On the Ermanno-Bernoulli and Quasi-Ermanno-Bernoulli constants for linearizing dynamical systems

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ABSTRACT

It is well known that the Ermanno-Bernoulli constants derived from the Laplace-Runge-Lenz vector of dynamical systems are efficiently used to reduce them to a system of harmonic oscillator(s) and conservation law in the context of point and nonlocal symmetries of dynamical systems. In this paper, we review Ermanno-Bernoulli constants and observe that one can also use analogous constants obtained from the Hamilton vector of dynamical systems to serve the same purpose. We report the generic natural variables for reducing such dynamical systems in two-dimensions and three-dimensions to a system of one harmonic oscillator and two harmonic oscillators respectively, and a conservation law with some examples. We also note that the symmetry groups obtained from the reduced systems using the alternative constants are realizations of those obtained from Ermanno-Bernoulli constants. We also report here that the symmetries of the original dynamical systems can be obtained from symmetries of the reduced systems.

Key words: Quasi, Constants, Symmetries, Vectors, Conservation, Oscillators, Laplace-Runge-Lenz, Hamilton, Coordinates, Dynamical.

1 Introduction

In the discussion of the Lie symmetries, first integrals and linearization of dynamical systems using the Kepler problem as a vehicle, Leach and Andriopoulos (2003) used the representation of the Cartesian components of the angular momentum $\mathbf{L}$ and Laplace-Runge-Lenz vector $\mathbf{J}$ in the polar (r,θ,ϕ) coordinate systems, and obtained the complete symmetry representation by the Nucci-reduced technique and the alternate derivation of the reduction through manipulation of the equation of motion and the realizations of the Lie symmetry algebra of a certain linear system. This suggested the Ermanno-Bernoulli constants obtained from the Laplace-Runge-Lenz vector as tools for reducing the equation of motion to systems of harmonic equations and a conservation law. This phenomenon simplified the reduction of the Kepler problem from the sixth-order nonlinear system to a fifth-order system comprising of the above linear system by virtue of using the conserved vector $\mathbf{J}$ and $\mathbf{L}$. This analysis was carried out by Leach and Nucci (2004) on the MICZ-Kepler problem which possesses instead of the conserved $\mathbf{L}$, the Poincaré vector constant of motion $\mathbf{P}$. The symmetry algebra of the reduced system is well known [1, 2, 3, 4]. In the sequel, we report herein that this idea can be extended to other dynamical systems describing motion in a plane and which admits Laplace-Runge-Lenz vector. We also report that alternative constants for reducing the dynamical system using the Hamilton vector exist. The symmetry algebraic structures of the reduced systems are well known in the Lie symmetry analysis of dynamical systems. The paper is organized as following. In section **2** we introduced the general technique for reducing general dynamical systems using the Ermanno-Bernoulli constants and the alternative constants. In sections **3** specific examples for two-dimensions are given. In section **4** we give examples of the computations of exact symmetries transformations using the Lie symmetries of the reduced system viz Kepler problem and the generalized Kepler problem. Section **5** outlines the same details for the general dynamical systems in three-dimensions. Section **6** contains concluding remarks.

2 General ideas of the Ermanno-Bernoulli constants

The most general form of a dynamical system describing motion in a plane is

$$\ddot{\mathbf{x}} = p_1\mathbf{x} + p_2(\mathbf{L}\wedge\mathbf{x}) = F(\mathbf{x},\dot{\mathbf{x}}), \qquad (1)$$

where $p_i = p_i(\mathbf{x},\dot{\mathbf{x}})$ are functions of their arguments and $\mathbf{L} = \mathbf{x}\wedge\dot{\mathbf{x}}$. We consider those systems (1) which possess a Laplace-Runge-Lenz (LRL) vector of the form

$$\mathbf{J} = g_1\mathbf{L}\wedge\dot{\mathbf{x}} + g_2\mathbf{x} + g_3(\mathbf{L}\wedge\mathbf{x}), \qquad (2)$$

where $g_1 = g_1(L),\ g_2 = g_2(\mathbf{x},\dot{\mathbf{x}}),$ and $g_3 = g_3(\mathbf{x},\dot{\mathbf{x}})$ are functions of their arguments, and $L = |\mathbf{L}|$. We note that $\mathbf{L}$ satisfies the equation of motion [7]

$$\dot{\mathbf{L}} = p_2\mathbf{x}\wedge(\mathbf{L}\wedge\mathbf{x}) = r^2p_2\mathbf{L}\ .$$

i.e. $\dot{\mathbf{L}} = r^2p_2\mathbf{L}$, (3)

and $\dot{L} = r^2p_2L$.

