Euler-Lagrange-Hamilton Dynamics with Fractional Action

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Abstract: Our aim is three-fold: to point out that the fractional integral actions are coming from Stieltjes actions, to find the roots and the geometry of some Euler-Lagrange or Hamilton ODEs or PDEs, to evidentiate some ideas that include the fractal theory of solids. Section 1 discusses the Euler-Lagrange ODEs associated to single-time Stieltjes actions. Teir dual Hamilton ODEs are analized in Section 2. Section 3 studies the geometry associated to single-time Euler-Lagrange or Hamilton operators. Section 4 analyzes the Euler-Lagrange PDEs associated to multitime Stieltjes actions (multiple or curvilinear integrals). Section 5 formulates the multitime perimetric problem of non-renewable resources. Section 6 studies the Hamilton PDEs associated to multitime Stieltjes actions. Section 7 describes the geometry associated to multitime Euler-Lagrange or Hamilton operators (dynamical connection and semi-spray, Poincaré-Cartan form, Hamilton-Poisson systems on jet bundle). Section 8 formulates a multitime Hamilton-Poisson systems theory on jet bundle.

Key–Words: fractional Stieltjes action, Euler-Lagrange or Hamilton equations, dynamic connection, symplectic manifold.

1 Euler-Lagrange ODEs associated to single-time Stieltjes actions

Two functions $f : R \to R$ and $g : R \times R_+ \to R$, $g(t,\tau) = g_\tau(t)$, $\tau > 0$ with suitable properties determine the simple Stieljes integral (generalized convolution) of $f(t)$ with respect to $g_\tau(t)$, on the interval $[0,\tau]$, denoted by $I_\tau f = \int_0^\tau f(t)dg_\tau(t)$. The best simple existence theorem states that if f is continuous and g is of bounded variation on $[0,\tau]$, then the integral exists. Note that g is of bounded variation if and only if it is the difference between two monotone functions. If the convolution is not desirable, the interval of integration can be taken independent of τ.

If the function $g_\tau(t)$ should happen to be everywhere differentiable, then the previous Stieltjes integral is reduced to a special Riemann integral, $I_\tau f = \int_0^\tau f(t)g'_\tau(t)dt$. The well-known situations appearing in applications are:

$$g : R\times R_+ \to R,\ g_\tau(t) = \frac{\tau^r - (\tau - t)^r}{\Gamma(1+r)},\ r \in (0,1], \tag{1}$$

where Γ is the Euler function; then

$$I_\tau f = \frac{1}{\Gamma(r)}\int_0^\tau f(t)(\tau - t)^{r-1}dt$$

which is known as the *fractional Riemann-Liouville integral of order* r;

$$g : R \times R_+ \to R,\ g_\tau(t) = -\frac{e^{-\tau t}}{\tau}; \tag{2}$$

then $I_\tau f = \int_0^\tau f(t)e^{-\tau t}dt$, an integral used in economics when we speak about discounted $f(t)$ at rate τ;

$$g : R \times R_+ \to R,\ g_\tau(t) = \frac{t^\tau}{\tau}; \tag{3}$$

then τ can be taken as a fractal dimension and $I_\tau f = \int_0^\tau f(t)t^{\tau-1}dt$ is a fractional integral used as a fractal action.

Now, let $(t, x, \dot{x})$ be a local system of coordinates on $J^1(R,M)$, where $x = (x^i), \dot{x} = (\dot{x}^i), i = 1, ..., n$. Any C^∞ real function $L = L(t, x(t), \dot{x}(t))$ defined on $J^1(R,M)$ is called *Lagrangian density of energy*.

The *single-time Stieltjes action* is defined via the Stieltjes integral of L with respect to $g_\tau(t)$ in the sense of functional

$$\mathcal{I}_\tau(x(\cdot)) = \int_0^\tau L(t, x(t), \dot{x}(t))dg_\tau(t).$$

Particularly, we define the *single-time action of* $L(t, x(t), x_\alpha(t))$ *with respect to the weight* $g'_\tau(t)$ by the Riemann integral

$$\mathcal{I}_\tau(x(\cdot)) = \int_0^\tau L(t, x(t), \dot{x}(t))g'_\tau(t)dt, \qquad (4)$$

where τ is fixed. The function $\mathcal{L}(t, x(t), \dot{x}(t)) = L(t, x(t), \dot{x}(t))g'_\tau(t)$ is called *Lagrangian*.

