Generalized Multitime Lagrangians and Hamiltonians

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Abstract: We establish a natural frame for affine Lagrangians and Hamiltonians. The focus is on the Hamiltonians applicable in classical fields and their generalizations. A unitary treatment of scalar and volume-valued Hamiltonians in a special class is obtained. Considering a variational problem of the action defined by a Hamiltonian in this class, one obtains informations about the multitime dynamical solutions of the classical variational problem for scalar and volume-valued Hamiltonians.

Key–Words: affine lagrangian, affine hamiltonian, variational equation, jet space.

1 Introduction

A large interest is currently manifested for Lagrangians and Hamiltonians on affine bundles. The most known examples of affine bundle used in differential geometry is the higher order tangent space and the jet space of a fibered manifold. These two classical cases were recently studied in many papers. The higher order spaces are studied from the affine point of view in [4]. The jet spaces are studied in the context of multitime Lagrangian and Hamiltonian geometry in [5]-[11] and in an affine setting in [1]-[3]. The purpose of our paper is to indicate a link between these two cases, and also to give a general setting for Lagrangians and Hamiltonians on affine bundles. An F-Hamiltonian (volume-valued) and an affine Hamiltonian (scalar valued) are sections in certains affine bundles; they both naturally lift to $\tilde{F}$-Hamiltonians.

In section 2 one analize the affine Lagrangians and Hamiltonians on affine spaces and on affine bundles. The general setting in the second Section is used in the Section 3, on the jet space of a fibered manifold, to prove the main results. Considering a Hamilton-Jacobi variational principle for $\tilde{F}$-Hamiltonians, one obtains (Theorem 1) the Hamilton-Jacobi equations (equations (HJ.1)- (HJ.3)). An $\tilde{F}$-Hamiltonian $\tilde{h}$ is the lift of an F-Hamiltonian h iff the equation (HJ.3) is an identity (Proposition 4). In general, we prove that the existence of a solution of equation (HJ.3) ensures, with some additional conditions, the posssibility that the solutions of the Hamilton-Jacobi PDEs of an $\tilde{F}$-Hamiltonian can be obtained from the solutions of PDEs produced by natural F-Hamiltonian h (Theorem 2).

Some examples of $\tilde{F}$-Hamiltonians that are quadratic in momenta are given; these examples include the lift of the electromagnetic Hamiltonian.

2 Affine Lagrangians and Hamiltonians

We breafly recall some facts about affine Lagrangians and Hamiltonian in an affine setting (see [4] for more details).

Let A be an affine space modeled on the real (finite dimensional) vector space V. A *Lagrangian* on A is a differentiable function $L : A \to I\!R$. An *affine Hamiltonian* on A is a differentiable map (nonnecessary linear) $h : V^* \to A^\dagger$ such that $\pi \circ h = 1_{V^*}$, where $A^\dagger = Aff(A, I\!R)$. Using coordinates, the affine Hamiltonian is $(p_i) \xrightarrow{h} (p_i, h_0(p_i))$ .If the coordinates change, then $h'_0(p_{i'}) = h_0(p_i) + p_i a^i$. For example, if $x_0(\alpha_i) \in A$, then $(p_i) \xrightarrow{h_{x_0}} (p_i, \alpha^i p_i)$ is an affine Hamiltonian. The *vertical Hessian* of a Lagrangian L is the bilinear form on the manifold A, defined by $g_{ij}(y^k) = \dfrac{\partial^2 L}{\partial y^i \partial y^j}(y^k)$. The *vertical Hessian* of an affine Hamiltonian h is the bilinear form on the manifold V^*, defined by $h^{ij}(p_k) = \dfrac{\partial^2 h_0}{\partial p_i \partial p_j}(p_k)$. The *Legendre map* defined by a Lagrangian $L : A \to I\!R$ is $\mathcal{L} : A \to V^*$, $\mathcal{L}(y^j) = \dfrac{\partial L}{\partial y^i}(y^j)e^i$ and the *co-Legendre* map defined by an affine Hamiltonian $h : V^* \to A^\dagger$ as above is $\mathcal{H} : V^* \to A, \mathcal{L}(p_i) = \left(\dfrac{\partial h_0}{\partial p_i}(p_j)\right)$.

The Lagrangian L is *regular* (*hyperregular*) if the Legendre map is a local diffeomorphism (global diffeomorphism). Analogous one say that an affine Hamiltonian h is *regular* (*hyperregular*) if its co-Legendre map is a local diffeomorphism (global diffeomorphism). A Lagrangian (affine Hamiltonian) is *singular* if it is not regular. For example, the image of the co-Legendre map of an affine Hamiltonian of the form h_{x_0} is $\{x_0\}$ and its vertical Hessian is null (degenerate; an extreme case.) Then L (or h) is regular iff the vertical Hessian is non-degenerate at every point (as a bilinear form).