The second equation in (3) implies that the unit vector $\hat{\mathbf{L}}$ parallel to the angular momentum $\mathbf{L}$ is constant, and that the system (1) describes motion in a plane perpendicular to $\hat{\mathbf{L}}$. We write $\mathbf{L} = L\mathbf{k}$, and $\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j}$ where $\mathbf{k} = \hat{\mathbf{L}}$ and $\mathbf{i},\mathbf{j}$ are two fixed orthogonal unit vectors in the plane of motion. The expression for $\mathbf{J}$ in (2) reduces to

$$\begin{aligned}\mathbf{J} = g_1L(-\dot{x}_2\mathbf{i} + \dot{x}_1\mathbf{j}) + g_2(x_1\mathbf{i} + x_2\mathbf{j})\\ + g_3L(-x_2\mathbf{i} + x_1\mathbf{j})\end{aligned} \qquad (4)$$

so that

$$J_\pm = J_1 \pm iJ_2 = \pm ig_1L(\dot{x}_\pm) + g_2(x_\pm) \pm ig_3L(x_\pm)\ , \qquad (5)$$

where

$$x_\pm = (x_1 \pm ix_2), \dot{x}_\pm = (\dot{x}_1 \pm i\dot{x}_2)\ .$$

Setting $x_1 = r\cos\theta, x_2 = r\sin\theta$ where (r,θ) are polar coordinates in the plane of motion, $J_\pm$ reduce to

$$\begin{aligned}&\pm ig_1L(\dot{r} \pm ir\dot{\theta})e^{\pm i\theta} + g_2re^{\pm i\theta} \pm ig_3Lre^{\pm i\theta}\\ &\quad= (\pm ig_1L(\dot{r} \pm ir\dot{\theta}) + g_2r \pm ig_3Lr)e^{\pm i\theta}\\ &\quad= (-g_1Lr\dot{\theta} \pm igL\dot{r} + g_2r \pm ig_3Lr)e^{\pm i\theta}\\ &\quad= [(-g_1Lr\dot{\theta} + g_2r) \pm i(g_1L\dot{r} + g_3Lr)]e^{\pm i\theta}\\ &\quad= [(-g_1L^2r^{-1} + g_2r) \pm i(g_1L\dot{r} + g_3Lr)]e^{\pm i\theta}\ .\end{aligned}$$

Hence we have that

$$J_\pm = (w \pm iw_*)e^{\pm i\theta}\ . \qquad (6)$$

Where

$w = -g_1L^2r^{-1} + g_2r$, and $w_* = g_1L\dot{r} + g_3Lr$.

So $\dot{J}_\pm = 0$ implies that

$$\dot{w} \pm i\dot{w}_* + (w \pm iw_*)i\dot{\theta} = 0\ . \qquad (7)$$

That is

$$\dot{w} - \dot{\theta}w_* = 0\ , \dot{w}_* + \dot{\theta}w = 0\ . \qquad (8)$$

Taking θ as new independent variable equation (8) gives

$$w' = w_*\ , w'_* = -w\ \Rightarrow w'' + w = 0\ . \qquad (9)$$

We assume also that the equation of motion for L in equation (3) possesses the solution

$$V(r,\theta,L) = const\ . \qquad (10)$$

Then the pair $(v_1, v_2) = (w, V)$ satisfies the equations

$$\begin{aligned}v_1'' + v_1 &= 0\\ v_2' &= 0\ .\end{aligned} \qquad (11)$$

The two-dimensional form of the dynamical systems (1) is equivalent to (11). The constants $J_\pm$ are called the Ermanno-Bernoulli constants.

The Hamilton vector $\mathbf{K} = \hat{\mathbf{L}}\wedge\mathbf{J}$ can also be used to obtain Quasi- Ermannor-Bernoulli constants given by $K_\pm = K_1 \pm iK_2 = (\omega_1 \pm i\omega_1')e^{\pm i\theta}$. This reduces (1) to

$$\begin{aligned}\omega_1'' + \omega_1 &= 0\\ v_2' &= 0\ .\end{aligned}$$

It turns out that in two dimensions $K_{\pm} = \pm iJ_{\pm}$ i.e. $\omega_1 = -v_1'$ and $\omega_1' = v_1$, this corresponds to the reduced system by the Ermanno-Bernoulli constants. However this is not the case when they are expressed in three- dimensional coordinates.

