Multitime Models of Optimal Growth*

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Abstract: Section 1 underline the limitations of standard multi-variable variational calculus and the sense of multitime. Section 1 formulates the controllability problem for a multiple integral functional or for a path independent curvilinear integral subject to a multitime evolution of flow type. Section 2 describes a two-time optimal economic growth modelled by Euler-Lagrange PDEs associated to a double integral functional or to a path independent curvilinear integral in two dimensions. Section 3 motivates the optimal economic growth by two-time maximum action.

Key–Words: multitime maximum principle, multitime optimal economic growth, bang-bang policy.

1 Multitime optimal control theory

The interval $[0, T] = \Omega_{0,T}$, in $R^m$ with product order, is called planning horizon. Geometrically, it is a hyperparallelepiped fixed by the diagonal opposite points 0 and T. Consider a dynamic system evolving over multi-time $t = (t^1, \ldots, t^m) \in \Omega_{0,T}$ and an agent (planner) who has the task to control the evolution of $m$-sheets. We assume $T = (T^1, \ldots, T^m)$ has finite norm, but sometimes we can relax this assumption. The dynamic behaviour of the system is described by the state variables $x = (x^1, \ldots, x^n) : \Omega_{0,T} \rightarrow R^n$, $x(t) \in SV$ (state variables). The planner knows the initial state of the system $x(0) = x_0$ and the final state of the system $x(T) = x_T$ (boundary conditions).

We accept that the state variables are affected through a set of control variables $c = (c_1, \ldots, c_q) : \Omega_{0,T} \rightarrow R^q$, $c(t) \in CV$ (control variables). The planner knows the relationship between the actions taken and evolution of the states, which are summarized by a "law of evolution" of the states, a (non-autonomous) PDEs system of the type

$$\frac{\partial x^i}{\partial t^\alpha}(t) = X^i_\alpha(x(t), c(t), t) \quad (PDE)$$

$i = 1, \ldots, n; \alpha = 1, \ldots, m$,

satisfying the complete integrability conditions

$$[X_\alpha, X_\beta] = \frac{\partial X_\alpha}{\partial c_a} \frac{\partial c_a}{\partial t^\beta} - \frac{\partial X_\beta}{\partial c_a} \frac{\partial c_a}{\partial t^\alpha} + \frac{\partial X_\alpha}{\partial t^\beta} - \frac{\partial X_\beta}{\partial t^\alpha},$$

$a = 1, \ldots, q$.

Fixing the control variables at a given multi-instant $t$, the evolution of the state variables at point $t$ are obtained as solutions of the previous (PDE). Also given the value of the state at point $t$, the future values are determined.

Controllability problem: We are allowed to act on the m-sheets of the (PDE) system by means of a suitable control (included in the right hand side, in the boundary conditions, etc). Then, given a multitime $t \in \Omega_{0,T}$, and initial and final states, we have to find a control such that the solution matches both the initial state at multi-time $t = 0$ and the final one at multitime $t = T$.

A way to choose properly the controls is to introduce:

1) either a multiple integral functional

$$I(c(\cdot)) = \int_{\Omega_{0,T}} L(x(t), c(t), t) dt^1 \ldots dt^m,$$

2) or a path independent objective functional

$$J(c(\cdot)) = \int_{\Gamma_{0,T}} L_\beta(x(t), c(t), t) dt^\beta,$$

where $\Gamma_{0,T}$ is a $C^1$ path joining the diagonal opposite points 0 and $T$. 

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2 Two-time optimal economic growth

The theory of optimal economic growth starts with the following question: how much should be consumed and how much should be invested for future consumption? To formulate an answer, we accept that the evolution is 2-dimensional. That is why we introduce the following variables and functions:

\[ t = (t^1, t^2) = 2 \cdot \text{moment of the economical effect}; \]

\[ K(t) \text{ capital}; \]

\[ L(t) \text{ labour force; with partial growing at a constant exogenous rate } n_{\alpha}, \text{ i.e., } \frac{\partial}{\partial t^\alpha} \ln L = n_{\alpha}, \]

\[ \alpha = 1, 2 \text{ or equivalently } L = c_1 e^{n_{\alpha} t^\alpha}; \]

\[ Y_{\alpha} = F_{\alpha}(K, L) \text{ homogeneous commodities (production functions).} \]