Let $L : A \to I\!R$ be a hyperregular Lagrangian. Then let us denote by $\mathcal{L}^{-1} : V^* \to A$ the inverse of the Legendre map; using coordinates, $\mathcal{L}^{-1}(p_i) = (\mathcal{L}^j(p_i))$. Then $h : V^* \to A^\dagger, h(p_i) = (p_i, h_0(p_i))$, $h_0(p_i) = p_j L^j(p_i) - L(\mathcal{L}^j(p_i))$, is an affine Hamiltonian.

Conversely, let $h : V^* \to A^\dagger$ be a hyperregular affine Hamiltonian and $\mathcal{H}^{-1} : A \to V^*$ the inverse of the co-Legendre map; using coordinates, $H^{-1}(y^i) = (\mathcal{H}_j(y^i))$. Then $L : A \to I\!R$, $L(y^i) = y^j\mathcal{H}_j(y^i) - h_0(\mathcal{H}_j(y^i))$, is an affine Lagrangian.

A surjective submersion $E \xrightarrow{\pi} M$ is usually called a *fibered manifold*. A locally trivial fibration $A \xrightarrow{\pi} M$ is an *affine bundle* if its fiber is modeled by a (real) affine space A_0 and the structural functions are affine transformations of A_0. We consider local coordinates adapted to the bundle structure: (x^i) on $U \subset M$ and (x^i, y^α) on $\pi^{-1}(U) \cong U \times A_0$, that change according to the rules

$$\begin{cases} x^{i'} = x^{i'}(x^i) \\ y^{\alpha'} = y^\alpha a_\alpha^{\alpha'}(x^i) + a^{\alpha'}(x^i). \end{cases} \tag{1}$$

A vector bundle is a particular case of an affine bundle ($a^{\alpha'} = 0$ in relation (1)). It is a locally trivial fibration with a fiber type a vector space. An affine bundle $\pi : A \to M$ gives rise to the vector bundles $\bar{\pi} : \bar{A} \to M$ (given by the director vector spaces at every point) and its dual vector bundle $\bar{\pi}' : \bar{A}^* \to M$, called the *dual vector bundle* of the given affine bundle and usually denoted by $\pi^* : A^* \to M$, or A^* for shortness.

Let $\pi_1 : F \to M$ be an affine bundle with the affine line $I\!R$ as typical fiber (i.e. with a one-dimensional fiber). The local coordinates on F change according to the rules

$$\begin{cases} x^{i'} = x^{i'}(x^i) \\ y' = y\sigma(x^i) + \tau(x^i). \end{cases} \tag{2}$$

If $\sigma = 1$ and $\tau = 0$, then $\pi_1 : F \to F = M \times I\!R \to M$ is the projection on the first factor, thus it is the trivial vector bundle. If only $\sigma(x^i) = 1$ (for every local chart), then the affine bundle is associated with the trivial vector bundle $M \times I\!R \to M$; we say that the affine bundle F has *structural translations*.

Let $\pi : A \to M$ be an affine bundle and $\pi_1 : F \to M$ be an affine bundle with a one-dimensional fiber. The $\bar{F}$-dual of A is $L(\bar{A}, \bar{F})$, denoted by A^{*F}. The local coordinates on A^{*F} change according to the rules

$$\begin{cases} x^{i'} = x^{i'}(x^i) \\ \sigma p_{\alpha'} a_\alpha^{\alpha'}(x^i) = p_\alpha. \end{cases} \tag{3}$$

Let us consider $\bar{F}_* \subset \bar{F}$, the fibered submanifold of the vector bundle $\pi_0 : \bar{F} \to M$, consisting in non-null vectors. Denote $\tilde{A} = A \times_M \bar{E}$. The natural projection $\bar{\pi} : \bar{A} \to \bar{F}_*$ is the canonical projection of an affine bundle. Let us denote also by $\tilde{F} = F \times_M \bar{F}_*$ and by $\tilde{\pi}_0 : \tilde{F} \to \bar{F}$ the canonical projection.

Proposition 1 *The projection $\tilde{\pi}_0 : \tilde{F} \to \bar{F}_*$ is the canonical projection of an affine bundle with structural translation (i.e. the associated vector bundle is the trivial vector bundle $M \times I\!R \to M$).*

If $\pi : E \to M$ is a fibered manifold, its first jet space $J^1\pi$ can be regarded as an affine bundle $J^1\pi \to E$. Using on E local coordinates that are adapted to the submersion, then coordinates on $J^1\pi$ have the form $(x^i, y^\alpha, y_i^\alpha)$. They change according to the rules:

$$\begin{cases} x^{i'} = x^{i'}(x^i) \\ y^{\alpha'} = y^{\alpha'}(x^i, y^\alpha) \\ y_{i'}^{\alpha'} \dfrac{\partial x^{i'}}{\partial x^i} = y_i^\alpha \dfrac{\partial y^{\alpha'}}{\partial y^\alpha} + \dfrac{\partial y^{\alpha'}}{\partial x^i}. \end{cases} \tag{4}$$

If $s : M \to E$ is a section (it can be a local one), then it lifts to a section $s' : M \to J^1E$ of the fibered manifold $J^1E \to M$. Using local coordinates, if s has the local form $(x^i) \to (x^i, s^\alpha(x^i))$, then s' is $(x^i) \to (x^i, s^\alpha(x^i), \dfrac{\partial s^\alpha}{\partial x^i})$.