3 Some well known examples in two-dimensions

3.1 The Kepler problem

The equation of the motion is

$$\ddot{\mathbf{x}} = -\mu r^{-3}\mathbf{x}\,;\ |\mathbf{x}| = r\,. \qquad (12)$$

The Laplace-Runge-Lenz vector is $\mathbf{J} = \dot{\mathbf{x}} \wedge \mathbf{L} - \mu r^{-1}\mathbf{x}$.

The Ermanno-Bernoulli constants $J_{\pm} = (v_1 \pm iv_1')e^{\pm i\theta}$ yield the expression $v_1 = \frac{L^2}{r} - \mu$, and equation (3) becomes $\dot{L} = 0$. That is L is constant and v_2 is simply $L = r^2\dot{\theta}$.

3.2 The Generalized Kepler problem

The equation of motion given by

$$\ddot{\mathbf{x}} - \tfrac{1}{2}\left(\frac{\dot{g}}{g} + 3\frac{\dot{r}}{r}\right)\dot{\mathbf{x}} + \mu g\mathbf{x} = 0\,. \qquad (13)$$

This possesses the Laplace-Runge-Lenz vector $\mathbf{J} = L^{-2}(\dot{\mathbf{x}} \wedge \mathbf{L}) - \mu A^{-2} r^{-1}\mathbf{x}$, where $A = \left(\frac{r}{g}\right)^{\frac{1}{2}}\dot{\theta}$. The corresponding Ermanno-Bernoulli constants are $J_{\pm} = (v_1 \pm iv_1')e^{\pm i\theta}$, where

$$v_1 = \frac{1}{r} - \mu A^{-2}\,. \qquad (14)$$

In the calculations to follow in sec. 4, we will use $v_1 = A^2 r^{-1} - \mu$ which is a constant multiple of (14) and which coincides with the expression for v_1 in the Kepler problem in **3.1** when $gr^3 = 1$.The corres-pondding equation (3) becomes

$$\dot{L} - \tfrac{1}{2}\left(\frac{\dot{g}}{g} + \frac{3\dot{r}}{r}\right)L = 0\,. \qquad (15)$$

Multiplying (15) by $(gr^3)^{-\frac{1}{2}}$ we obtain the equation $\left[(gr^3)^{-\frac{1}{2}}L\right]^{\cdot} = 0$. That is $(gr^3)^{-\frac{1}{2}}L = const$. This constant is A, and hence we make the choice

$$A = v_2 = \left(\frac{r}{g}\right)^{\frac{1}{2}}\dot{\theta}\,. \qquad (16)$$

3.3 Dynamical systems with symmetry groups of the inverse cube law equation

The most general dynamical system which possesses the Lie symmetry group of the equation $\ddot{\mathbf{x}} = Gr^{-4}\mathbf{x}$ and which also describes motion in a plane is [5, 6]

$$\ddot{\mathbf{x}} = P_1 r^{-4}\mathbf{x} + P_2 r^{-4}(\mathbf{L} \wedge \mathbf{x})\,, \qquad (17)$$

where P_1 and P_2 are functions of L. Using (3), equation of motion for L in polar coordinates (r,θ) is

$$\dot{L} = P_2 r^{-2} L = P_2\dot{\theta}\,. \qquad (18)$$

That is $P_2^{-1}\dot{L} = \dot{\theta}.$

So defining

$$H(L) = \int^L P_2^{-1}(L')dL'\,, \qquad (19)$$

where $'$ denotes that the integrand is evaluated at time variable t'. Then the expression (18) implies that $(H(L) - \theta)^{\cdot} = 0$, therefore we take as $V(r,\theta,L)$ in (10) the expression

$$H(L) - \theta = const. \qquad (20)$$

The system (17) is known to possess the Hamilton vector $\mathbf{K}$ and the Laplace-Runge-Lenz vector $\mathbf{J}$ given by [5]

$$\mathbf{K} = (L^{-\frac{1}{2}}\mathbf{x})^{\cdot} = L^{-\frac{1}{2}}(\dot{\mathbf{x}} - \tfrac{1}{2}P_2 r^{-1}\mathbf{x})\,,$$

