) \ge 0$, we obtain

$$E(\rho,t) \le \varepsilon K_4 (\rho-\rho_0)^{\mu} + K_5 \left(t^{\frac{1-\theta}{p-1}} \frac{\partial E(\rho,t)}{\partial \rho}\right)^{\frac{1}{\xi}}, \quad (90)$$

where $K_4 = 2K_3^{-1}$, $K_5 = 2K_2 K_3^{-1} C(\epsilon)$ and

$$\xi = \frac{p}{p-1}(1-\eta) = 1 + \left(1 - \gamma p \frac{m}{m+1}\right) \frac{1-\theta}{p-1}. \quad (91)$$

From $(60)_1$ and (81), we have $0 < \xi < 1$. Rising both sides of (90) to the power ξ and defining

$$\mu := \frac{1}{1-\xi} = \frac{(p-1)(m+1)}{[\gamma pm - (m+1)](1-\theta)}, \quad (92)$$

we obtain the ordinary differential inequality in the variable ρ

$$E(\rho,t)^{\xi} \le \varepsilon K_6 (\rho-\rho_0)^{\frac{\xi}{1-\xi}} + K_7 \frac{\partial E(\rho,t)}{\partial \rho}, \quad (93)$$

where $K_6 = (2\varepsilon K_4)^{\xi}$ and $K_7 = (2K_5)^{\xi} t^{\frac{1-\theta}{p-1}}$. Notice that, from (91) and (92), $\mu > 0$. The analysis of (93) shows us that its solutions are of the form $E(\rho,t) = K(\rho-\rho_0)^{1/(1-\xi)}$, where the constant K should satisfies to

$$\left(K^{\xi} - K_6 - \frac{K_7}{1-\xi} K\right) (\rho-\rho_0)^{\frac{\xi}{1-\xi}} \le 0.$$

Finally, once that $\rho_0 \le \rho$, we obtain $E(\rho,t) = 0$ if and only if $K^{\xi} - K_6 - K_7/(1-\xi)K \ge 0$. Writing this inequality in terms of t and ε, we obtain $E(\rho,t) = 0$ for all t such that

$$t \le \left(\frac{K^{\xi} - (2\varepsilon K_4)^{\xi}}{(2K_5)^{\xi} K}(1-\xi)\right)^{\frac{p-1}{1-\theta}} := t^*$$

and ε is chosen such that $\varepsilon \le K/(2K_4) := \varepsilon^*$. $\blacksquare$

The results of Theorem 3 and 4 are still valid for a global non-zero accumulation/ablation rate function a. Indeed, finite speed of propagations property (Theorem 3) holds, provided we assume $a = 0$ in $B_{\rho_0}(\mathbf{x}_0) \times [0,t^*]$. As for the waiting time property (Theorem 4), it holds if we assume $a = 0$ in $B_{\rho_0}(\mathbf{x}_0) \times [0,T]$. Since $H = h - b$, we can analyze the properties of finite speed of propagation and waiting time, established in Theorem 3 and 4, in terms of the ice top surface $z = h$. As discussed in Fowler [11, 12] the slope of the surface $z = h$ is singular in advance but finite in retreat. This distinction causes the finite speed of propagations and the waiting time behaviors. Indeed, after a local retreat, the margin slope must rebuilt itself before another advance it is possible.

7 Conclusions

Ice sheets are vast and slow-moving edifices of solid ice, which flow under their own weight by solid state creep processes such as the creep of dislocation in the crystalline lattice structure of the ice. Experimental results have showed different creep curves (graph of strain versus time) when a polycrystalline aggregate of ice is subjected to a constant stress. An elastic deformation is followed by a small period of transient or primary creep in which the strain rate decreases continuously until a minimum value. Then the secondary creep is reached and it remains for a long period as the strain rate is approximately constant. After that, the strain rate increases and tertiary creep is reached. If the test is carried on for long enough, a steady value is reached. In good approximation, we can assume that in deforming ice masses like ice sheets, secondary creep prevails for low temperatures (below -10$^{\text{o}}$C), whereas tertiary creep prevails for higher temperatures. Numerous laboratory experiments have shown that, for secondary creep, the relation between shear strain rate and shear stress is given by Glen's law (11), where n varies from about 1.9 to 4.8. However, most of evidence from either laboratory tests and deformation measurements in Antarctic and Greenland have showed that $n = 3$ is more appropriated (see Hutter [14] and Paterson [19]). The results of Theorem 3 and 4 hold provided $n > (1 + \sqrt{17})/4 \simeq 1.3$ and from experiments $n \in [1.9, 4.8]$, being $n = 3$ usually accepted as the best choice in ice-sheet modeling.

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References:

[1] H.W. Alt and S. Luckhaus. Quasilinear elliptic-parabolic differential equations. Math. Z. **183** (1983), no. 3, 311-341.

[2] S. N. Antontsev, J. I. Díaz and S. I. Shmarev. *Energy methods for free boundary problems*, Progr. Nonlinear Differential Equations Appl. **48**, Birkhäuser, 2002.

[3] S. N. Antontsev and H. B. de Oliveira. Localization of weak solutions for non-Newtonian fluid flows. *Proceedings of the CMCE Congress*. LNEC, Lisbon (2004), 15 pp.

[4] S. N. Antontsev and H. B. de Oliveira. On a mathematical model in ice sheet dynamics. Proceedings of the 5th IASME/WSEAS Conference, WSEAS, Athens (2007), 1-8.

[5] S. N. Antontsev, J.I. Díaz and H. B. de Oliveira. Mathematical models in dynamics of non-Newtonian fluids and in glaciology. *Proceedings of the CMNE/CILAMCE Congress*. Universidade do Porto, Porto (2007), 20 pp.

[6] Ph. Benilan and P. Wittbold. On mild and weak solutions of elliptic-parabolic problems. *Adv. Differential Equations* **1** (1996), no. 6, 1053-1073.

[7] N. Calvo, J.I. Díaz, J. Durany, E. Schiavi and C. Vázquez. On a doubly nonlinear parabolic obstacle problem modelling ice sheet dynamics. *SIAM J. Appl. Math.* **63**, 2 (2002), 683-707.

[8] J. Carrillo and P. Wittbold. Uniqueness of renormalized solutions of degenerate elliptic-parabolic problems. *J. Differential Equations* **156** (1999), no. 1, 93-121.

[9] J.I. Díaz, A.I. Muñoz and E. Schiavi. Existence of weak solutions to a system of nonlinear partial differential equations modelling ice streams. *Nonlinear Anal. Real World Appl.* **8** (2007), no. 1, 267-287.

[10] J.I. Díaz and L. Veron. Local vanishing properties of solutions of elliptic and parabolic quasilinear equations. *Trans. Amer. Math. Soc.* **290** (1985), no. 2, 787-814

[11] A.C. Fowler. Modelling ice sheet dynamics. *Geophys. Astrophys. Fluid Dynam.* **63**, 1-4 (1992), 29-65.

[12] A.C. Fowler. Glaciers and ice sheets. *NATO ASI Ser. Ser. I Glob. Environ. Change*, **48**, Springer, Berlin, 1997, 301-336.

[13] R. Greve. A continuum-mechanical formulation for shallow polythermal ice sheets. *Philos. Trans. R. Soc. Lond., A* **355** (1997), 921-974.

[14] K. Hutter. *Theoretical Glaciology.* D. Reidel Publishing Company, Dordrecht, 1982.

[15] P. Huybrechts. A 3-D model for the Antartctic ice sheet: A sensitivy study on the glacial-interglacial contrast. *Clim. Dyn.* **5** (1990), 79-92.

[16] A.V. Ivanov and J.F. Rodrigues. Existence and uniqueness of a weak solution to the initial mixed boundary value problem for quasilinear elliptic-parabolic equations. English translation in *J. Math. Sci.* **109** (2002), no. 5, 1851-1866.

[17] A.M. Meirmanov, V.V. Pukhnachov and S.I. Shmarev. *Evolution equations and Lagrangian coordinates.* Walter de Gruyter, Berlin, 1997.

[18] F. Otto. L^1-contraction and uniqueness for quasilinear elliptic-parabolic equations. *J. Differential Equations* **131** (1996), no. 1, 20-38.

[19] H. Paterson. *The Physics of Galciers.* Third Edition. Pergamon, Oxford, 1994.

[20] F. Patyn. A new three-dimensional higher-order thermomechanical ice sheet model: Basic sensitivity, ice stream development, and ice flow across subglacial lakes. *J. Geophys. Res.* **108** (B8), 2382, doi:10.1029/2002JB002329, 2003.

[21] J.F. Rodrigues and L. Santos. Some free boundary problems in theoretical glaciology. *NATO ASI Ser. Ser. I Glob. Environ. Change*, **48**, Springer, Berlin, 1997, 337-364.