Let us consider the vector bundle $J^1\pi^* = V^*E \otimes \pi^*TM \to E$; the coordinates on $J^1\pi^*$ have the form $(x^i, y^\alpha, p^i_\alpha)$, The coordinates (x^i) and (y^α) change as in relations (4), while

$$p^{i'}_{\alpha'} \frac{\partial y^{\alpha'}}{\partial y^\alpha} = p^i_\alpha \frac{\partial x^{i'}}{\partial x^i}. \tag{5}$$

If $E = M \times T$, where T is a manifold, then $x^{i'} = x^{i'}(x^i)$, $y^{\alpha'} = y^{\alpha'}(y^\alpha)$ and the coordinates (y^α_i) on $J^1\pi$ change in a tensor manner, thus $J^1\pi = VE \otimes \pi^*T^*M$ is a vector bundle and $J^1\pi^*$ is its dual vector bundle. This vector bundle is used in a systematic way in the study of multi-time Lagrangians and Hamiltonians (see [9] and the references therein). Another particular case, considered below, is when $\pi_1 : F \to M$ is an affine bundle with a one dimensional fiber. In this case the formulas (4) have the form:

$$\begin{cases} x^{i'} = x^{i'}(x^i) \\ y' = y\sigma(x^i) + \tau(x^i) \\ y_{i'} \dfrac{\partial x^{i'}}{\partial x^i} = y_i \sigma(x^i) + y \dfrac{\partial \sigma}{\partial x^i} + \dfrac{\partial \tau}{\partial x^i}. \end{cases} \tag{6}$$

If $\pi_1 : F \to M$ is a vector bundle, then $\tau = 0$.

Let us suppose that $\pi_1 : F \to M$ is an affine bundle with structural translations. If (x^i) and (x^i, y) are local coordinates on M and on F respectively, then the local change of coordinates are $x^{i'} = x^{i'}(x^i)$, $y' = y + f(x^i)$. The first jet bundle $J^1\pi_1$ has the coordinates (x^i, y, u_i) and the coordinates (u_i) change following the rule: $u'_i = u_i + \dfrac{\partial f}{\partial x^i}$. There is an affine bundle $\nu : F_1 \to M$ that has as coordinates (x^i, u_i) and the affine bundle $J^1\pi_1$ is canonically isomorphic with the induced bundle $\pi_1^*\nu$; we write $J^1\pi_1 = \pi_1^*\nu$.

A section $s \in \Gamma(\pi_1)$ lifts naturally to a section $s' \in \Gamma(J^1F_1 \to M)$, given locally by $(x^i) \to (x^i, s(x^i), \dfrac{\partial s}{\partial x^i})$. It induces a section $s'' \in \Gamma(\nu)$ and implicitely an affine section $s^J \in \Gamma(J^1F \to F)$ that has the local form $(x^i, y) \to (x^i, y, \dfrac{\partial s}{\partial x^i})$. The section s^J defines a connection in the bundle π that has a null curvature. If a connection on π is defined by a section $\xi \in \Gamma(J^1F \to F)$, $(x^i, y) \to (x^i, y, \xi_i(x^i, y^\alpha))$, it has the curvature given locally by $R_{ij} = \dfrac{\partial \xi_i}{\partial x^j} - \dfrac{\partial \xi_j}{\partial x^i}$. The curvature vanishes iff locally ξ has the form $\xi = s^J$, i.e. it is a lift of a local section $s \in \Gamma(\pi_1)$.

We are going to prove in what follows that one can associate with every affine bundle with one dimensional fiber an affine bundle with one dimensional fiber and with structural translations. Let $\pi : A \to M$ be an affine bundle and $\pi_1 : F \to M$ be an affine bundle with a one-dimensional fiber. An F-lagrangian on E is a fibered manifold map $L : A \to F$ (i.e. $\pi_1 \circ L = \pi$). Since every affine map induces a linear map on the director vector space, there is a canonical projection $\Pi : Aff(A, F^*) \to A^{*F}$. An F-Hamiltonian on E is a fibered manifold map $h : A^{*F} \to Aff(A, F^*)$ such that $\Pi \circ h = 1_{A^{*F}}$. For example, let us consider $F = M \times I\!R$ and $p_1 : M \times I\!R \to M$ be the projection on the first factor. The F-dual of A is just A^*. An F-Lagrangian has the form $L(e) = (\pi(e), L_0(e))$, where $L_0 : A \to I\!R$ is usually called a *Lagrangian*. An *F-Hamiltonian* on A has the form $h : A^* \to Aff(A, M \times I\!R)$. This is the case considered in [3] for affine Hamiltonians of higher order. Another example, more elaborated, is given in the next section, on jet spaces.