and $\mathbf{J} = -\hat{\mathbf{L}} \wedge \mathbf{K}$, (21)

provided that

$$P_1 = \tfrac{1}{2}P_2(LP_2' + \tfrac{1}{2}P_2)\,. \qquad (22)$$

The expressions for the constants

$$K_{\pm} = K_1 \pm iK_2\,, \qquad (23)$$

$$J_{\pm} = J_1 \pm iJ_2\,, \qquad (24)$$

turn out to be

$$K_{\pm} = L^{-\frac{1}{2}}[(\dot{r} - \tfrac{1}{2}P_2 r^{-1}) \pm ir^{-1}L]e^{\pm i\theta}\,, \qquad (25)$$

$$J_{\pm} = L^{-\frac{1}{2}}[r^{-1}L \mp i(\dot{r} - \tfrac{1}{2}P_2 r^{-1})]e^{\pm i\theta}\,. \qquad (26)$$

Using (10), (20) and (26) the expression for (v_1, v_2) in (11) are given by

$$v_1 = r^{-1}L^{\frac{1}{2}},\ v_2 = H(L) - \theta\ .$$

4 Exact symmetries of the Kepler problem and the generalized Kepler problem in two-dimensions

We report here that the symmetries of dynamical systems can be accurately calculated from the Lie symmetries of the reduced systems. We shall only give few actual calculations. The Lie symmetry generators of the reduced dynamical system (11) are as follows:

$V_1 = v_2\partial_2$; $V_2 = \partial_\theta$; $V_3 = v_1\partial_1$; $V_{4\pm} = e^{\pm i\theta}\partial_1$; $V_{6\pm} = e^{\pm 2i\theta}[\partial_\theta \pm iv_1\partial_1]$; $V_{8\pm} = e^{\pm i\theta}[v_1\partial_\theta \pm iv_1^2\partial_1]$, where $\partial_i = \partial/\partial v_i$.

4.1 The Kepler Problem in 2-dimensions

We now proceed to find the symmetry transformations generated by the vector field $\alpha V_1 = \alpha v_2\partial_2$ in two-dimensions. The flow of the vector field is the function

$$f(v_1, v_2, \theta) = (\bar{v}_1, \bar{v}_2, \bar{\theta})$$

$$\text{where } \frac{d\bar{v}_1}{d\lambda} = 0\ ;\ \frac{d\bar{v}_2}{d\lambda} = \alpha\bar{v}_2\,; \frac{d\bar{\theta}}{d\lambda} = 0\,. \qquad (26)$$

Solving system (26) we have the following

$$\bar{v}_1 = v_1\ \ ;\ \ \bar{v}_2 = e^{\alpha\lambda}v_2\,;\bar{\theta} = \theta\,. \qquad (27)$$

The second equation in (27) implies $\bar{L} = CL$ while the first equation implies that

$$\bar{L}^2\bar{r}^{-1} - \mu = L^2r^{-1} - \mu$$

i.e. $C^2L^2\bar{r}^{-1} - \mu = L^2r^{-1} - \mu$

where $C = e^{\alpha\lambda}$, then

$$\frac{\bar{r}}{r} = C^2 \Rightarrow \bar{r} = C^2r \qquad (28)$$

$\bar{L} = CL$ implies that

$$\dot{\bar{\theta}}\bar{r}^2 = C\dot{\theta}r^2 \Rightarrow \bar{r}^2\frac{d\bar{\theta}}{d\bar{t}} = Cr^2\frac{d\theta}{dt},$$

this implies that

$$\frac{d\bar{t}}{dt} = C^3, \qquad (29)$$

i.e. $\bar{t} = d + C^3t$, where d is an arbitrary constant.

Consequently the exact symmetry transformations generated by the vector field above for the Kepler problem is given by equations (28) and (29). If $(x_1, x_2) = (r\cos\theta, r\sin\theta)$ denotes the Cartesian coordinates of $\mathbf{x}$ in the plane of motion then $\bar{\theta} = \theta$ implies that

$$\bar{\mathbf{x}} = C^2\mathbf{x}\,, \qquad (30)$$

where $\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j}$ is the two dimensional Cartesian vector. The transformation defined by (29) and (30) is also a three-dimensional symmetry transformation of the Kepler problem when $\mathbf{x}$ is made three-dimensional. We note that the vector fields αV_2 and $V_4 = (\alpha_1\cos\theta + \alpha_2\sin\theta)\partial_1$ also generate symmetry transformations with $\bar{\theta} = \theta$, $\bar{v}_2 = v_2$ i.e $\bar{L} = L$.