Each commodity \( Y_{\alpha}(t) = F_{\alpha}(K(t), L(t)) \) decomposes as sum of consumed part \( c_{\alpha}(t) \), partial velocity of capital \( \frac{\partial K}{\partial t^\alpha}(t) \) (further capital) and depreciation capital \( \mu_{\alpha} K(t) \), where \( \mu_{\alpha} \) is a constant rate:

\[ Y_{\alpha}(t) = c_{\alpha}(t) + \frac{\partial K}{\partial t^\alpha}(t) + \mu_{\alpha} K(t), \quad \alpha = 1, 2. \]

The production functions \( Y_{\alpha} = F_{\alpha}(K, L) \), assumed homogeneous of degree one, could be written \( Y_{\alpha} = L F_{\alpha} \left( \frac{K}{L}, 1 \right) = L f_{\alpha}(k), \quad k = \frac{K}{L}. \) Putting \( y_{\alpha} = \frac{Y_{\alpha}}{L} \), it follows \( y_{\alpha} = f_{\alpha}(k) \), where each function \( f_{\alpha}(k) \) is a strictly concave monotonically increasing function of \( k \), with slope \( f'_{\alpha}(k) \) decreasing from \( \lim_{k \to 0} f'_{\alpha}(k) = \infty \) to \( \lim_{k \to \infty} f'_{\alpha}(k) = 0. \) In this way we obtain a two-time evolution

\[
\frac{\partial k}{\partial t^\alpha}(t) = f_{\alpha}(k(t)) - (\mu_{\alpha} + n_{\alpha}) k(t) - c_{\alpha}(t), \quad \alpha = 1, 2.
\]

Also we accept that this PDEs system satisfies the complete integrability conditions.

Let us apply the multi-time Euler-Lagrange theory: let \( D = (D_1, D_2) \) be a constant positive rate vector of future discount; let \( \lambda_{\alpha} = \mu_{\alpha} + n_{\alpha} \) and \( g_{\alpha}(k) = f_{\alpha}(k) - \lambda_{\alpha} k. \)

### 2.1 Case of double integral functional

Let \( u(c) \) be the utility function which obeys the law of diminishing marginal utility \( d^2 u(c) < 0 \) (concave function), \( \frac{\partial u}{\partial c_{\gamma}} > 0. \) Maximize the functional

\[ I(c(\cdot)) = \int_{t_0, T} e^{-D_{\lambda} t^1} u(c(t)) dt^1 dt^2, \quad c = (c_1, c_2), \]

subject to

\[ c_{\alpha}(t) = g_{\alpha}(k(t)) - \frac{\partial k}{\partial t^\alpha}(t), \]

\[ k(0) = k_0, \quad k(T) = k_T, \quad 0 = (0, 0), \quad T = (T^1, T^2). \]

Eliminating \( c_{\alpha}(t) \), we find the Lagrangian

\[
L(k(t), k_{\gamma}(t), t) = e^{-D_{\lambda} t^1} u(c(t)) =
\]

\[
e^{-D_{\lambda} t^1} \left( g_1(k(t)) - \frac{\partial k}{\partial t^1}(t), g_2(k(t)) - \frac{\partial k}{\partial t^2}(t) \right).
\]

The extremals are solutions of the multi-time Euler-Lagrange equation

\[
\frac{\partial L}{\partial k} - \frac{\partial}{\partial t^\gamma} \frac{\partial L}{\partial k_{\gamma}} = 0.
\]

It follows the PDEs system

\[
\frac{\partial^2 u}{\partial c_{\alpha} \partial c_{\gamma}} + \frac{\partial u}{\partial c_{\gamma}} \left( \frac{dg_{\gamma}}{dk} - D_{\gamma} \right) = 0
\]

\[
\frac{\partial k}{\partial t^\alpha}(t) = g_{\alpha}(k(t)) - c_{\alpha}(t).
\]