Then an F-Lagrangian L has the local form $(x^i, y^\alpha) \xrightarrow{L} (x^i, L_0(x^i, y^\alpha))$ and the local functions L_0 change according to the rules given by (2):

$$L'_0(x^{i'}, y^{\alpha'}) = L_0(x^i, y^\alpha)\sigma(x^i) + \tau(x^i). \tag{7}$$

Since $\dfrac{\partial L_0}{\partial y^\alpha} = \sigma \dfrac{\partial L'_0}{\partial y^{\alpha'}} \dfrac{\partial y^{\alpha'}}{\partial y^\alpha} = \sigma \dfrac{\partial L'_0}{\partial y^{\alpha'}} a^{\alpha'}_\alpha$, the formula $(x^i, y^\alpha) \to (x^i, \dfrac{\partial L_0}{\partial y^\alpha})$ defines a *Legendre map* $\mathcal{L} : A \to A^{*F}$ of L. The local form of a map $\Omega \in Aff(A, F^*)$ is $(y^\alpha) \xrightarrow{\Omega} \begin{pmatrix} y^\alpha p_\alpha & b \end{pmatrix}$ and $\Pi(\Omega)$ has the local form $(y^\alpha) \xrightarrow{\Pi(\Omega)} (y^\alpha p_\alpha)$.

The local form of $\Pi : Aff(A, F^*) \to A^{*F}$ is $\begin{pmatrix} p_\alpha & p \end{pmatrix} \xrightarrow{\Pi} (p_\alpha)$. An F-Hamiltonian $h : A^{*F} \to Aff(A, F^*)$ has the local form $(x^i, p_\alpha) \xrightarrow{h} (x^i, p_\alpha, h_0(x^i, p_\alpha))$. The change rules of local coordinates are $\begin{pmatrix} p_\alpha & p \end{pmatrix} = \sigma \cdot \begin{pmatrix} p_{\alpha'} & p' \end{pmatrix} \begin{pmatrix} a^{\alpha'}_\alpha & a^{\alpha'} \\ 0 & 1 \end{pmatrix}$, or $\begin{pmatrix} p_{\alpha'} & p' \end{pmatrix} = \sigma' \cdot \begin{pmatrix} p_\alpha & p \end{pmatrix} \begin{pmatrix} a^\alpha_{\alpha'} & a^\alpha \\ 0 & 1 \end{pmatrix}$, where $(a^\alpha_{\alpha'}) = (a^{\alpha'}_\alpha)^{-1}$, $\sigma = (\sigma')^{-1}$ and $a^\alpha =$

$-a^{\alpha'}a^{\alpha}_{\alpha'}$. Thus $h'_0(x^{i'},p_{\alpha'}) = \sigma^{-1}(x^i)\cdot(p_\alpha a^\alpha(x^i) + h_0(x^i,p_\alpha))$. It is easy to see that the formula $(x^i,p_\alpha) \to (x^i, -\frac{\partial h_0}{\partial p_\alpha})$ defines a *co-Legendre map* $\mathcal{H}^* : A^{*F} \to A$ of h. A Lagrangian $L : A \to F$ is *regular* if its Legendre map is a local diffeomorphism; it is equivalent with the fact that the *vertical hessian*, given by the local matrix $\left(g_{\alpha\beta} = \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta}\right)$ is non-singular. The Lagrangian is *hyperregular* if its Legendre map is a (global) diffeomorphism.

If $L : A \to F$ is an F-Lagrangian, then $\tilde{L} : \tilde{A} \to \tilde{F}$ defined locally by $\tilde{L}(x^i,y^\alpha,\bar{y}) = \frac{L(x^i,y^\alpha)}{\bar{y}}$ is an $\tilde{F}$-Lagrangian on $\tilde{A}$. We say that $\tilde{L}$ is the *lift* of L from A to $\tilde{A}$. It is easy to see that the following statement is true.

Proposition 2 *The lift $\tilde{L}$ is regular (hyperregular) iff L is regular (hyperregular).*

Analogously, if $h : A^{*F} \to Aff(A,F)$ is an F-Hamiltonian, then one can consider an $\tilde{F}$-Hamiltonian $\tilde{h} : \tilde{A}^{*\tilde{F}} \to Aff(\tilde{A},\tilde{F})$ defined by $\tilde{h}(x^i,\bar{y},\tilde{p}_\alpha) = \frac{1}{\bar{y}}h(x^i,\frac{1}{\bar{y}}\tilde{p}_\alpha)$. We say that $\tilde{h}$ is the *lift* of h (from A^{*F} to $\tilde{A}$). It is easy to see that the following statement is true.

Proposition 3 *The lift $\tilde{h}$ is regular (hyperregular) iff h is regular (hyperregular).*

There are natural maps $\Phi : A^* \times_M \bar{F} = \tilde{A}^* \to A^{*F}$ and $\Psi : Aff(A,I\!R) \times_M \bar{F} = Aff(\tilde{A},I\!R) \to Aff(A,F)$ given in local coordinates by

$$\Phi : (x^i,\tilde{p}_\alpha,\bar{z}) \to (x^i, p_\alpha = \bar{z}^{-1}\tilde{p}_\alpha),$$
$$\Psi : (x^i,\tilde{p}_\alpha,\bar{z},\tilde{p}) \to (x^i, p_\alpha = \bar{z}^{-1}\tilde{p}_\alpha, \bar{z}^{-1}\tilde{p}).$$

One can consider also some natural maps $\tilde{\Pi} : Aff(\tilde{A},I\!R) \to \tilde{A}^*$ and $\Pi : Aff(A,F) \to A^{*F}$.