Applying the same manner of calculations we obtain the symmetry transformations f_2, f_4 given by

$$f_i(\mathbf{x}, t) = (\bar{\mathbf{x}}, \bar{t})$$

$$\bar{\mathbf{x}} = H_i^{-1}\mathbf{x}\,;\ \frac{d\bar{t}}{dt} = H_i^{-2}$$

where

$$H_2 = \mu L^{-2}r + C(1 - \mu L^{-2}r)$$

$$H_4 = 1 + \lambda L^{-2}\boldsymbol{\alpha}\cdot\mathbf{x}\,,\ C = e^{\alpha\lambda}\,,$$

$$\boldsymbol{\alpha}\cdot\mathbf{x} = \alpha_1x_1 + \alpha_2x_2\ . \qquad (31)$$

The transformations f_2, f_4 are again symmetry transformations when $\mathbf{x}$ and $\boldsymbol{\alpha}\cdot\mathbf{x}$ are made three-dimensional.

4.2 Generalized Kepler Problem in 2-dimensions

Using the expressions for v_1 and v_2 just after (14) and (16) i.e. $v_1 = A^2r^{-1} - \mu, v_2 = A$, the symmetry transformation generated by the vector field $\alpha V_1 = \alpha v_2\partial_2$ is given by

$$\bar{\mathbf{x}} = C^2\mathbf{x}\,,\quad \frac{d\bar{t}}{dt} = \sqrt{\frac{g(r)}{g(C^2r)}} \qquad (32)$$

where $C = e^{\alpha\lambda}$.

The vector fields αV_i $i = 2,...,9$ generate transformation where $\bar{v}_2 = v_2$ which implies that

$$\frac{d\bar{t}}{dt} = \left\{\frac{\bar{r}}{r}\frac{g(r)}{g(C^2r)}\right\}^{\frac{1}{2}} \qquad (33)$$

In particular the transformations f_2, f_4 generated by αV_2, and $V_4 = (\alpha_1 \cos\theta + \alpha_2 \sin\theta)\partial_1$ are given by $f_i(\mathbf{x},t) = (\bar{\mathbf{x}},\bar{t})$ where $\bar{t}$ satisfies (33) and $\bar{\mathbf{x}}$ is given by $\bar{\mathbf{x}} = H_i'^{-1}\mathbf{x}$; $i = 2,4$ (34)

where H_2', H_4' are obtained from H_2, H_4 in (31) by replacing L by A.

5 The case of three-dimension motion

The reduction of Kepler problem and the MICZ problem to a system of harmonic oscillators and a conservation law is well known [1, 2]. But it is not reported in the literature the generic natural variables for the reduction of the general dynamical systems of the form (1) with a LRL vector (2). We report in this section the generic natural variables for reducing such systems. In the three dimensional case when expressed in spherical coordinates (r,θ,ϕ) the Ermanno-Bernoulli constants are given by

$$J_\pm = J_1 \pm iJ_2 = (u_1 \pm iu_1')e^{\pm i\phi}. \quad (35)$$

Similarly, the Quasi- Ermanno-Bernoulli constants are of the form

$$K_\pm = K_1 \pm iK_2 = (\upsilon_1 \pm i\upsilon_1')e^{\pm i\phi}. \quad (36)$$

The direction of the angular momentum $\hat{\mathbf{L}}$ in the Cartesian coordinates is given by

$$\hat{\mathbf{L}} = \hat{L}_1\mathbf{i} + \hat{L}_2\mathbf{j} + \hat{L}_3\mathbf{k}.$$

Similar to $\mathbf{J}$, the constancy of $\hat{\mathbf{L}}$ i.e. $(\hat{\mathbf{L}})^{\cdot} = 0$ implies that there exists a function u_2 (of $\mathbf{x}$ and $\dot{\mathbf{x}}$) such that

$$\hat{L}_\pm = \hat{L}_1 \pm i\hat{L}_2 = (u_2 \pm iu_2')e^{\pm i\phi}. \quad (37)$$

Consequently, we have that

$$L_\pm = L_1 \pm iL_2 = L(u_2 \pm iu_2')e^{\pm i\phi}. \quad (38)$$

We assume that in terms of spherical coordinates the solution to the second equation in (3) (correspondding to (10)) takes the form $W(r,\theta,\phi,L) = const.$ i.e. W depends on $\dot{\mathbf{x}}$ only through L (the expression for L is given by $L^2 = r^4(\dot\theta^2 + \sin^2\theta\dot\phi^2)$).