First we obtain an equilibrium point \((k^*, c_{\gamma}^*)\) at which \( \frac{\partial k}{\partial t^\alpha} = 0, \quad \frac{\partial c_{\alpha}}{\partial t^\alpha} = 0. \) It follows

\[
\frac{\partial u}{\partial c_{\gamma}} \left( \frac{dg_{\gamma}}{dk} - D_{\gamma} \right) = 0, \quad g_{\alpha}(k(t)) - c_{\alpha}(t) = 0,
\]

which must produce \( k^* \) and \( c_{\gamma}^* = g_{\alpha}(k^*). \)
Second, an analytical solution is possible when \( f_\alpha(k) \) and \( u(c) \) are explicitly given. For example, \( f_\alpha(k) = a_\alpha k \), i.e., \( g_\alpha(k) = (a_\alpha - \lambda_\alpha)k \), and \( u(c) = c_1^2 + c_2^2 \). Then the previous PDEs system is reduced to

\[
\frac{\partial c_1}{\partial t^1} + \frac{\partial c_2}{\partial t^2} + c_1(a_\alpha - \lambda_\alpha - D_1) + c_2(a_\beta - \lambda_\beta - D_2) = 0
\]

\[
\frac{\partial k}{\partial t^\alpha}(t) = (a_\alpha - \lambda_\alpha)k - c_\alpha(t).
\]

A particular solution of the first PDE is

\[ c_1(t) = c_2(t) = e^{-(a_\alpha - \lambda_\alpha - D_\alpha)}t^\alpha. \]

In the complete integrability conditions of the second PDEs,

\[ 2a_1 - 2\lambda_1 - 2a_2 + 2\lambda_2 + D_2 - D_1 = 0, \]

we obtain the corresponding solution \( k(t) \).

### 2.2 Case of path independent integral functional

Let \( u_\beta(c) \) be the utility 1-form whose elements obey the law of diminishing marginal utility \( \frac{\partial^2 u_\beta(c)}{\partial c_\gamma \partial c_\delta} < 0 \) (concave functions), \( \frac{\partial u_\beta}{\partial c_\gamma} > 0 \). Maximize the functional

\[ J(c(\cdot)) = \int_{t_0}^{t_T} e^{-D_\alpha t^\alpha} u_\beta(c(t))dt^\beta, \quad c = (c_1, c_2), \]

subject to

\[ c_\alpha(t) = g_\alpha(k(t)) - \frac{\partial k}{\partial t^\alpha}(t), \]

\[ k(0) = k_0, \quad k(T) = k_T, \quad 0 = (0, 0), \quad T = (T^1, T^2). \]

Eliminating \( c_\alpha(t) \), we find the Lagrangian 1-form

\[ L_\beta(k(t), \gamma(t), t) = e^{-D_\alpha t^\alpha} u_\beta(c(t)) = (g_1(k(t)) - \frac{\partial k}{\partial t^1}(t), g_2(k(t)) - \frac{\partial k}{\partial t^2}(t)) \]

that must satisfy the complete integrability conditions. The extremals are solutions of the multi-time Euler-Lagrange equations

\[ \frac{\partial L_\beta}{\partial k} - \frac{\partial}{\partial t^\gamma} \frac{\partial L_\beta}{\partial \dot{k}_\gamma} = a_\beta. \]

It follows the PDEs system

\[ \frac{\partial^2 u_\beta}{\partial c_\gamma \partial t^\alpha} \frac{\partial c_\alpha}{\partial t^\gamma} + \frac{\partial u_\beta}{\partial c_\gamma} \left( \frac{\partial g_\gamma}{\partial k} - D_\gamma \right) = a_\beta. \]

First we obtain an equilibrium point \((k^*, c^*)\) at which

\[ \frac{\partial k}{\partial t^\alpha} = 0, \quad \frac{\partial c_\alpha}{\partial t^\gamma} = 0. \]

It follows

\[ \frac{\partial u_\beta}{\partial c_\gamma} \left( \frac{\partial g_\gamma}{\partial k} - D_\gamma \right) = a_\beta, \quad g_\alpha(k(t)) - c_\alpha(t) = 0, \]

which must produce \( k^* \) and \( c^*_\alpha = g_\alpha(k^*) \).