If $\bar{h} : A^* \to Aff(A,I\!R)$ is an affine Hamiltonian, then one can consider an $\tilde{F}$-Hamiltonian $\tilde{h} : \tilde{A}^{*\tilde{F}} \to Aff(\tilde{A},\tilde{F}^*)$ defined by $\tilde{h}(x^i,\bar{y},\tilde{p}_\alpha) = \bar{h}(x^i,\tilde{p}_\alpha)$, that we call the *lift* of $\bar{h}$ (from A^* to $\tilde{A}$).

3 Related Hamiltonians on a jet space

Let $\pi : E \to M$ be a fibered manifold (or a bundle). We consider the vector bundle $\Lambda^m(TM) \to M$, $m = \dim M$, with a one-dimensional fiber, that has as sections the top forms (or volume densities) on M. For our purpose we consider also the induced vector bundles with one dimensional fibers $\pi_1 : F = \pi^*\Lambda^m(TM) \to E$, $\pi'_1 : F^* = \pi^*\Lambda^m(T^*M) \to E$. A Hamiltonian considered in [1, 2, 3] is called, in our terminology, as an *F-Hamiltonian* on E. It is a section $h : J^1\pi^{*F} \to J^1\pi^{\dagger F}$ and it has the local form

$$(x^i,y^\alpha,p^i_\alpha) \to (x^i,y^\alpha,p^i_\alpha,h(x^i,y^\alpha,p^i_\alpha)). \qquad (8)$$

The local coordinates (p^i_α) and the local functions h change according to the rules $p^{i'}_{\alpha'}\frac{\partial y^{\alpha'}}{\partial y^\alpha} = \sigma^{-1}p^i_\alpha\frac{\partial x^{i'}}{\partial x^i}$ and $h' = \sigma^{-1}\left(h + p^i_\alpha\frac{\partial y^\alpha}{\partial y^{\alpha'}}\frac{\partial y^{\alpha'}}{\partial x^i}\right)$. An *affine Hamiltonian* on $J^1\pi^*$ is a section $\bar{h} : J^1\pi^* \to J^1\pi^\dagger$ and it has the local form

$$(x^i,y^\alpha,\tilde{p}^i_\alpha) \to (x^i,y^\alpha,\tilde{p}^i_\alpha,\bar{h}(x^i,y^\alpha,\tilde{p}^i_\alpha)). \qquad (9)$$

The local coordinates $(\tilde{p}^i_\alpha)$ and the local functions $\bar{h}$ change according to the rules $\tilde{p}^{i'}_{\alpha'}\frac{\partial y^{\alpha'}}{\partial y^\alpha} = \tilde{p}^i_\alpha\frac{\partial x^{i'}}{\partial x^i}$ and $\bar{h}' = \bar{h} + \tilde{p}^i_\alpha\frac{\partial y^\alpha}{\partial y^{\alpha'}}\frac{\partial y^{\alpha'}}{\partial x^i}$.We are going to put together F-Hamiltonians and affine Hamiltonians. In order to do this we consider $\tilde{F}$-Hamiltonians. In order to simplify notations and the exposition, we consider F^*_* instead of $\bar{F}_*$ in the previous section. We denote $\tilde{F} = F \times_M \bar{F}^*_*$ and we use the canonical projection $\tilde{\pi}_0 : \tilde{F} \to \bar{F}^*_*$. Also, $\tilde{J} = J^1E \times_M \bar{F}^*_*$ and $\tilde{\pi} : \tilde{J} \to \tilde{E} = E \times_M \bar{F}^*_*$ (a canonical projection of a fibered manifold). An *$\tilde{F}$-Hamiltonian* on E is a section $\tilde{h} : \tilde{J}^* \to \tilde{J}^\dagger$ and it has the local form

$$(x^i,y^\alpha,\bar{\omega},\tilde{p}^i_\alpha) \to (x^i,y^\alpha,\bar{\omega},\tilde{p}^i_\alpha,\tilde{h}(x^i,y^\alpha,\bar{\omega},\tilde{p}^i_\alpha)). \qquad (10)$$

The local functions $\tilde{h}$ change according to the rules $\tilde{h}' = \tilde{h} + \tilde{p}^i_\alpha\frac{\partial y^\alpha}{\partial y^{\alpha'}}\frac{\partial y^{\alpha'}}{\partial x^i}$. According to the previous section, an F-Hamiltonian, as well as an affine Hamiltonian, lifts to an $\tilde{F}$-Hamiltonian. More specifically,

– if h is an F-Hamiltonian that has the local form (8), then its lift $\tilde{h}$ has the local form (10), with $\tilde{h}(x^i, y^\alpha, \bar{\omega}, \tilde{p}^i_\alpha) = \frac{1}{\bar{\omega}} h(x^i, y^\alpha, \bar{\omega}\tilde{p}^i_\alpha)$;

– if $\bar{h}$ is an affine Hamiltonian that has the local form (9), then its lift $\tilde{h}$ has the local form (10), with $\tilde{h}(x^i, y^\alpha, \bar{\omega}, \tilde{p}^i_\alpha) = \bar{h}(x^i, y^\alpha, \tilde{p}^i_\alpha)$. An important tool in the study of F-Hamiltonians (Hamiltonians in the classical terminology can be found in the multisymplectic formalism developped in [1, 2, 3] (see also the bibliography therein). In [1] one define the action of an F-Hamiltonian h on sections on $E \to M$ and one deduce the equation of a critical section of this action, deduced from a Hamilton-Jacobi principle. We intend to define an action for an $\tilde{F}$-Hamiltonian, in order to recover the same action for the lift of an F-Hamiltonian.

Let $m = \dim M$. The Hamilton-Cartan forms $\Theta = p^i_\alpha dy^\alpha \wedge d^{m-1}x_i - p d^m x$, $\Omega = -d\Theta = dp^i_\alpha \wedge dy^\alpha \wedge d^{m-1}x_i + dp \wedge d^m x$ on $J^1E^{\dagger F} = Aff(J^1E, F)$ gives the pull-back forms $\Psi^*\Theta$ and $\Psi^*\Omega$ on $\tilde{J}^\dagger = Aff(\tilde{J}, I\!R)$. We consider an $\tilde{F}$-Hamiltonian $\tilde{h} : \tilde{J}^* \to \tilde{J}^\dagger$. It defines the pull-back form on $\tilde{J}^*$, given by $\Theta_{\tilde{h}}=\tilde{h}^*\Theta$ and $\Omega_{\tilde{h}}=\tilde{h}^*\Omega$, that has the local forms

$$\Psi^*\Theta_{\tilde{h}} = \bar{\omega}(\tilde{p}^i_\alpha dy^\alpha \wedge d^{m-1}x_i - \tilde{h}(x^i, y^\alpha, \bar{\omega}, \tilde{p}^i_\alpha) d^m x)$$

and

$$\Psi^*\Omega_{\tilde{h}} = -d\bar{\omega} \wedge (\tilde{p}^i_\alpha dy^\alpha \wedge d^{m-1}x_i - \tilde{h} d^m x) - \bar{\omega}(d\tilde{p}^i_\alpha \wedge dy^\alpha \wedge d^{m-1}x_i - d\tilde{h} \wedge d^m x)$$

respectively. (Here $\Phi : J^1\pi^* \times_M F^*_* = \tilde{J}^* \to J^1\pi^{*F}$ and $\Psi : \tilde{J}^\dagger \to J^1\pi^{\dagger F}$ are defined in the general case in the previous section, $A = J^1\pi$.) Let us consider the natural fibered manifold $\tilde{\tau} : \tilde{J}^* \to M$ and denote by $\Gamma_0(M, \tilde{J}^*)$ the set of its sections $\psi : M \to \tilde{J}^*$ such that the pull-back m-form $s^*\Theta_{\tilde{h}}$ on M has a compact support; a section ψ has the local form

$$\psi : (x^i) \to (x^i, y^\alpha(x^i), \bar{\omega}(x^i), \tilde{p}^i_\alpha(x^i)). \tag{11}$$

It follows a map $\mathbf{H} : \Gamma_0(M, \tilde{J}^*) \to I\!R$, $\psi \to \int_M \psi^*\Theta_{\tilde{h}}$. For a local compact-supported vector field $X \in \mathcal{X}(\tilde{J}^*)$ that has a local one-parameter group σ_t, one can consider a variation $\psi_t = \sigma_t \circ \psi$. The *variational problem*, is the search of a section ψ, called a *critical section*, such that

$$\left.\frac{d}{dt}\right|_{t=0} \int_M \psi^*_t \Theta_{\tilde{h}} = 0. \tag{12}$$

(*Hamilton-Jacobi principle*).

Theorem 4 *A section ψ that has the local form (11) is a critical section iff it satisfies the following system of equations*

$$\frac{\partial \tilde{h}}{\partial \tilde{p}^i_\alpha} = \frac{\partial y^\alpha}{\partial x^i}, \tag{HJ.1}$$

$$\bar{\omega}\frac{\partial \tilde{h}}{\partial y^\alpha} = -\bar{\omega}\frac{\partial \tilde{p}^i_\alpha}{\partial x^i} - \tilde{p}^i_\alpha \frac{\partial \bar{\omega}}{\partial x^i}, \tag{HJ.2}$$

$$\tilde{p}^i_\alpha \frac{\partial \tilde{h}}{\partial \tilde{p}^i_\alpha} - \tilde{h} - \bar{\omega}\frac{\partial \tilde{h}}{\partial \bar{\omega}} = 0. \tag{HJ.3}$$

As an example, let us consider a section $s : E \to J^1E$, a two covariant tensor G on the fibers of $J^1\pi^* \to E$, a one covariant tensor K and a function N on E, having local forms $(x^i, y^\alpha) \to s^\alpha_i(x^i, y^\alpha)$, $G = G^{\alpha\beta}_{ij}(x^i, y^\alpha)\tilde{p}^i_\alpha \otimes \tilde{p}^j_\beta$, $K = K^\alpha_i(x^i, y^\alpha)\tilde{p}^i_\alpha$ and $N(x^i, y^\alpha)$ respectively. Let σ be a volume form on M and $F = \Lambda^m(M)$.

