Determining a Pair of Metrics by Boundary Energy Associated to a Multitime PDE System

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Abstract: Our theory of determining a tensor by boundary energy of a multitime first order PDE system is similar to those developed by Sharafutdinov. Section 1 refines the theory of potential maps determined by a first order multitime PDE system and a vertical metric. Section 2 defines the boundary energy of a first order PDE system and proves that the problem of determining a vertical metric from the boundary energy of a multitime PDE system cannot have a unique solution. Section 3 linearizes the above mentioned problem and defines the notion of multi-ray transform.

Key Words: potential map, least squares Lagrangian, boundary energy, multi-ray transform, extremals.

1 Potential maps

Let (M, g) be a compact Riemannian manifold of dimension n with the boundary ∂M, $x = (x^i)$ be local coordinates on (M, g) and $(G_{jk,i})$ (G^i_{jk}) be the Christoffel symbols of (M, g) of the first and second type respectively.

Let (T, h) be an oriented compact Riemannian manifold of dimension p, with the boundary ∂T, $t = (t^\alpha)$ local coordinates on (T, h) and $(H_{\beta\gamma,\alpha})$, $(H^\alpha_{\beta\gamma})$ its Christoffel symbols of the first type and second type, respectively.

Consider $\varphi\colon T \to M$, $\varphi(t) = x$, $t = (t^1, \dots, t^p)$, $x = (x^1, \dots, x^n)$, a C^∞-map. We want to approximate the Jacobian matrix $\left(\dfrac{\partial x^i}{\partial t^\alpha}\right)$ by a matrix of gradients $(X^i_\alpha(t, x))$ associated to a C^∞ - distinguished tensor field X_α (n gradients) on $T \times M$, in the sense of least squares. For that we build the PDE system

$$\frac{\partial x^i}{\partial t^\alpha}(t) = X^i_\alpha(t, x(t)), \quad x|_{\partial T} = \chi,$$

and the least squares Lagrangian

$$L(t, x^i(t), x^i_\alpha(t)) = \frac{1}{2} h^{\alpha\beta}(t) g_{ij}(x(t))[x^i_\alpha(t) - X^i_\alpha(t, x(t))][x^j_\beta(t) - X^j_\beta(t, x(t))]\sqrt{h},$$

where $x^i_\alpha = \dfrac{\partial x^i}{\partial t^\alpha}$, and $h = \det(h_{\alpha\beta})$.

The Euler-Lagrange prolongation of the PDE system describes the potential map in the multitime geometric dynamics.

The geometric dynamics (ODEs or PDEs) is a Lagrangian dynamics (ODEs or PDEs) determined by a least squares Lagrangian attached to a first order (single-time or multitime) ODE or PDE system and a pair of Riemannian metrics, one in the source space and other in the target space [1], [2], [4]-[13].

Theorem 1.1 *The extremals of L are described by the PDEs*

$$h^{\alpha\beta} x^i_{\alpha\beta} = g^{iq} h^{\alpha\beta} g_{kj} (\nabla_q X^k_\alpha) X^j_\beta + h^{\alpha\beta} F_j{}^i{}_\alpha x^j_\beta + h^{\alpha\beta} D_\beta X^i_\alpha,$$

$$x|_{\partial T} = \chi,$$

where

$$\frac{\delta}{\partial t^\beta} x^i_\alpha = x^i_{\alpha\beta} = \frac{\partial^2 x^i}{\partial t^\alpha \partial t^\beta} - H^\gamma_{\alpha\beta} x^i_\gamma + G^i_{jk} x^j_\alpha x^k_\beta, \quad (1)$$

$$F_j{}^i{}_\alpha = \nabla_j X^i_\alpha - g^{iq} g_{kj} \nabla_q X^k_\alpha, \quad (2)$$

$$\nabla_j X^i_\alpha = \frac{\partial X^i_\alpha}{\partial X^j} + G^i_{jk} X^k_\alpha, \; D_\beta X^i_\alpha = \frac{\partial X^i_\alpha}{\partial t^\beta} - H^\gamma_{\alpha\beta} X^i_\gamma. \quad (3)$$

Proof. If we write $L = E\sqrt{h}$, where E is the energy density, then the Euler-Lagrange equations of extremals

$$\frac{\partial L}{\partial x^k} - \frac{\partial}{\partial t^\alpha}\frac{\partial L}{\partial x^k_\alpha} = 0$$

can be written

$$\frac{\partial E}{\partial x^k} - \frac{\partial}{\partial t^\alpha}\frac{\partial E}{\partial x^k_\alpha} - H^\gamma_{\gamma\alpha}\frac{\partial E}{\partial x^k_\alpha} = 0. \qquad (4)$$

We compute

$$\frac{\partial E}{\partial x^k} = \frac{1}{2}h^{\alpha\beta}\frac{\partial g_{ij}}{\partial x^k}x^i_\alpha x^j_\beta - h^{\alpha\beta}\frac{\partial g_{ij}}{\partial x^k}x^i_\alpha X^j_\beta + \frac{1}{2}h^{\alpha\beta}\frac{\partial g_{ij}}{\partial x^k}X^i_\alpha X^j_\beta - h^{\alpha\beta}g_{ij}x^i_\alpha\frac{\partial X^j_\beta}{\partial x^k} + h^{\alpha\beta}g_{ij}\frac{\partial X^i_\alpha}{\partial x^k}X^j_\beta,$$

$$\frac{\partial E}{\partial x^k_\alpha} = h^{\alpha\beta}g_{kj}x^j_\beta - h^{\alpha\beta}g_{kj}X^j_\beta,$$

$$-\frac{\partial}{\partial t^\alpha}\frac{\partial E}{\partial x^k_\alpha} = -\frac{\partial h^{\alpha\beta}}{\partial t^\alpha}g_{kj}x^j_\beta - h^{\alpha\beta}\frac{\partial g_{kj}}{\partial x^\ell}x^\ell_\alpha x^j_\beta - h^{\alpha\beta}g_{kj}\frac{\partial^2 x^j}{\partial t^\alpha\partial t^\beta} + \frac{\partial h^{\alpha\beta}}{\partial t^\alpha}g_{kj}X^j_\beta + h^{\alpha\beta}\frac{\partial g_{kj}}{\partial x^\ell}x^\ell_\alpha X^j_\beta + h^{\alpha\beta}g_{kj}\left(\frac{\partial X^j_\beta}{\partial t^\alpha} + \frac{\partial X^j_\beta}{\partial x^\ell}x^\ell_\alpha\right).$$

We replace into (4) taking into account the formulas (1), (3) and

$$\frac{\partial g_{ij}}{\partial x^k} = G^\ell_{ki}g_{\ell j} + G^\ell_{kj}g_{\ell i}$$

$$\frac{\partial h^{\alpha\beta}}{\partial t^\gamma} = -H^\alpha_{\gamma\lambda}h^{\lambda\beta} - H^\beta_{\gamma\lambda}h^{\alpha\lambda}. \qquad (5)$$

We find

$$h^{\alpha\beta}g_{kj}x^j_{\alpha\beta} = h^{\alpha\beta}g_{ij}(\nabla_k X^i_\alpha)X^j_\beta + h^{\alpha\beta}g_{kj}(\nabla_\ell X^j_\beta)x^\ell_\alpha - h^{\alpha\beta}g_{ij}x^i_\alpha\nabla_k X^j_\beta + h^{\alpha\beta}g_{kj}D_\alpha X^j_\beta.$$

Transvecting by g^{ik} and using the formula (2), we obtain

$$h^{\alpha\beta}x^i_{\alpha\beta} = g^{ik}h^{\alpha\beta}g_{\ell j}(\nabla_k X^\ell_\alpha)X^j_\beta + h^{\alpha\beta}F_j{}^i{}_\alpha x^j_\beta + h^{\alpha\beta}D_\alpha X^i_\beta \blacksquare$$

Definition 1.1 *The map $\varphi \in C^\infty(T, M)$, $\varphi(t) = x$, which verifies the PDEs from the above-mentioned theorem is called potential map associated to the d-tensor X_α (n gradients).*

Definition 1.2 *Suppose that ∂M is foliated by submanifolds of type σ. The pair (h, g) of Riemannian metrics or the vertical metric $h^{-1}\otimes g$ is called simple if there is a unique potential map $\varphi \in C^\infty(T, M)$, $\varphi(t) = x$, $2 \le p \le n$, fixed by a closed border σ of dimension $p - 1$, included in ∂M.*

2 Determining a pair of metrics by boundary energy associated to a first order PDE system

Starting from the boundary energy, we study the recovering of a tensor from the centered moments which determine the vertical metric $h^{-1}\otimes g$. In this sense we continue the research in [14], [15], generalizing the theory of Sharafutdinov [3].

Let (h, g) be a pair of simple metrics and $\varphi \in C^\infty(T, M)$, $\varphi(t) = x$, $2 \le p \le n$, $\dim T = p$, the corresponding potential map fixed by a closed border σ of dimension $p - 1$, $\sigma \subset \partial M$. Let $\mathcal{M}$ be the set of the closed borders σ of dimension $p-1$, $\sigma \subset \partial M$.

Definition 2.1 *Let $\sigma \in \mathcal{M}$. The function $E_{(h,g)}\colon \mathcal{M} \to \mathbb{R}$, $\sigma \mapsto E_{(h,g)}(\sigma)$,*

$$E_{(h,g)}(\sigma) = \frac{1}{2}\int_T h^{\alpha\beta}(t)g_{ij}(x(t))[x^i_\alpha(t) - X^i_\alpha(t, x(t))][x^j_\beta(t) - X^j_\beta(t, x(t))]\, dv_h,$$

is called the boundary energy of the multi-time PDE system

$$\frac{\partial x^i}{\partial t^\alpha}(t) = X^i_\alpha(t, x(t)),\ x|_{\partial T} = \chi$$

along the potential map $\varphi \in C^\infty(T, M)$, $\varphi(t) = x$.

Problem1 Given an energy function E, is there a pair of simple metrics (h, g) that realizes that energy? How can these metrics be found?

Let us show that the existence problem of the metrics with the property that $E\colon \mathcal{M} \to \mathbb{R}$ represents the boundary energy cannot have a unique solution.

Let $\Phi\colon T \times M \to T \times M$, $\Phi(t^1,\dots,t^p; x^1,\dots,x^n) = (\psi(t),\varphi(x))$ be a diffeomorphism with the properties $\psi|_{\partial T} = \mathrm{id}$, $\varphi|_{\partial M} = \mathrm{id}$. The diffeomorphism transforms the simple metrics h^0, g^0 into the simple metrics $h^1 = \psi^* h^0$ and $g^1 = \varphi^* g^0$, because we have

$$h^1(t)(\mu,\nu) = h^0((d_t\psi)\mu,(d_t\psi)\nu)_{\psi(t)},$$

where $d_t\psi\colon T_tT \to T_{\psi(t)}T$ is the differential of ψ and

$$g^1(x)(\xi,\eta) = g^0((d_x\varphi)\xi,(d_x\varphi)\eta)_{\varphi(x)},$$

$d_x\varphi\colon T_xM \to T_{\varphi(x)}M$ is the differential of φ.

It can be noticed that

$$X'^i_\alpha = \frac{\partial x'^i}{\partial x^j}\,\frac{\partial t^\gamma}{\partial t'^\alpha} X^j_\gamma,$$

that is

$$d_t\psi X' = d_x\varphi X,$$

where $t' = \psi(t)$, $x' = \varphi(x)$ and X'_γ represents the distinguished tensor field X_γ with respect to t', x'.

The pairs (h^0,g^0) and (h^1,g^1) give different families of potential maps with the same boundary energy E.

Problem 2 The problem of finding a pair of metrics by the boundary energy can be changed into the following problem. Let (h^0,g^0) and (h^1,g^1) be pairs of simple metrics, h^0, h^1 on T, respectively g^0, g^1 on M. Does the equality $E_{(h^0,g^0)} = E_{(h^1,g^1)}$ imply the existence of a diffeomorphism $\Phi\colon T \times M \to T \times M$, $\Phi = (\psi,\varphi)$, $\psi|_{\partial T} = \mathrm{id}$, $\varphi|_{\partial M} = \mathrm{id}$, $h^1 = \psi^* h^0$ and $g^1 = \varphi^* g^0$?

3 Linearization of the problem of finding a pair of metrics by the boundary energy

Let us linearize the problem **2**. Let (h^τ, g^τ) be a family of simple metrics which depends smoothly on the parameter $\tau \in (-\varepsilon,\varepsilon)$, $\varepsilon > 0$. Let σ be a closed border of dimension $p-1$, included in ∂M and $a = E(\sigma)$, where $E\colon \mathcal{M} \to \mathbb{R}$ is the given frontier energy. Consider $x^\tau\colon T \to M$ the potential map corresponding to σ, $T = [0,a]^p$,

$$t' = (t^{\tau,\alpha}), \quad t^{\tau,\alpha} = t^\alpha(\tau),$$
$$x^{\tau,i} = x^i(t'^\alpha,\tau), \quad i = \overline{1,n}.$$

Let $x' = (x^{\tau,i})$ and X'^i_α be the representation of $X^i_\alpha(t',x'(t'),\tau)$. We denote by $h_\tau = (h^{\alpha\beta}_\tau)$ and $g^\tau = (g^\tau_{ij})$.

The energy of the deformation x^τ is

$$E_{(h^\tau,g^\tau)}(\sigma) = \frac{1}{2}\int_T h^{\alpha\beta}_\tau(t')g^\tau_{ij}(x'(t'))[x^i_\alpha(t',\tau) - X^i_\alpha(t',x',\tau)][x^j_\beta(t',\tau) - X^j_\beta(t',x',\tau)]\,dv_{h^\tau}.$$

Differentiating with respect to τ, we obtain

$$\frac{\partial}{\partial\tau}E_{(h^\tau,g^\tau)}(\sigma) = \int_T \Big\{ h^{\alpha\beta}_\tau(t')\frac{\partial g^\tau_{ij}}{\partial\tau}(x^\tau(t')) + \Big[\frac{\partial h^{\alpha\beta}_\tau}{\partial\tau}(t') + \frac{\partial h^{\alpha\beta}_\tau}{\partial t'^\gamma}(t')\Big]\frac{dt'^\gamma}{d\tau}(\tau)g^\tau_{ij}(x^\tau(t'))\Big\}[x^i_\alpha(t',\tau)x^j_\beta(t',\tau) -$$
$$2x^i_\alpha(t',\tau)X^j_\beta(t',x^\tau(t'),\tau) + X^i_\alpha(t',x^\tau(t'),\tau)X^j_\beta(t',x^\tau(t'),\tau)]$$
$$+h^{\alpha\beta}_\tau(t')\frac{\partial g^\tau_{ij}}{\partial x^k}(x^\tau(t'))\Big[\frac{1}{2}x^i_\alpha(t',\tau)x^j_\beta(t',\tau) - x^i_\alpha(t',\tau) X^j_\beta(t',x^\tau(t'),\tau) + \frac{1}{2}X^i_\alpha(t',x^\tau(t'),\tau)X^j_\beta(t',x^\tau(t'),\tau)\Big]$$
$$\Big[\frac{\partial x^k}{\partial\tau}(t',\tau) + \frac{\partial x^k}{\partial t'^\gamma}(t',\tau)\frac{dt'^\gamma}{d\tau}(\tau)\Big] + h^{\alpha\beta}_\tau(t')g^\tau_{ij}(x^\tau(t'))$$
$$\Big\{\Big[\frac{\partial x^i_\alpha}{\partial\tau}(t',\tau) + \frac{\partial x^i_\alpha}{\partial t'^\gamma}(t',\tau)\frac{dt'^\gamma}{d\tau}\Big]x^j_\beta(t',\tau) - \frac{\partial x^i_\alpha}{\partial\tau}(t',\tau)$$
$$X^j_\beta(t',x^\tau(t'),\tau)(\tau) - x^i_\alpha(t',\tau)\frac{\partial X^j_\beta}{\partial x^k}(t',x^\tau(t'),\tau)$$
$$\frac{\partial x^k}{\partial\tau}(t',\tau) + \frac{\partial X^i_\alpha}{\partial x_k}(t',x^\tau(t'),\tau)(\tau)\Big[\frac{\partial x^k}{\partial\tau}(t',\tau)$$
$$+\frac{\partial x^k}{\partial t'^\gamma}(t',\tau)\frac{dt'^\gamma}{d\tau}(\tau)\Big]X^j_\beta(t',x^\tau(t'),\tau) - x^i_\alpha(t',\tau)$$
$$\frac{\partial X^j_\beta}{\partial\tau}(t',x^\tau(t'),\tau) + \frac{\partial X^j_\beta}{\partial\tau}(t',x^\tau(t'),\tau)X^i_\alpha(t',x^\tau(t'),\tau)\Big\}dv_{h^\tau}$$
$$= \int_T \Big\{ h^{\alpha\beta}_\tau(t')\frac{\partial g^\tau_{ij}}{\partial\tau}(x^\tau(t')) + \Big[\frac{\partial h^{\alpha\beta}_\tau}{\partial t'^\gamma}\frac{dt'^\gamma}{d\tau} + \frac{\partial h^{\alpha\beta}_\tau}{\partial\tau}(t')\Big]g^\tau_{ij}(x^\tau(t'))\Big\}[x^i_\alpha(t',\tau)x^j_\beta(t',\tau)$$
$$-2x^i_\alpha(t',\tau)X^j_\beta(t',x^\tau(t'),\tau) + X^i_\alpha(t',x^\tau(t'),\tau)$$
$$X^j_\beta(t',x^\tau(t'),\tau)]dv_{h^\tau} + \int_T \Big\{ h^{\alpha\beta}_\tau(t')\frac{\partial g^\tau_{ij}}{\partial x^k}(x^\tau(t'))$$
$$\Big[\frac{1}{2}x^i_\alpha(t',\tau)x^j_\beta(t',\tau) - x^i_\alpha(t',\tau)X^j_\beta(t',x^\tau(t'),\tau)$$
$$+\frac{1}{2}X^i_\alpha(t',x^\tau(t'),\tau)X^j_\beta(t',x^\tau(t'),\tau)\Big]$$
$$-h^{\alpha\beta}_\tau(t')g^\tau_{ij}(x^\tau(t'))\Big[x^i_\alpha(t',\tau)\frac{\partial X^j_\beta}{\partial x^k}(t',x^\tau(t'),\tau)$$

$$-\frac{\partial X^i_\alpha}{\partial x^k}(t', x^\tau(t'), \tau) X^j_\beta(t', x^\tau(t'), \tau)\Big]\Big\}\left[\frac{\partial x^k}{\partial \tau}(t', \tau)\right.$$
$$\left.+\frac{\partial x^i_\alpha}{\partial t'^\gamma}(t', \tau)\frac{dt'^\gamma}{d\tau}\right] dv_{h^\tau} + \int_T h^{\alpha\beta}_\tau(t') g^\tau_{ij}(x^\tau(t'))$$
$$[x^j_\beta(t', \tau) - X^j_\beta(t', x^\tau(t'), \tau)]\left[\frac{\partial x^i_\alpha}{\partial \tau}(t', \tau)\right.$$
$$\left.+\frac{\partial x^i_\alpha}{\partial t'^\gamma}(t', \tau)\frac{dt'^\gamma}{d\tau}(\tau)\right] dv_{h^\tau} + \int_T h^{\alpha\beta}_\tau(t') g^\tau_{ij}(x^\tau(t'))$$
$$[x^i_\alpha(t', \tau) - X^i_\alpha(t', x^\tau(t'), \tau)]$$
$$\frac{\partial X^j_\beta}{\partial \tau}(t', x^\tau(t'), \tau)\, dv_{h^\tau}. \quad (6)$$

Integrating by parts then considering $\tau = 0$ and using the fact that the total derivative of x^i with respect to τ is zero on ∂T, the third integral becomes:

$$I_3 = -\int_T \left\{ \frac{\partial h^{\alpha\beta}_0}{\partial t^\alpha}(t^0) g^0_{ij}(x^0(t^0))[x^j_\beta(t^0, 0)\right.$$
$$-X^j_\beta(t^0, x^0(t^0), 0)] + h^{\alpha\beta}_0(t^0)\frac{\partial g_{ij}}{\partial x^k}(x^0(t^0)) x^k_\alpha(t^0, 0)$$
$$[x^j_\beta(t^0, 0) - X^j_\beta(t^0, x^0(t^0), 0)] + h^{\alpha\beta}_0(t^0) g^0_{ij}(x^0(t^0))$$
$$\left[\frac{\partial^2 x^j}{\partial t^\alpha \partial t^\beta}(t^0, 0) - \frac{\partial X^j_\beta}{\partial t^\alpha}(t^0, x^0(t^0), 0)\right.$$
$$\left.-\frac{\partial X^j_\beta}{\partial x^k}(t^0, x^0(t^0), 0) x^k_\alpha(t^0, 0)\right] + h^{\alpha\beta}_0(t^0) g^0_{ij}(x^0(t^0))$$
$$\left.[x^j_\beta(t^0, 0) - X^j_\beta(t^0, x^0(t), 0)]\frac{1}{2} h^{\gamma\delta}_0(t^0)\frac{\partial h^0_{\gamma\delta}}{\partial t^\alpha}(t)\right\}$$
$$\left[\frac{\partial x^i}{\partial \tau}(t^0, 0) + \frac{\partial x^i}{\partial t'^\gamma}(t^0, 0)\frac{dt'^\gamma}{d\tau}(0)\right] dv_{h^0}. \quad (7)$$

Making the sum of the second and third integrals of (6) and using (7), we obtain:

$$I_2 + I_3 = \frac{1}{2}\int_T \left\{ h^{\alpha\beta}_0(t^0)\frac{\partial g^0_{ij}}{\partial x^k}(x^0(t^0))[x^i_\alpha(t^0, 0) x^j_\beta(t^0, 0)\right.$$
$$-2x^i_\alpha(t^0, 0) X^j_\beta(t^0, x^0(t^0), 0) + X^i_\alpha(t^0, x^0(t^0), 0)$$
$$X^j_\beta(t^0, x^0(t^0), 0)] - 2h^{\alpha\beta}_0(t^0) g^0_{ij}(x^0(t^0))\Big[x^i_\alpha(t^0, 0)$$
$$\left.\frac{\partial X^j_\beta}{\partial x^k}(t^0, x^0(t^0), 0) - \frac{\partial X^i_\alpha}{\partial x^k}(t^0, x^0(t^0), 0) X^j_\beta(t^0, x^0(t^0), 0)\right]$$
$$-2\frac{\partial h^{\alpha\beta}_0}{\partial t'^\alpha}(t^0) g^0_{kj}(x^0(t^0))[x^j_\beta(t^0, 0) - X^j_\beta(t^0, x^0(t^0), 0)]$$
$$-2h^{\alpha\beta}_0(t^0)\frac{\partial g^0_{kj}}{\partial x^\ell}(x^0(t^0)) x^\ell_\alpha(t^0, 0)[x^j_\beta(t^0, 0)$$
$$-X^j_\beta(t^0, x^0(t^0), 0)] - 2h^{\alpha\beta}_0(t^0) g^0_{kj}(x^0(t^0))$$
$$\left[\frac{\partial^2 x^j}{\partial t'^\alpha \partial t'^\beta}(t^0, 0) - \frac{\partial X^j_\beta}{\partial t'^\alpha}(t^0, x^0(t^0), 0)\right.$$
$$\left.-\frac{\partial X^j_\beta}{\partial x^\ell}(t^0, x^0(t^0), 0) x^\ell_\alpha(t^0, 0)\right] - h^{\alpha\beta}_0(t^0) h^{\gamma\delta}_0(t^0) g^0_{kj}(x^0(t^0))$$
$$\left.\frac{\partial h^0_{\gamma\delta}}{\partial t'^\alpha}(t^0)[x^j_\beta(t^0, 0) - X^j_\beta(t^0, x^0(t^0), 0)]\right\}$$
$$\left[\frac{\partial x^k}{\partial \tau}(t^0, 0) + \frac{\partial x^k}{\partial t'^\gamma}(t^0, 0)\frac{dt'^\gamma}{d\tau}(0)\right] dv_{h^0}$$
$$= \int_T h^{\alpha\beta}_0(t^0) x^j_\beta(t^0, 0)\Big[-g^0_{ki}(x^0(t^0))$$
$$(\nabla_j X^i_\alpha)(t^0, x^0(t^0), 0) + g^0_{\ell j}(x^0(t^0))$$
$$(\nabla_k X^\ell_\alpha)(t^0, x^0(t^0), 0) - g^0_{jq}(x^0(t^0))(\nabla_k X^q_\alpha)(x^0(t^0))$$
$$+g^0_{qk}(x^0(t^0)) G^q_{\ell j}(x^0(t^0), 0) X^\ell_\alpha(t^0, x^0(t^0), 0)$$
$$\left.+g^0_{k\ell}(x^0(t^0))\frac{\partial X^\ell_\alpha}{\partial x^j}(t^0, x^0(t^0), 0)\right]$$
$$\left[\frac{\partial x^k}{\partial \tau}(t^0, 0) + \frac{\partial x^k}{\partial t'^\gamma}(t^0, \tau)\frac{dt'^\gamma}{d\tau}(0)\right] dv_{h^0} = 0.$$

Consequently,

$$\left.\frac{\partial}{\partial \tau}\right|_{\tau=0} E_{(h^\tau, g^\tau)}(\sigma) = \int_T F^{\alpha\beta}_{ij}(t^0, x^0(t^0), 0)[x^i_\alpha(t^0, 0)$$
$$-X^i_\alpha(t^0, x^0(t^0), 0)][x^j_\beta(t^0, 0) - X^j_\beta(t^0, x^0(t^0), 0)]\, dv_{h^0}$$
$$+\int_T h^{\alpha\beta}_0(t^0) g^0_{ij}(x^0(t^0))\frac{\partial X^j_\beta}{\partial \tau}(t^0, x^0(t^0), 0)\ [x^i_\alpha(t^0, 0)$$
$$-X^i_\alpha(t^0, x^0(t^0), 0)] dv_{h^0},$$

where

$$F^{\alpha\beta}_{ij}(t^0, x^0(t^0), 0) = h^{\alpha\beta}_0(t^0) f_{ij}(x^0(t^0))$$
$$+\left[k^{\alpha\beta}(t^0) + \frac{\partial h^{\alpha\beta}_0}{\partial t'^\gamma}(t^0)\frac{dt'^\gamma}{d\tau}(0)\right] g^0_{ij}(x^0(t^0)).$$

Let us consider that

$$\frac{\partial X^j_\beta}{\partial \tau}(t^0, x^0(t^0), 0) = 0.$$

Considering the d-tensor field $F = (F^{\alpha\beta}_{ij})$ and using the functional

$$I_F(x^0) = \int_T \left\{ F^{\alpha\beta}_{ij}(t^0, x^0(t^0), 0)[x^i_\alpha(t^0, 0)\right.$$
$$-X^i_\alpha(t^0, x^0(t^0), 0)][x^j_\beta(t^0, 0) - X^j_\beta(t^0, x^0(t^0), 0)] dv_{h^0}$$

the previous relation becomes

$$\frac{\partial}{\partial\tau}\Big|_{\tau=0} E_{(h^\tau,g^\tau)}(\sigma) = I_F(x^0),$$

where x^0 is a potential map corresponding to the closed border σ of dimension $p-1$, included in ∂M.

The function I_F, determined by this equality is called *the multi-ray transform of the tensor field* F.

The existence of solutions of problem **1** for the family (h^τ, g^τ) implies the existence of a one parameter group of diffeomorphisms $\Phi^\tau(t,x) = (\psi^\tau(t), \varphi^\tau(x))$, such that $g^\tau = (\varphi^\tau)^* g^0$ and $h^\tau = (\psi^\tau)^* h^0$. Explicitly

$$h^\tau_{\alpha\beta} = (h^0_{\mu\nu} \circ \psi^\tau) \frac{\partial t'^\mu}{\partial t^\alpha} \frac{\partial t'^\nu}{\partial t^\beta}, \tag{8}$$

where $\psi^\tau(t) = (\psi^1(t,\tau), \ldots, \psi^p(t,\tau))$, $t' = \psi^\tau(t)$,

$$g^\tau_{ij} = (g^0_{k\ell} \circ \varphi^\tau) \frac{\partial x'^k}{\partial x^i} \frac{\partial x'^\ell}{\partial x^j}, \tag{9}$$

where $\varphi^\tau(x) = (\varphi^1(x,\tau), \ldots, \varphi^n(x,\tau))$, $x' = \varphi^\tau(x)$.

Theorem 3.1 *Let* $v^k(x) = \frac{\partial}{\partial\tau}\Big|_{\tau=0}(x'^k)(x,\tau)$, $k = \overline{1,n}$, $v_i = g^0_{ij} v^j$ *and* $v_{i;j}$ *the covariant derivative of* (v_i). *Also, we consider* $u^\alpha = \frac{\partial}{\partial\tau}\Big|_{\tau=0}(\psi^\alpha)(t,\tau)$, $\alpha = \overline{1,p}$, $u_\alpha = h^0_{\alpha\mu} u^\mu$, $\alpha = \overline{1,p}$, $u_{\alpha;\beta}$ *is the covariant derivative of* (u^α) *and* $u^{\alpha;\beta} = -h_0^{\gamma\alpha} h_0^{\mu\beta} u_{\gamma;\mu}$, $\alpha, \beta = \overline{1,p}$.

Then, the following relations hold

$$f_{ij} = \frac{1}{2}(v_{i;j} + v_{j;i}), \quad i,j = \overline{1,n}, \tag{10}$$

$$k^{\alpha\beta} = \frac{1}{2}(u^{\alpha;\beta} + u^{\beta;\alpha}), \quad \alpha,\beta = \overline{1,p}. \tag{11}$$

Proof. Differentiating the relation (9) with respect to τ and then considering $\tau = 0$, we find

$$2f_{ij} = \frac{\partial}{\partial\tau}\Big|_{\tau=0} g^\tau_{ij} = \frac{\partial g^0_{k\ell}}{\partial x^m} v^m \frac{\partial x'^k}{\partial x^i} \frac{\partial x'^\ell}{\partial x^j}$$

$$+ g^0_{k\ell} \frac{\partial}{\partial x^i} \left(\frac{\partial}{\partial\tau}\Big|_{\tau=0} x'^k \right) \frac{\partial x'^\ell}{\partial x^j}$$

$$+ g^0_{k\ell} \frac{\partial x'^k}{\partial x^i} \frac{\partial}{\partial x^j} \left(\frac{\partial}{\partial\tau}\Big|_{\tau=0} x'^\ell \right)$$

$$= \frac{\partial g^0_{ij}}{\partial x^q} v^q + g^0_{jq} \frac{\partial v^q}{\partial x^i} + g^0_{iq} \frac{\partial v^q}{\partial x^j}.$$

On the other hand

$$v_{i;j} + v_{j;i} = \frac{\partial v^i}{\partial x^j} - G^m_{ij} v_m + \frac{\partial v_j}{\partial x^i} - G^m_{ji} v_m$$

$$= \frac{\partial g^0_{ij}}{\partial x^q} v^q + g^0_{jq} \frac{\partial v^q}{\partial x^i} + g^0_{iq} \frac{\partial v^q}{\partial x^j},$$

and relation (10) is proved.

Because of the equality $h^{\mu\nu}_\tau h^\tau_{\mu\gamma} = \delta^\nu_\gamma$, the differentiation with respect to τ leads to

$$\frac{\partial}{\partial\tau}\Big|_{\tau=0}(h^{\mu\nu}_\tau) h^0_{\mu\gamma} + h^{\mu\nu}_0 \frac{\partial}{\partial\tau}\Big|_{\tau=0}(h^\tau_{\mu\gamma}) = 0,$$

that is

$$\frac{\partial}{\partial\tau}\Big|_{\tau=0}(h^{\alpha\nu}_\tau) = -h_0^{\gamma\alpha} h_0^{\mu\nu} \frac{\partial}{\partial\tau}\Big|_{\tau=0}(h^\tau_{\mu\gamma}). \tag{12}$$

Differentiating the relation (8) with respect to τ, it can be proved an equality similar to (10),

$$\frac{\partial}{\partial\tau}\Big|_{\tau=0}(h^\tau_{\mu\nu}) = u_{\mu;\nu} + u_{\nu;\mu}.$$

By replacing into relation (12), we obtain the equality (11) ∎

Therefore, the following generalization of open problem **2** appears. To what extent do the integrals

$$I_F(x) = \int_T \Big\{ F^{\alpha\beta}_{ij}(t,x(t))[x^i_\alpha(t)$$

$$-X^i_\alpha(t,x(t))][x^j_\beta(t) - X^j_\beta(t,x(t))] dv_h$$

determine the tensor field $(F^{\alpha\beta}_{ij})$?

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