Defining $u_3 = W$, the triple (u_1,u_2,u_3) satisfies the coupled equations,

$$u_1'' + u_1 = 0$$
$$u_2'' + u_2 = 0 \quad (39)$$
$$u_3' = 0.$$

The expressions for $L_\pm$, are given in [1], they are as follows

$$L_\pm = L_1 \pm iL_2 = (w_2 \pm iw_2')e^{\pm i\phi}. \quad (40)$$

where $w_2 = r^2\dot\phi\sin\theta\cos\theta$, $w_2' = -r^2\dot\theta$.

Thus the expressions for u_2, u_3 in (38) and (39) are as follows

$$\hat{L}_\pm = \hat{L}_1 \pm i\hat{L}_2 = L^{-1}(L_1 \pm iL_2) = (u_2 \pm iu_2')e^{\pm i\phi}, \quad (41)$$

where $u_2 = L^{-1}r^2\dot\phi\sin\theta\cos\theta$, $u_2' = -L^{-1}r^2\dot\theta$.

In ref. 1, which deals with the Kepler problem, Leach et al reported that $(u_2,u_3) = (w_2, L_3)$ where $L_3 = L\hat{L}_3$. For this problem there are several choices of these variables viz $u_2 = w_2$ or $L^{-1}w_2$, $u_3 = L_3, \hat{L}_3$ or L. This is not the case with dynamical systems describing motion in a plane where L is not constant. We note that the choice $u_3 = \hat{L}_3$ analogous to (38) cannot be made along with $u_2 = L^{-1}w_2$ since in this case $\hat{L}_3^{\,2} = 1 - \hat{L}_1^{\,2} - \hat{L}_2^{\,2} = 1 - (u_2^2 + u_2'^2)$ (u_3 must be functionally independent of u_1, u_1', u_2, u_2'). Using the expression for $\mathbf{J}$ just after (13) the generalized Kepler problem possesses the three-dimensional Ermanno-Bernoulli constants $J_\pm = (u_1 \pm iu_1')e^{i\phi}$ where $u_1 = \left(\frac{1}{r} - \mu A^{-2}\right)\sin\theta - L^{-2}r^2\dot r\dot\theta\cos\theta$

and $u_1' = -L^{-2}r^2\dot r\dot\phi\sin\theta$. (42)

By replacing A by L in (42), we obtain the corresponding variables for the Kepler problem as

$$u_1 = \left(\frac{1}{r} - L^{-2}\mu\right)\sin\theta - L^{-2}r^2\dot r\dot\theta\cos\theta$$

and $u_1' = -L^{-2}r^2\dot r\dot\phi\sin\theta$. (43)

5.1 Exact symmetry transformations of the Kepler problem in 3-dimensions

We report here also that the symmetries of dynamical systems in three-dimensions can be obtained from the Lie symmetries of the reduced systems as in section **4**. We list here the Lie symmetry generators of the reduced system (39).

They consist of sixteen generators, one viz Γ_1 for the conservation law $u_3' = 0$ and the fifteen Lie symmetry generators for the pair of harmonic oscillators in (39). They are as follows

$\Gamma_1 = u_3\partial_3$, $\Gamma_2^{jk} = u_j\partial_k$,

$\Gamma_3 = \partial_\phi$, $\Gamma_4^j = e^{i\phi}\partial_j$,

$\Gamma_5 = e^{\pm 2i\phi}(\partial_\phi + i\mathbf{u}\cdot\partial)$,

$$\Gamma_6^j = e^{\pm i\phi}u_j(\partial_\phi + i\mathbf{u}\cdot\partial) \quad (44)$$

where $j,k = 1,2; \partial_j = \partial/\partial u_j$ and $\mathbf{u}\cdot\partial = u_1\partial_1 + u_2\partial_2$.