The formula $h_1(x^i, y^\alpha, p^i_\alpha) = s^\alpha_i p^i_\alpha + \frac{1}{2\sigma} p^i_\alpha p^j_\beta G^{\alpha\beta}_{ij} + K^\alpha_i p^i_\alpha + \sigma N$ defines an F-Hamiltonian on J^1E. It is polynomial, of second degree in volume-valued momenta (p^i_α). The corresponding lift is the $\tilde{F}$-Hamiltonian given by the formula

$$\tilde{h}_1(x^i, y^\alpha, \bar{\omega}, \tilde{p}^i_\alpha) = s^\alpha_i \tilde{p}^i_\alpha + \frac{\bar{\omega}}{2\sigma}\tilde{p}^i_\alpha \tilde{p}^j_\beta G^{\alpha\beta}_{ij} + K^\alpha_i \tilde{p}^i_\alpha + \frac{\sigma}{\bar{\omega}} N. \tag{13}$$

It is also polynomial, of second degree in scalar momenta $(\tilde{p}^i_\alpha)$. It is also easy to see that $\tilde{p}^i_\alpha \frac{\partial \tilde{h}_1}{\partial \tilde{p}^i_\alpha} - \tilde{h}_1 - \bar{\omega}\frac{\partial \tilde{h}_1}{\partial \bar{\omega}} = 0$. Considering the variational problem for this $\tilde{F}$-Hamiltonian (Theorem 4), the relation (HJ.3) is automatically fulfilled, thus only equations (HJ.1) and (HJ.2) must be satisfied.

A particular case is a global form of the electromagnetism equation, as follows. We consider $E = T^*M \xrightarrow{\pi} M$ and g a (pseudo)riemannian metric on M. Then $J^1\pi^* = \Lambda_2(T^*M)$ is the vector bundle of 2-contravariant tensors $\tilde{p} = (\tilde{p}^{ij})$ on M. Then $H(x^i, \tilde{p}^{ij}) = \frac{1}{2}\tilde{\xi}_{ij}\tilde{\xi}^{ij}$, where $\tilde{\xi}^{ij} = \tilde{p}^{ij} - \tilde{p}^{ji}$ and $\tilde{\xi}_{ij} = g_{ir}g_{js}\tilde{\xi}^{rs}$, is an well-defined real function on $J^1\pi^*$. If (Γ^i_{jk}) are the Christoffel coefficients of the metric g, then the local functions $\tilde{\Gamma}_{ij}(x^r, p_s) = \Gamma_{ij}^k(x^r)p_k$ define a global section $\tilde{\Gamma} : T^*M \to J^1\pi^*$. Denoting $\sigma = \sqrt{|g|}$, where $g = \det(g_{ij})$, we obtain a volume form on M.

Considering $F = \pi^*\Lambda^m(TM)$ and the F-dual $(T^*M)^{*F}$, then the formula $h(x^i, p_j, p^{ij}) = \Gamma_{ij}p^{ij} + \frac{1}{2\sigma}\xi^{ij}\xi_{ij}$ defines an F-Hamiltonian on $J^1(T^*M)$, called the *electromagnetic Hamiltonian*. The corresponding $\tilde{F}$-Hamiltonian is $\tilde{h}(x^i, p_j, \bar{\omega}, \tilde{p}_{ij}) = \Gamma_{ij}\tilde{p}^{ij} + \frac{\bar{\omega}}{2\sigma}\tilde{\xi}^{ij}\tilde{\xi}_{ij}$.

Proposition 5 *An $\tilde{F}$-Hamiltonian $\tilde{h}$ is the lift of an F-Hamiltonian h if and only if $\tilde{h}$ verifies the conditions (HJ.3).*

In the case considered in Proposition 5 (i.e. $\tilde{h}$ is the lift of an F-Hamiltonian h), then equation (HJ.3) is satisfied identically, while the other two equations (HJ.1) and (HJ.2) follows from the classical Hamilton-Jacobi equation of h (see [1, 2]):

$$\frac{\partial h}{\partial \tilde{p}_\alpha^i} = \frac{\partial y^\alpha}{\partial x^i}, \qquad (HJ'.1)$$

$$\frac{\partial h}{\partial y^\alpha} = -\frac{\partial p_\alpha^i}{\partial x^i}. \qquad (HJ'.2)$$

In that follows we focus on the case when equation (HJ.3) *is not* satisfied identically.