Second, an analytical solution is possible when \( f_\alpha(k) \) and \( u_\beta(c) \) are explicitly given. For example, \( f_\alpha(k) = a_\alpha k \), i.e., \( g_\alpha(k) = (a_\alpha - \lambda_\alpha)k \), and

\[ u_\beta(c) = \begin{cases} 
\frac{c_\gamma^{1-\nu}}{1-\nu} & \text{if } \nu > 0, \nu \neq 1 \\
\ln c_\beta & \text{if } \nu = 1.
\end{cases} \]

### 3 Reformulation as an optimal control

Let us formulate the optimal growth as a multi-time optimal control model (see [2], [6]-[9]) starting with \( \lambda_\alpha = n_\alpha + \mu_\alpha \) (constant population growth rates + constant depreciation rates).

#### 3.1 Case of double integral functional

For that we choose a rate of per capita consumption \( c(t) = (c_1(t), c_2(t)) \) which satisfies the multi-time growth law

\[ \frac{\partial k}{\partial t^\alpha}(t) = f_\alpha(k(t)) - \lambda_\alpha k(t) - c_\alpha(t), \quad \alpha = 1, 2 \]

and which minimizes the functional

\[ I(c(\cdot)) = \int_{t_0}^{t_T} e^{-D_\alpha t^\alpha} u(c(t))dt^\beta. \]

The nonautonomous control Hamiltonian is

\[ H = e^{-D_\alpha t^\alpha} \left( u(c) + q^\alpha(f_\alpha(k) - \lambda_\alpha k - c_\alpha) \right), \]

where the co-states variables \( p^\alpha(t) = q^\alpha(t)e^{-D_\alpha t^\alpha} \) mean the discounted values of additional investment. For an interior maximum with respect to the control \( c \) we must have \( \frac{\partial H}{\partial c_\gamma} = 0 \), i.e., \( \frac{\partial u}{\partial c_\gamma} = p^\gamma \). The adjoint equation

\[ \frac{\partial p^\alpha}{\partial t^\alpha} = -\frac{\partial H}{\partial k} = -(f_\alpha' - \lambda_\alpha)p^\alpha \]

and transversality condition

\[ p^1(t)n^1(t) + p^2(t)n^2(t)\big|_{t_0}^{t_T} = 0 \]
are equivalent to
\[
\frac{\partial q^\alpha}{\partial t^\alpha} = -(f^\alpha_k - \lambda^\alpha - D_\alpha)q^\alpha,
\]
\[
q^1(t)n^1(t) + q^2(t)n^2(t)|_{\partial\Omega_{0,T}} = 0.
\]
These PDEs produce the same information as those in the previous paragraph.

### 3.2 Case of path independent integral functional

For that we choose a rate of per capita consumption \(c(t) = (c_1(t), c_2(t))\) which satisfies the multi-time growth law
\[
\frac{\partial k}{\partial t^\alpha}(t) = f^\alpha_k(k(t)) - \lambda^\alpha k(t) - c^\alpha(t), \quad \alpha = 1, 2
\]
and which minimizes the functional
\[
J(c(\cdot)) = \int_{\Gamma_{0,T}} e^{-D_\lambda t^\lambda} u_\beta(c(t))dt^\beta.
\]
The nonautonomous control 1-form is
\[
S_\alpha = e^{-D_\lambda t^\lambda} (u_\alpha(c) + q(f^\alpha_k(k) - \lambda^\alpha k - c^\alpha))
\]
where the co-states variable \(p(t) = q(t)e^{-D_\lambda t^\lambda}\) means the discounted value of additional investment. For an interior maximum with respect to the control \(c\) we must have \(\frac{\partial S_\alpha}{\partial c_\gamma} = 0\), i.e., \(\frac{\partial S_\alpha}{\partial c_\gamma} = p\delta^\alpha_\gamma\). The adjoint equation
\[
\frac{\partial p}{\partial t^\alpha} = -\frac{\partial S_\alpha}{\partial k} = -(f^\alpha_k - \lambda^\alpha)p, \quad p(T) = 0
\]
is equivalent to
\[
\frac{\partial q^\alpha}{\partial t^\alpha} = -(f^\alpha_k - \lambda^\alpha - D_\alpha)q^\alpha, \quad q(T) = 0.
\]
Of course, here we need the complete integrability conditions. These PDEs produce the same information as those in the previous paragraph.