We now compute the symmetry transformation generated by the vector field $\Gamma_2^{11} = u_1\partial_1$ for the Kepler problem. The symmetry transformation generated by this vector field is the transformation f given by $(\bar{u}_j, \bar{\phi}) = f(u_j, \phi)$ where

$$\bar{u}_1 = Cu_1, \bar{u}_2 = u_2,\ \bar{u}_3 = u_3,\ \bar{\phi} = \phi, C = e^{\alpha\lambda} \quad (45)$$

from which it follows that

$$\bar{u}_1' = Cu_1',\ \bar{u}_2' = u_2',\ \bar{L} = L\ . \quad (46)$$

The relations in (46) imply that

$u_2^2 \sec^2\theta + (u_2')^2 = 1$.

Thus from the invariance of u_2 and u_2' in (44) and (45) we deduce $\sec\bar{\theta} = \sec\theta$,

$$\text{i.e. } \bar{\theta} = \theta. \quad (47)$$

The relations in (43) imply that

$$u_1 = \left(\frac{1}{r} - \mu L^{-2}\right)\sin\theta - u_1'\theta'\cot\theta\ . \quad (48)$$

Since L, θ' and $\cot\theta$ are invariants of this transformation, the first relation in (45) becomes

$$\left(\frac{1}{\bar{r}} - \mu L^{-2}\right)\sin\theta - \bar{u}_1'\theta'\cot\theta =$$

$$C\left(\frac{1}{r} - \mu L^{-2}\right)\sin\theta - Cu_1'\theta'\cot\theta, \quad (49)$$

which reduces to

$$\left(\frac{1}{\bar{r}} - \mu L^{-2}\right) = C\left(\frac{1}{r} - \mu L^{-2}\right); \quad (50)$$

i.e. $\bar{r} = H_2^{-1}r$,

where H_2 is given in (31). The relation $\bar{u}_2' = u_2$ in (46) implies that

$\bar{L}^{-1}\bar{r}^2\dot{\bar{\theta}} = L^{-1}r^2\dot{\theta}$

$$\text{i.e. } \frac{d\bar{t}}{dt} = \left(\frac{\bar{r}}{r}\right)^2 = H_2^{\ 2} \quad (51)$$

In view of the (50),(51) and the relations $\bar{\theta} = \theta$, $\bar{\phi} = \phi$ in (47) and (45) the required transformation is the transformation f_2 in (31). We now consider the Hamilton vector $\mathbf{K}$ which for the Kepler problem is given by [1] $\mathbf{K} = \dot{\mathbf{x}} - \mu L^{-2}r^{-1}(\mathbf{L}\wedge\mathbf{x})$ (This is a constant multiple of the expression for $\mathbf{K}$ given just after (11)). This expression for $\mathbf{K}$ yields the relation

$K_+ = K_1 + iK_2 = (\upsilon_1 \pm i\upsilon_1')e^{\pm i\phi}$,

where

$$\upsilon_1 = \dot{r}\sin\theta + r(1 - \mu L^{-2}r)\cos\theta\dot{\theta},$$
$$\upsilon_1' = (1 - \mu L^{-2}r)\sin\theta\dot{\phi}. \quad (52)$$

We note that one could consider instead of (39), the same system of equations with u_1 replaced with υ_1, and its Lie symmetries to obtain symmetries of the original system. We report without proof that the symmetry transformation generated by $\upsilon_1\partial_1$ (where $\partial_1 = \partial/\partial\upsilon_1$) is also given by (50), (51) and (47) i.e. the transformation f_2 in (31).

6 Concluding remarks

In this paper we note the following:

1) The Hamilton vector can be used to reduce the dynamical system to coupled systems of oscillator(s) and a conservation law just as the Laplace-Runge-Lenz vector is used.
2) The Lie point symmetry groups of the reduced systems are widely known in the literature and the backward transformation from the symmetries of the reduced systems to symmetries in original variables of the dynamical systems is schematically available. We note that the symmetry groups from the reduced systems using the Quasi-Ermanno-Bernoulli constants are isomorphic although, their forms in the original variables may defer (realizations). One can obtain other nonlocal symmetries of the dynamical system.
3) The generating algebras are not altered consequence of 2) above.

4) We obtained here symmetry transformations of dynamical systems without any reference to their generators in the original variables.

We have shown the generic natural variables for reducing dynamical systems in two-dimensions and three-dimensions to systems of harmonic oscillator(s) and a conservation law.

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