Let us consider an $\tilde{F}$-Hamiltonian given by the formula $\tilde{h}_1(x^i, y^\alpha, \bar{\omega}, \tilde{p}_\alpha^i) = s_i^\alpha \tilde{p}_\alpha^i + \frac{1}{2} f\left(\frac{\bar{\omega}}{\sigma}\right)\tilde{p}_\alpha^i\tilde{p}_\beta^j G_{ij}^{\alpha\beta} + g\left(\frac{\bar{\omega}}{\sigma}\right)K_i^\alpha\tilde{p}_\alpha^i + u\left(\frac{\bar{\omega}}{\sigma}\right)N$, where f, g and u are some real functions. Then by a straightforward computation one has

$$\tilde{p}_\alpha^i\frac{\partial \tilde{h}_1}{\partial \tilde{p}_\alpha^i} - \tilde{h}_1 - \bar{\omega}\frac{\partial \tilde{h}_1}{\partial \bar{\omega}} = \frac{1}{2}\left(f - \frac{\bar{\omega}}{\sigma}f'\right)\tilde{p}_\alpha^i\tilde{p}_\beta^j G_{ij}^{\alpha\beta} - \frac{\bar{\omega}}{\sigma}g'K_i^\alpha\tilde{p}_\alpha^i + \left(u + \frac{\bar{\omega}}{\sigma}u'\right)N,$$

where the real functions f, g, u and their derivatives have as variable $\frac{\bar{\omega}}{\sigma}$.

Since the above expression is a polinomial in $\tilde{p}_\alpha^i$, the $\tilde{F}$-Hamiltonian $\tilde{F}_1$, given by (13), is the only $\tilde{F}$-Hamiltonian that fulfills the relation (HJ.3).

Proposition 6 *Let us suppose that the equation (HJ.3) can be solved with respect with $\bar{\omega}$. Then:*

1) The solution $\bar{\omega} = s_0(xi, y^\alpha, \tilde{p}_\alpha^i)$ defines a volume-valued Hamiltonian $h_0 : J^1\pi^* \to F^*$.

2) The local functions $h_1(x^i, y^\alpha, \tilde{p}_\alpha^i) = \tilde{h}(x^i, y^\alpha, s_0(x^i, y^\alpha, \tilde{p}_\alpha^i), \tilde{p}_\alpha^i)$ define an affine Hamiltonian $h_1 : J^1\pi^* = J^1\pi^\dagger$.

We say that an $\tilde{F}$-Hamiltonian $\tilde{h}$ has an F-Hamiltonian h as a *Hamilton-Jacobi projection* (or *HJ-projection* for short) if there is a non-null volume-valued hamiltonian $s_0 : J^1\pi^* \to F^*$ such that the solutions of the Hamilton-Jacobi equations of $\tilde{h}$ (equations (HJ.1) – (HJ.3)) have the property that $\bar{\omega} = s_0$ verify (HJ.3) and the local functions $(x^i) \to (x^i,\ y^\alpha(x^i),\ p_\alpha^i(x^i) = s_0(x^i)\tilde{p}_\alpha^i(x^i))$ are solutions of (classical) Hamilton-Jacobi equations of h, i.e. (HJ'.1) and (HJ'.2).

Theorem 7 *Let us suppose that the equation (HJ.3) of an $\tilde{F}$-Hamiltonian $\tilde{h}$ can be solved with respect to $\bar{\omega} = h_0$. Then the local functions $h(x^i, y^\alpha, p_\alpha^i) = h_0\tilde{h}(x^i, y^\alpha, h_0, \frac{p_\alpha^i}{h_0})$ define an F-Hamiltonian $h : J^1\pi^{*F} \to J^1\pi^{\dagger F}$ that is an HJ-projection of $\tilde{h}$.*

Let us consider that an $\tilde{F}$-Hamiltonian $\tilde{h}$ is the lift of an affine Hamiltonian $\bar{h}$, thus $\tilde{h}(x^i, y^\alpha, \bar{\omega}, \tilde{p}_\alpha^i) = \bar{h}(x^i, y^\alpha, \tilde{p}_\alpha^i)$ does not depend on $\bar{\omega}$. The relation $\frac{\partial \tilde{h}}{\partial \bar{\omega}} = 0$ (i.e. $\tilde{h}$ does not depend on $\bar{\omega}$) is the condition that $\tilde{h}$ is the lift of an affine Hamiltonian $\bar{h}$. The equations (HJ.1) – (HJ.3) become in this case

$$\frac{\partial \bar{h}}{\partial \tilde{p}_\alpha^i} = \frac{\partial y^\alpha}{\partial x^i}, \qquad \text{(HJ.1}'\text{)}$$

$$\bar{\omega}\frac{\partial \bar{h}}{\partial y^\alpha} = -\bar{\omega}\frac{\partial \tilde{p}_\alpha^i}{\partial x^i} - \tilde{p}_\alpha^i\frac{\partial \bar{\omega}}{\partial x^i}, \qquad \text{(HJ.2}'\text{)}$$

$$\tilde{p}_\alpha^i\frac{\partial \bar{h}}{\partial \tilde{p}_\alpha^i} - \bar{h} = 0. \qquad \text{(HJ.3}'\text{)}$$

In the setting of Proposition 6, the solutions of Hamilton-Jacobi equations of the $\tilde{F}$-Hamiltonians $\tilde{h}$ and of the lift $\tilde{h}_1$ (of h_1) are not the same.

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