### 4 Optimal economic growth with bang-bang policy

In this section we adapt the multi-time controllability, observability and bang-bang principle [8] to the context of this paper. For that, let us accept that \(u_\beta(c) = c_\beta, a_\alpha^\alpha = \text{const}, c^\alpha_{0}(t) = \text{per capita consumption}, \|T\| = \infty\) and that we use the path independent curvilinear integral. Then
\[
\text{maximize } J(c(\cdot)) = \int_{\Gamma_{0,T}} e^{-D_\lambda t^\lambda} c_\beta(t)dt^\beta
\]
subject to
\[
\frac{\partial k}{\partial t^\alpha}(t) = f^\alpha_k(k(t)) - \lambda^\alpha k(t) - c^\alpha(t),
\]
where \(k\) = capital, and \(k(0) = k_0, D_\alpha, \lambda^\alpha\) are positive constants.

The nonautonomous control 1-form is
\[
S_\alpha = e^{-D_\lambda t^\lambda} c_\alpha + p(f^\alpha_k(k) - \lambda^\alpha k - c^\alpha)
\]
or, with the definition \(p(t) = q(t)e^{-D_\lambda t^\lambda}\), we can write
\[
S_\alpha = e^{-D_\lambda t^\lambda} (1 - q)c_\alpha + e^{-D_\lambda t^\lambda} q(f^\alpha_k(k) - \lambda^\alpha k).
\]
We remark that the control tensor is linear in the control variables \(c^\alpha(\cdot)\). Also we accept \(\bar{c}_\alpha \leq c^*_\alpha \leq f^\alpha_k(\cdot), \text{i.e., } \bar{c}_\alpha\text{ is the minimum level and } f^\alpha_k\text{ is the maximum level}. The switching functions \(\sigma_\alpha = e^{-D_\lambda t^\lambda} (1 - q)c_\alpha\text{ shows that the optimal policy is to choose}
\[
c^*_\alpha = \begin{cases} 
\bar{c}_\alpha (= 0) & \text{if } q = \begin{cases} > 1 \\ = 1 \\ < 1 \end{cases} 
\end{cases}
\]
The dynamic state and adjoint systems are
\[
\frac{\partial k}{\partial t^\alpha} = f^\alpha_k(k) - \lambda^\alpha k - c^*_\alpha(q),
\]
\[
\frac{\partial q^\alpha}{\partial t^\alpha} = -q^\beta(f^\alpha_k - \lambda^\alpha - D_\alpha),
\]
which are solved after substituting the optimal control \(c^* = (c^*_1, c^*_2)\).

Cases:
1) The first bang-bang policy should be used when \(q < 1\), \(c^*_\alpha = f^\alpha_k(k) = c^\alpha_{\text{max}}\). The above dynamic system becomes
\[
\frac{\partial k}{\partial t^\alpha} = -\lambda^\alpha k, \quad \frac{\partial q^\alpha}{\partial t^\alpha} = (\lambda^\alpha + D_\alpha - f^\alpha_k(k))q.
\]
In this way the capital stock decreases at 2-rate \((\lambda_1, \lambda_2), \text{i.e., } k(t) = e^{-\lambda_1 t^\lambda}\).

2) The second bang-bang policy should be used when \(q > 1\), \(c^*_\alpha = \bar{c}_\alpha = c^\alpha_{\text{min}} = 0\). The multi-time dynamic system is
\[
\frac{\partial k}{\partial t^\alpha} = f^\alpha_k(k) - \lambda^\alpha k, \quad \frac{\partial q^\alpha}{\partial t^\alpha} = (\lambda^\alpha + D_\alpha - f^\alpha_k(k))q.
\]
3) A singular control would be the appropriate policy if \( q \equiv 1, \sigma_\alpha = 0 \).


References:


