Determining a Metric by Boundary Single-Time Flow Energy

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Abstract: Our theory of determining a tensor by single-time flow energy is similar to those developed by Sharafutdinov. Section 1 refines the theory of potential curves determined by a flow and a Riemannian metric. Section 2 defines the boundary energy of a potential curve and proves that the problem of determining a metric from a single-time flow boundary energy cannot have a unique solution. Section 3 linearizes the above-mentioned problem and defines the notion of single-ray transform.

Key Words: potential curves, least squares Lagrangian, single-ray transform, boundary energy.

1 Potential curves

Let (M, g) be a compact Riemannian manifold with the boundary ∂M and of dimension n. We consider $x = (x^i)$ local coordinates on the manifold (M, g), $(G_{jk,\ell})$ and (G^i_{jk}) its Christoffel symbols of the first and of the second type, respectively.

Let $\varphi\colon [0,1] \to M$, $\varphi(t) = x$, $x(t) = (x^1(t), \dots, x^n(t))$ be a C^∞-curve. We want to approximate the velocity $\dfrac{dx}{dt}$ of components $\dfrac{dx^i}{dt}$ by a C^∞-distinguished vector field X of components $X^i(x)$, in the sense of least squares. For that we build the flow $\dfrac{dx^i}{dt}(t) = X^i(x(t))$, $\overline{1,n}$, $x(0) = p$, $x(1) = q$, where p and q are two points from the boundary ∂M of the manifold M, and the least squares Lagrangian (flow energy density),

$$L(t, x(t), \dot{x}(t)) = \frac{1}{2} g_{ij}(x(t))[\dot{x}^i(t) - X^i(x(t))] [\dot{x}^j(t) - X^j(x(t))],$$

where $\dot{x}^i(t) = \dfrac{dx^i}{dt}(t)$.

The geometric dynamics (ODEs or PDEs) is a Lagrangian dynamics (ODEs or PDEs) determined by a least squares Lagrangian attached to a (single-time or multi-time) flow and a pair of Riemannian metrics, one in the source space and other in the target space.

Let us look for the Euler-Lagrange prolongation of the flow obtained as Euler-Lagrange ODEs produced by L. The extremals of L are called *potential curves*. These curves are geodesics [4].

Theorem 1.1 *The extremals of L are described by the ODEs:*

$$\frac{\delta}{dt}\dot{x}^i = g^{i\ell} g_{kj} (\nabla_\ell X^k) X^j + F^i_j \dot{x}^j, \quad i = \overline{1,n}$$
$$x(0) = p, \quad x(1) = q,$$

where:

$$\frac{\delta}{dt}\dot{x}^i = \ddot{x}^i + G^i_{jk}\,\dot{x}^j\,\dot{x}^k, \quad i = \overline{1,n},$$

$$F^i_j = \nabla_j X^i - g^{i\ell} g_{kj} \nabla_\ell X^k, \quad i, j = \overline{1,n}; \quad (1)$$

$$\nabla_j X^i = \frac{\partial X^i}{\partial x^j} + G^i_{jk} X^k, \quad i, j = \overline{1,n}. \quad (2)$$

Proof. We compute

$$\frac{\partial L}{\partial x^k} = \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\dot{x}^i\,\dot{x}^j - \frac{\partial g_{ij}}{\partial x^k}\dot{x}^i X^j + \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k} X^i X^j - g_{ij}\dot{x}^j\frac{\partial X^i}{\partial x^k} + g_{ij}\frac{\partial X^i}{\partial x^k} X^j;$$

$$\frac{\partial L}{\partial \dot{x}^k} = g_{ik}\dot{x}^i - g_{ik} X^i;$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right) = \frac{\partial g_{ik}}{\partial x^\ell}\dot{x}^\ell \dot{x}^i + g_{ik}\ddot{x}^i - \frac{\partial g_{ik}}{\partial x^\ell}\dot{x}^i X^\ell - g_{ik}\left(\frac{\partial X^i}{\partial t} + \frac{\partial X^i}{\partial x^\ell}\dot{x}^\ell\right).$$

We take into account formula (1) and

$$\frac{\partial g_{ij}}{\partial x^k} = G^{\ell}_{ki} g_{\ell j} + G^{\ell}_{kj} g_{\ell i}, \quad i,j,k = \overline{1,n}.$$

If we replace these relations in Euler-Lagrange equations,

$$\frac{\partial L}{\partial x^k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right) = 0, \quad k = \overline{1,n},$$

we find

$$g_{kj}\frac{\delta \dot{x}^j}{dt} = g_{ij}(\nabla_k X^i)X^j + g_{kj}(\nabla_\ell X^j)\dot{x}^\ell$$
$$-g_{ij}\dot{x}^i \nabla_k X^j, \quad k = \overline{1,n}.$$

Transvecting by $g^{\ell k}$ and using formula (2), we obtain

$$\frac{\delta \dot{x}^i}{dt} = g^{ik} g_{\ell j}(\nabla_k X^\ell)X^j + F^i_j \dot{x}^j, \quad i = \overline{1,n} \ \blacksquare$$

Definition 1.1 *The map $\varphi \in C^\infty([0,1], M)$, $\varphi(t) = x$, which verifies the ODEs from the previous theorem is called potential curve associated to the d-vector field X.*

Definition 1.2 *The Riemannian metric g is called simple if there is a unique potential curve $x\colon [0,1] \to M$, $x(0) = p$, $x(1) = q$, $p, q \in \partial M$.*

2 Determining a metric by boundary flow energy

Starting from the boundary flow energy, we study the recovering of a tensor from the centered moments which determine the metric g.

Let g be a simple metric and $x\colon [0,1] \to M$ the corresponding potential curve, $x(0) = p$, $x(1) = q$, $p, q \in \partial M$.

Definition 2.1 *Let p and q be two points on the boundary ∂M of the manifold M. The function $E_g\colon \partial M \times \partial M \to \mathbb{R}$, $(p,q) \mapsto E_g(p,q)$,*

$$E_g(p,q) = \frac{1}{2}\int_0^1 g_{ij}(x(t))[\dot{x}^i(t) - X^i(x(t))][\dot{x}^j(t)$$
$$-X^j(x(t))]\, dt$$

is called the boundary energy of the flow along the potential curve $x\colon [0,1] \to M$, $x(0) = p$, $x(1) = q$, $p, q \in \partial M$.

Open problem 1. Given an energy E, is there a metric g that realizes this energy? How can these metrics be found?

Let us show that the existence problem of the metrics with the property that $E\colon \partial M \times \partial M \to \mathbb{R}$ represents the boundary energy cannot have a unique solution.

Let $\varphi\colon M \to M$ be a diffeomorphism with the property $\varphi|_{\partial M} = \mathrm{id}$. This diffeomorphism transforms the simple metric g^0 into a simple metric $g^1 = \varphi^* g^0$, because we have

$$g^1(x)(\xi, \eta) = g^0((d_x\varphi)\xi, (d_x\varphi)\eta)_{\varphi(x)},$$

where $d_x\varphi\colon T_x M \to T_{\varphi(x)}M$ is the differential of φ.

It can be noticed that

$$X'^i = \frac{\partial x'^i}{\partial x^i} X^i,$$

where $X^{i'}$ represents the distinguished vector field X with respect to $x' = \varphi(x)$.

The metrics g^0 and g^1 give different families of potential curves with the same boundary energy E.

Open problem 2. The problem of recovering a metric by boundary energy can be changed in the following way. Let g^0 and g^1 be simple metrics on M. Does the equality $E_{g^0} = E_{g^1}$ impliy the existence of a diffeomorphism $\varphi\colon M \to M$, $\varphi|_{\partial M} = \mathrm{id}$ and $g^1 = \varphi^* g^0$?

3 Linearization of the problem of finding a metric from the boundary energy

Let us linearize the open problem **1**. Let (g^τ) be a family of simple metrics which depends smoothly on parameter $\tau \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$. Let p and q be two points of the border ∂M of the manifold M and $a = E(p,q)$, where $E\colon \partial M \times \partial M \to \mathbb{R}$ is the given boundary energy. Consider $x^\tau\colon [0,a] \to M$ the potential curve corresponding to the pair (p,q), $x' = (x^{\tau,i})$, $x^{\tau,i}(t) = x^i(t,\tau)$, $i = \overline{1,n}$. We denote $g^\tau = (g^\tau_{ij})$.

The energy of deformation x^τ is

$$E_{g^\tau}(p,q) = \frac{1}{2}\int_0^a g^\tau_{ij}(x^\tau(t))[\dot{x}^i(t,\tau) - X^i(x(t,\tau),\tau)]$$
$$[\dot{x}^j(t,\tau) - X^j(x(t,\tau),\tau)]\, dt.$$

Differentiating with respect to τ we obtain

$$\frac{\partial}{\partial\tau}E_{g^\tau}(p,q)=\int_0^a\left\{\frac{\partial g^\tau_{ij}}{\partial\tau}(x^\tau(t))[\dot{x}^i(t,\tau)-X^i(x^\tau(t),\tau)]\right.$$
$$[\dot{x}^j(t,\tau)-X^j(x^\tau(t),\tau)+\frac{\partial g^\tau_{ij}}{\partial x^k}(x^\tau(t))\Big[\frac{1}{2}\dot{x}^i(t,\tau)\dot{x}^j(t,\tau)]$$
$$-\dot{x}^i(t,\tau)X^j(x^\tau(t),\tau)+\frac{1}{2}X^i(x^\tau(t),\tau)$$
$$X^j(x^\tau(t),\tau)\Big]\frac{\partial x^k}{\partial\tau}(t,\tau)+g^\tau_{ij}(x^\tau(t))\Big[\frac{\partial\dot{x}^i}{\partial\tau}(t,\tau)\dot{x}^j(t,\tau)$$
$$-\frac{\partial\dot{x}^i}{\partial\tau}(t,\tau)X^j(x^\tau(t),\tau)-\dot{x}^i(t,\tau)\frac{\partial X^j}{\partial\tau}(x^\tau(t),\tau)$$
$$-\dot{x}^i(t,\tau)\frac{\partial X^j}{\partial x^k}(x^\tau(t),\tau)\frac{\partial x^k}{\partial\tau}(t,\tau)+X^i(x^\tau(t),\tau)$$
$$\frac{\partial X^j}{\partial\tau}(x^\tau(t),\tau)+X^i(x^\tau(t),\tau)\frac{\partial X^j}{\partial x^k}(x^\tau(t),\tau)$$
$$\left.\frac{\partial x^k}{\partial\tau}(t,\tau)\Big]\right\}dt=\int_0^a\frac{\partial g^\tau_{ij}}{\partial\tau}(x(t,\tau))\Big[\dot{x}^i(t,\tau)\dot{x}^j(t,\tau))$$
$$-2\dot{x}^i(t,\tau)X^j(x^\tau(t),\tau+X^i(x^\tau(t),\tau)X^j(x^\tau(t),\tau)\Big]dt$$
$$+\int_0^a\left\{\frac{\partial g^\tau_{ij}}{\partial x^k}(x^\tau(t))\Big[\frac{1}{2}\dot{x}^i(t,\tau)\dot{x}^j(t,\tau)\right.$$
$$-\dot{x}^i(t,\tau)X^j(x^\tau(t),\tau)+\frac{1}{2}X^i(x^\tau(t),\tau)X^j(x^\tau(t),\tau)\Big]$$
$$-g^\tau_{ij}(x^\tau(t))[\dot{x}^i(t,\tau)-X^i(x^\tau(t),\tau)]$$
$$\left.\frac{\partial X^j}{\partial x^k}(x^\tau(t),\tau)\right\}\frac{\partial x^k}{\partial\tau}(t,\tau)\,dt+\int_0^a g^\tau_{ij}(x^\tau(t))[\dot{x}^j(t,\tau)$$
$$-X^j(x^\tau(t),\tau)]\frac{\partial\dot{x}^i}{\partial\tau}(t,\tau)\,dt-\int_0^a g^\tau_{ij}(x^\tau(t))[\dot{x}^i(t,\tau)$$
$$-X^i(x^\tau(t),\tau)]\frac{\partial X^j}{\partial x^k}(x^\tau(t),\tau)\Big\}\frac{\partial x^k}{\partial\tau}(t,\tau)\,dt+\int_0^a g^\tau_{ij}(x^\tau(t))$$
$$[\dot{x}^i(t,\tau)-X^i(x^\tau(t),\tau)]\frac{\partial X^j}{\partial\tau}(x^\tau(t),\tau)\,dt. \tag{3}$$

Integrating by parts, then considering $\tau=0$ and using the fact that

$$\left.\frac{\partial x^i}{\partial\tau}\right|_0^a=0,$$

the third integral from relation (3) becomes

$$I_3=-\int_0^a\left.\frac{d}{dt}\right|_{\tau=0}g^\tau_{ij}(x^\tau(t))[\dot{x}^j(t,\tau)-X^j(x^\tau(t),\tau)]\frac{\partial x^i}{\partial\tau}(t,0)\,dt$$
$$=-\int_0^a\left\{\frac{\partial g^0_{ij}}{\partial x^\ell}(x^0(t))\dot{x}^\ell(t,0)[\dot{x}^j(t,0)-X^j(x^0(t),0)]\right.$$
$$\left.+g^0_{ij}(x^0(t))\Big[\ddot{x}^j(t,0)-\frac{\partial X^j}{\partial x^\ell}(x^0(t),0)\Big]\dot{x}^\ell(t,0)\right\}\frac{\partial x^i}{\partial\tau}(t,0)\,dt.$$

Relation (3) becomes, at $\tau=0$:

$$\left.\frac{\partial}{\partial\tau}\right|_{\tau=0}E_{g^\tau}(p,q)=\int_0^a f_{ij}(x^0(t))[\dot{x}^i(t,0)x^j(t,0)$$
$$-2\dot{x}^i(t,0)X^j(x^0(t),0)+X^i(x^0(t),0)X^j(x^0(t),0)]\,dt$$
$$+\int_0^a\left\{\frac{\partial g^0_{ij}}{\partial x^k}(x^0(t))\Big[\frac{1}{2}\dot{x}^i(t,0)\dot{x}^j(t,0)\right.$$
$$-\dot{x}^i(t,0)X^j(x^0(t),0)+\frac{1}{2}X^i(x^0(t),0)X^j(t,x^0(t),0)\Big]$$
$$\left.-g^0_{ij}(x^0(t))[\dot{x}^i(t,0)-X^i(x^0(t),0)]\frac{\partial X^j}{\partial x^k}(x^0(t),0)\right\}$$
$$\frac{\partial x^k}{\partial\tau}(t,0)\,dt+\left\{-\int_0^a\frac{\partial g^0_{ij}}{\partial x^\ell}(x^0(t))\dot{x}^\ell(t,0)[\dot{x}^j(t,0)\right.$$
$$-X^j(x^0(t),0)]+g^0_{ij}(x^0(t))\Big[\ddot{x}^j(t,0)-\frac{\partial X^j}{\partial x^\ell}(x^0(t),0)$$
$$\left.\dot{x}^\ell(t,0)\Big]\frac{\partial x^i}{\partial\tau}(t,0)\,dt\right\}-\int_0^a g^0_{ij}(x^0(t))[\dot{x}^i(t,0)$$
$$-X^i(x^0(t),0)]\frac{\partial X^j}{\partial\tau}(x^0(t),0)\,dt, \tag{4}$$

where $f_{ij}=\frac{1}{2}\left.\frac{\partial}{\partial\tau}\right|_{\tau=0}g^\tau_{ij}$.

Making the sum of the second and third integrals of (4), we obtain

$$I_2+I_3=-2\frac{\partial g^0_{kj}}{\partial x^i}(x^0(t))\dot{x}^i(t,0)\dot{x}^j(t,0)+\frac{\partial g^0_{ij}}{\partial x^k}(t,0)$$
$$[-2\dot{x}^i(t,0)X^j(x^0(t),0)+X^i(x^0(t),0)X^j(x^0(t),0)]$$
$$-2g^0_{ij}(x^0(t))\Big[\dot{x}^i(t,0)\frac{\partial X^j}{\partial x^k}(x^0(t),0)-\frac{\partial X^i}{\partial x^k}(x^0(t),0)$$
$$X^j(x^0(t),0)\Big]+2\frac{\partial g^0_{kj}}{\partial x^\ell}(x^0(t))\dot{x}^\ell(t,0)X^j(x^0(t),0)$$
$$-2g^0_{kj}(x^0(t))\Big[\ddot{x}^j(t,0)-\frac{\partial X^j}{\partial x^\ell}(x^0(t),0)\dot{x}^\ell(t,0)\Big]\Big\}\frac{\partial x^k}{\partial\tau}(t,0)dt$$
$$=\frac{1}{2}\int_0^a\Big\{-2g^0_{\ell k}(x^0(t))G^\ell_{ij}(x^0(t),0)\dot{x}^i(t,0)\dot{x}^j(t,0)$$
$$+[G_{ki,j}(x^0(t),0)+G_{kj,i}(x^0(t),0)][-2\dot{x}^i(t,0)X^j(x^0(t),0)$$
$$+X^i(x^0(t),0)X^j(t,x^0(t),0)]-2g^0_{ij}(x^0(t))$$
$$\Big[\dot{x}^i(t,0)\frac{\partial X^j}{\partial x^k}(x^0(t),0)-\frac{\partial X^i}{\partial x^k}(x^0(t),0)X^j(x^0(t),0)\Big]$$
$$+2[G_{k\ell,j}(x^0(t),0)+G_{\ell j,k}(x^0(t),0)]\dot{x}^\ell(t,0)X^j(x^0(t),0)$$
$$-2g^0_{kj}(x^0(t))\Big[\ddot{x}^j(t,0)-\frac{\partial X^j}{\partial t}(x^0(t),0)-\frac{\partial X^j}{\partial x^\ell}(x^0(t),0)$$

$$\dot{x}^\ell(t,0)\Big]\Big\}\frac{\partial x^k}{\partial\tau}(t,0)dt=\frac{1}{2}\int_0^a\Big\{-2g^0_{\ell k}(x^0(t))G^\ell_{ij}(x^0(t),0)$$
$$\dot{x}^i(t,0)\dot{x}^j(t,0)-2g^0_{iq}(x^0(t))\dot{x}^i(t,0)\nabla_k X^q(x^0(t),0)$$
$$+2g^0_{jq}(x^0(t))\Big[G^q_{k\ell}(x^0(t),0)X^\ell(x^0(t),0)X^j(x^0(t),0)$$
$$+\frac{\partial X^q}{\partial x^k}(x^0(t),0)X^j(x^0(t),0)\Big]+2g^0_{qk}(x^0(t))$$
$$G^q_{j\ell}(x^0(t),0)\dot{x}^\ell X^j(x^0(t),0)+2g^0_{kj}(x^0(t))$$
$$\frac{\partial X^j}{\partial x^\ell}(x^0(t),0)\dot{x}^\ell(t,0)\Big\}\frac{\partial x^k}{\partial\tau}(t,0)\,dt$$
$$=\int_0^a\Big\{-g^0_{jk}(x^0(t))[\ddot{x}^j(t,0)-g^0_{iq}(x^0(t))$$
$$\dot{x}^i(t,0)\nabla_k X^q(x^0(t),0)+g^0_{qk}(x^0(t))G^q_{j\ell}(x^0(t),0)$$
$$\dot{x}^\ell(t,0)X^j(x^0(t),0)+g^0_{kj}(x^0(t))\frac{\partial X^j}{\partial x^\ell}(x^0(t),0)$$
$$\dot{x}^\ell(t,0)\Big\}\frac{\partial x^k}{\partial\tau}(t,0)\,dt=\int_0^a\Big[-g^0_{\ell j}(x^0(t))$$
$$(\nabla_k X^\ell(x^0(t),0)X^j(x^0(t),0)-g^0_{ki}(x^0(t))F^i_j(x^0(t),0)$$
$$\dot{x}^j(t,0)-g^0_{iq}(x^0(t))\dot{x}^i(t,0)\nabla_k X^q(x^0(t),0)$$
$$+g^0_{jq}(t)\nabla_k X^q(x^0(t),0)X^j(x^0(t),0)$$
$$+g^0_{qk}(x^0(t))G^q_{j\ell}(x^0(t),0)\dot{x}^\ell(t,0)$$
$$X^j(x^0(t),0)\Big]\frac{\partial x^k}{\partial\tau}(t,0)\,dt=\int_0^a\Big[-g^0_{ki}(x^0(t))$$
$$F^i_j(x^0(t),0)\dot{x}^j(t,0)-g^0_{iq}(x^0(t))\dot{x}^i(t,0)$$
$$(\nabla_k X^q)(x^0(t),0)+g^0_{qk}(x^0(t))G^q_{j\ell}(x^0(t),0)\dot{x}^\ell(t,0)$$
$$X^j(x^0(t),0)+g^0_{kj}(x^0(t))\frac{\partial X^j}{\partial x^\ell}(x^0(t),0)$$
$$\dot{x}^\ell(t,0)\Big]\frac{\partial x^k}{\partial\tau}(t,0)\,dt=\int_0^a\dot{x}^j(t,0)\Big\{-g^0_{ki}(x^0(t))$$
$$[(\nabla_j X^i)(x^0(t),0)-g_0^{iq}(x^0(t))g^0_{\ell j}(x^0(t))$$
$$(\nabla_q X^\ell(x^0(t),0))]-g^0_{jq}(x^0(t))(\nabla_k X^q)(x^0(t),0)$$
$$+g^0_{qk}(x^0(t))G^q_{\ell j}(x^0(t),0)X^\ell(x^0(t),0)$$
$$+g^0_{k\ell}(x^0(t))\frac{\partial X^\ell}{\partial x^j}(x^0(t),0)\Big\}\frac{\partial x^k}{\partial\tau}(t,0)\,dt$$
$$=\int_0^a\dot{x}^j(t,0)\Big[-g^0_{ki}(x^0(t))(\nabla_j X^i)(x^0(t),0)$$
$$+g^0_{\ell j}(x^0(t))(\nabla_k X^\ell)(x^0(t),0)-g^0_{jq}(x^0(t))$$
$$(\nabla_k X^q)(x^0(t),0)+g^0_{qk}(x^0(t))G^q_{\ell j}(x^0(t))X^\ell(x^0(t),0)$$
$$+g^0_{k\ell}(x^0(t))\frac{\partial X^\ell}{\partial x^j}(x^0(t),0)\Big]\frac{\partial x^k}{\partial\tau}(t,0)\,dt=0.$$

We have obtained that

$$\frac{\partial}{\partial\tau}\Big|_{\tau=0}E_{g^\tau}(p,q)=\int_0^a f_{ij}(x^0(t))[\dot{x}^i(t,0)-X^i(x^0(t),0)]$$
$$[\dot{x}^j(t,0)(-X^j(x^0(t),0)]\,dt-\int_0^a g^0_{ij}(x^0(t))[\dot{x}^i(t,0)$$
$$-X^i(x^0(t),0)]\frac{\partial X^j}{\partial\tau}(x^0(t),0)\,dt,$$

where $f_{ij}(x^0(t))=\frac{1}{2}\frac{\partial}{\partial\tau}\Big|_{\tau=0}g^\tau_{ij}$.

Considering that $\frac{\partial X^j}{\partial\tau}(x^0(t),0)=0$, and using the functional

$$I_f(x^0)=\int_0^a f_{ij}(x^0(t))[\dot{x}^i(t,0)-X^i(x^0(t),0)]$$
$$[\dot{x}^j(t,0)-X^j(x^0(t),0)]dt$$

the previous relation becomes

$$\frac{\partial}{\partial\tau}\Big|_{\tau=0}E_{g^\tau}(p,q)=I_f(x^0), \qquad (5)$$

where x^0 is the potential curve corresponding to the points p, q from the border ∂M of the manifold M.

The function I_f is called *the single-ray transform of the tensor field* (f_{ij}).

The existence of solutions of the open problem **1** for the family (g^τ) implies the existence of a one parameter group of diffeomorphisms $\varphi^\tau(x)$ such that $g^\tau=(\varphi^\tau)^*g^0$. Explicitly

$$g^\tau_{ij}=(g^0_{k\ell}\circ\varphi^\tau)\frac{\partial x'^k}{\partial x^i}\,\frac{\partial x'^\ell}{\partial x^j}, \qquad (6)$$

where $\varphi^\tau(x)=(\varphi^1(x,\tau),\dots,\varphi^n(x,\tau))$, $x'=\varphi^\tau(x)$.

Theorem 3.1 *Let* $v^k(x)=\frac{\partial}{\partial\tau}\Big|_{\tau=0}(x'^k)(x,\tau)$, $k=\overline{1,n}$, $v_i=g^0_{ij}v^j$ *and* $v_{i;j}$ *be the covariant derivative of* (v_i). *Then the following relation holds*

$$f_{ij}=\frac{1}{2}(v_{i;j}+v_{j;i}), \quad i,j=\overline{1,n}. \qquad (7)$$

Proof. Differentiating the relation (6) with respect to τ and then considering $\tau=0$, we find

$$2f_{ij}=\frac{\partial}{\partial\tau}\Big|_{\tau=0}g^\tau_{ij}=\frac{\partial g^0_{k\ell}}{\partial x^m}v^m\frac{\partial x'^k}{\partial x^i}\,\frac{\partial x'^\ell}{\partial x^j}$$
$$+g^0_{k\ell}\frac{\partial}{\partial x^i}\Big(\frac{\partial}{\partial\tau}\Big|_{\tau=0}x'^k\Big)\frac{\partial x'^\ell}{\partial x^j}$$

$$+g^0_{k\ell}\frac{\partial x'^k}{\partial x^i}\frac{\partial}{\partial x^j}\Big(\frac{\partial}{\partial \tau}\Big|_{\tau=0} x'^\ell\Big)$$

$$=\frac{\partial g^0_{k\ell}}{\partial x^m}v^m\delta^k_i\delta^\ell_j+g^0_{k\ell}\frac{\partial v^k}{\partial x^i}\delta^\ell_j+g^0_{k\ell}\delta^k_i\frac{\partial v^\ell}{\partial x^j}$$

$$=\frac{\partial g^0_{ij}}{\partial x^q}v^q+g^0_{jq}\frac{\partial v^q}{\partial x^i}+g^0_{iq}\frac{\partial v^q}{\partial x^j}.$$

On the other hand,

$$v_{i;j}+v_{j;i}=\frac{\partial v^i}{\partial x^j}-G^m_{ij}v_m+\frac{\partial v_j}{\partial x^i}-G^m_{ji}v_m=\frac{\partial g^0_{im}}{\partial x^j}v^m$$

$$+g^0_{im}\frac{\partial v^m}{\partial x^j}+\frac{\partial g^0_{jm}}{\partial x^i}v_m+g^0_{jm}\frac{\partial v^m}{\partial x^i}$$

$$-g^{mq}_0\Big(\frac{\partial g^0_{jq}}{\partial x^i}+\frac{\partial g^0_{iq}}{\partial x^j}-\frac{\partial g^0_{ij}}{\partial x^q}\Big)g^0_{ms}v^s$$

$$=\frac{\partial g^0_{ij}}{\partial x^q}v^q+g^0_{jq}\frac{\partial v^q}{\partial x^i}+g^0_{iq}\frac{\partial v^q}{\partial x^j},$$

and relation (7) is proved ■

Therefore, the following generalization of the open problem **1** appears. To what extent do the integrals

$$I_f(x)=\int_0^a f_{ij}(x(t))[\dot{x}^i(t)-X^i(x(t))]$$

$$[\dot{x}^j(t)-X^j(x(t))]dt$$

determine the tensor (f_{ij})?

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