Examples. 1) The *fractional action* from physics

$$\mathcal{I}_\tau(x(\cdot)) = \frac{1}{\Gamma(r)}\int_0^\tau L(t, x(t), \dot{x}(t))(\tau - t)^{r-1}dt$$

obtained for the function $g_\tau(t)$ in (1). Particularly, for $r = 1$ we obtain the classical action.

2) The *discounted action at rate* τ from economics

$$\mathcal{I}_\tau(x(\cdot)) = \int_0^\tau L(t, x(t), \dot{x}(t))e^{-\tau t}dt$$

obtained for $g_\tau(t)$ in (2).

3) The *fractal action* from physics [11]

$$\mathcal{I}_\tau(x(\cdot)) = \int_0^\tau L(t, x(t), \dot{x}(t))t^{\tau-1}dt$$

obtained for $g_\tau(t)$ in (3).

1.1. Proposition. *The single-time Euler-Lagrange ODEs associated to the action (4) are*

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = \frac{g''_\tau(t)}{g'_\tau(t)}\frac{\partial L}{\partial \dot{x}^i},\ i = 1, ..., n, \qquad (5)$$

where the symbol $\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{x}^i\frac{\partial}{\partial x^i} + \ddot{x}^i\frac{\partial}{\partial \dot{x}^i}$ *stands for the total derivative.*

Examples. 1) Let $g = (g_{ij})$ be a metric on the manifold M and Γ^i_{jk} the associated Christofell symbols. The Euler-Lagrange ODEs associated to the Lagrangian $\mathcal{L} = \frac{1}{2}g_{ij}(x(t))\dot{x}^i(t)\dot{x}^j(t)g'_\tau(t)$ are

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = \frac{g''_\tau(t)}{g'_\tau(t)}\frac{dx^i}{dt}, i = 1, ..., n.$$

For $g_\tau(t)$ in (1), this is just the *fractional Newton second law* from Physics.

2) Now, for $a \in (-1, 1)$, we use the Lagrangian density

$$L : R \times R \to R,\ L(x, \dot{x}) = -\frac{1}{2}\dot{x}^2 - ax\dot{x} - \frac{1}{2}x^2$$

and a differentiable function $g_\tau : R \to R$. Then the Euler-Lagrange ODE associated to the Lagrangian $\mathcal{L} = L(x, \dot{x})g'_\tau(t)$ is

$$\ddot{x} - x - \frac{g''_\tau(t)}{g'_\tau(t)}(\dot{x} - ax) = 0.$$

If the function g_τ is given by (1), then

$$\ddot{x} - (\frac{r-1}{\tau - t}a + 1)x - \frac{r-1}{\tau - t}\dot{x} = 0;$$

if g_τ is given by (2), then $\ddot{x} - \tau\dot{x} + (1 + a\tau)x = 0$.

Remarks. 1) A particular weight $g'_\tau(t)$ can be obtained taking the Riemannian manifold $(R, h_\tau(t) > 0)$ instead the Euclidean manifold $(R, 1)$. In this case the Lagrangian is $\mathcal{L} = L(t, x(t), \dot{x}(t))\sqrt{h_\tau(t)}$ and $g'_\tau(t) = \sqrt{h_\tau(t)}$.

2) If we have in mind only the Lagrangian density L, then the term $F_i = \frac{g''_\tau(t)}{g'_\tau(t)}\frac{\partial L}{\partial \dot{x}^i}$ in Euler-Lagrange ODEs (4) stands for an *external force*.

3) If the function $g_\tau(t)$ is given by (1), then the ODEs (4) reduce to

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = \frac{1-r}{\tau - t}\frac{\partial L}{\partial \dot{x}^i},\ i = 1, ..., n.$$

In particular, for $r = 1$ we obtain the classical Euler-Lagrange ODEs.

4) If the function $g_\tau(t)$ is given by (2), then the ODEs (4) reduce to

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = -\tau\frac{\partial L}{\partial \dot{x}^i},\ i = 1, ..., n.$$

2 Hamilton ODEs associated to single-time Stieltjes actions

To pass from Euler-Lagrange ODEs of second order to Hamilton ODEs of first order, suppose that the moment system $p_i = \frac{\partial L}{\partial \dot{x}^i}(t, x, \dot{x}),\ i = 1, ..., n$, define a bijection $\dot{x} \leftrightarrow p$. A sufficient condition is that the Lagrangian density of energy L to be regular, i.e., $\det\left(\frac{\partial^2 L}{\partial \dot{x}^i\partial \dot{x}^j}\right) \neq 0$. Then we introduce the Hamiltonian function

$$H : J^1(R, M)^* \to R,\ H = p_i\dot{x}^i - L(t, x, \dot{x}).$$

Remark. In the geometrical theories [1]-[4], [13]-[20], the d-tensor field

$$g_{ij}(t, x, \dot{x}) = \frac{\partial^2 L}{\partial \dot{x}^i\partial \dot{x}^j}(t, x, \dot{x})$$

is used like a vertical metric. A very important case for geometry and field theory is that of Kronecker decomposability $g_{ij}(t, x, \dot{x}) = g_{ij}(t, x, \dot{x})h(t)$.

2.1. Proposition. *The Euler-Lagrange ODEs (5) are equivalent to the Hamilton ODEs*

$$\dot{x}^i(t) = \frac{\partial H}{\partial p_i}(t, x(t), p(t))$$

$$\dot{p}_i(t) = -\frac{\partial H}{\partial x^i}(t, x(t), p(t)) + F_i(t, p(t)) \qquad (6)$$

$$F_i(t, p(t)) = \frac{g''_\tau(t)}{g'_\tau(t)} p_i(t).$$

Single-time Hamilton-Poisson systems on dual jet bundle. Let $f,\, h : J^1(R, M)^* \to R$ be differentiable functions. The Poisson bracket is defined by

$$\{f,\, h\} = \frac{\partial f}{\partial p_i}\frac{\partial h}{\partial x^i} - \frac{\partial f}{\partial x^i}\frac{\partial h}{\partial p_i}. \qquad (7)$$

From (6) and (7), it follows

$$\{H, p_i\} = \dot{p}_i - F_i,\ \{H, x^i\} = \dot{x}^i.$$

Also, for any differentiable function

$$\ell : J^1(R, M)^* \to R,$$

we have

$$\frac{d\ell}{dt} = \frac{\partial \ell}{\partial t} + \{H, \ell\} - \frac{g''_\tau(t)}{g'_\tau(t)} p_i \frac{\partial \ell}{\partial p_i}.$$

3 Geometry associated to single-time Euler-Lagrange derivative

We consider the jet bundle $J^1(R, M)$ and the local chart $(t, x, \dot{x})$. A natural local basis for the 1-forms on $J^1(R, M)$ is given by the 1-forms $\theta^i = dx^i - \dot{x}^i dt$. These 1-forms and the vertical vector fields $\frac{\partial}{\partial \dot{x}^i}$ defines the endomorphism $S = \theta^i \otimes \frac{\partial}{\partial \dot{x}^i}$, with the properties $S(\frac{\partial}{\partial t}) = -\dot{x}^i \frac{\partial}{\partial \dot{x}^i},\ S(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial \dot{x}^i}$. The vector valued 1-form S is used in the classical Hamilton-Cartan formalism for problems in the calculus of variations.

A C^∞ vector field Γ on $J^1(R, M)$ is called semi-spray (*time-dependent second order vector field or field of second order ODEs*), if it satisfies the conditions

$$dt(\Gamma) = 1,\ \theta^i(\Gamma) = 0,\ i = 1, ..., n.$$

Locally,

$$\Gamma = \frac{\partial}{\partial t} + \dot{x}^i \frac{\partial}{\partial x^i} + f^i \frac{\partial}{\partial \dot{x}^i},\ f^i \in C^\infty(J^1(R, M)).$$

The semi-spray is used in the study of time-dependent mechanics on $R \times TM$.

Any Lagrangian density of energy $L : J^1(R, M) \to R$ generates a Poincarè-Cartan 1-form

$$\theta_L = Ldt + S(L),\ \theta_L = (L - \dot{x}^i \frac{\partial L}{\partial \dot{x}^i})dt + \frac{\partial L}{\partial \dot{x}^i} dx^i.$$

Let $\omega_L = -d\theta_L$. If the Lagrangian density of energy L is nondegenerate, i.e., $\det \left(\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} \right) \neq 0$, then there exists a semi-spray Γ as solution of the equation $i_\Gamma \omega_L = 0$, called *Lagrangian spray*. Locally,

$$\Gamma = \frac{\partial}{\partial t} + \dot{x}^i \frac{\partial}{\partial x^i} + g^{ij}(-\frac{\partial L}{\partial x^j} + \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^j})\frac{\partial}{\partial \dot{x}^i}$$

$$(g^{ij}) = (\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j})^{-1}.$$

Commentary. 1) The single-time Stieltjes actions of type (3) are studied in the papers [11].

2) Similar techniques can be applied to the Lie algebroids [5].

4 Euler-Lagrange PDEs associated to multitime Stieltjes actions

The functions $f : R^m \to R$ and $g_\alpha : R \times R_+ \to R,\ g_\alpha(t^\alpha, \tau^\alpha) = g_{\tau^\alpha}(t^\alpha),\ \tau^\alpha > 0,\ \alpha = 1, ..., m$, with suitable properties, determine the multiple Stieljes integral (generalized convolution) of $f(t)$ with respect to the functions $g_{\tau^\alpha}(t^\alpha)$, on the hyperparallelipiped $\Omega_{0\tau}$ in R^m_+ (fixed by the diagonal opposite points $0 = (0, ..., 0)$ and $\tau = (\tau^1, ..., \tau^m)$), denoted by

$$I_\tau f = \int_{\Omega_{0\tau}} f(t) dg_{\tau^1}(t^1)...dg_{\tau^m}(t^m).$$

If the convolution is not desirable, the hyperparallelipiped of integration can be taken independent of τ.

If all the functions $g_{\tau^\alpha}(t^\alpha)$ should happen to be everywhere differentiable, then the Stieltjes integral is reduced to a special Riemann integral,

$$I_\tau f = \int_{\Omega_{0\tau}} f(t) g'_{\tau^1}(t^1)...g'_{\tau^m}(t^m) dt^1...dt^m.$$

Let us extend the fractional action theory from single-time case to the multitime case. For that we introduce the jet bundle of order one $J^1(T,M)$ and a local chart (t,x,x_α) on it defined by a local chart $t=(t^\alpha)$, $\alpha=1,...,m$, ("multitime") on the manifold T, a local chart $x=(x^i)$, $i=1,...,n$, on the manifold M and a local chart $x^i_\alpha=\dfrac{\partial x^i}{\partial t^\alpha}$, $i=1,...,n;$, $\alpha=1,...,m$, on the vertical fibre.

Any C^∞ real function $L=L(t,x(t),x_\alpha(t))$ defined on $J^1(R,M)$ is called *Lagrangian density of energy*. The *multi-time Stieltjes action* is defined via a multiple Stieljes integral of L with respect to the functions $g_{\tau^\alpha}(t^\alpha)$, $\alpha=1,...,m$ in the sense of functional

$$\mathcal{I}_\tau(x(\cdot))=\int_{\Omega_{0\tau}}L(t,x(t),x_\alpha(t))dg_{\tau^1}(t^1)...dg_{\tau^m}(t^m)$$

or, particularly, as *multitime Riemann action*

$$\mathcal{I}_\tau(x(\cdot))=\int_{\Omega_{0\tau}}L(t,x(t),x_\alpha(t))G_\tau(t)dt^1...dt^m,$$

where $G_\tau(t)=\prod_{\alpha=1}^m g'_{\tau^\alpha}(t^\alpha)$. We define the *multitime action of the Lagrangian density* $L(t,x(t),x_\alpha(t))$ *with respect to the weight* $G_\tau(t)$ by

$$\mathcal{I}_\tau(x(\cdot))=\int_{\Omega_{0\tau}}L(t,x(t),x_\alpha(t))G_\tau(t)dt^1...dt^m. \tag{8}$$

The function

$$\mathcal{L}(t,x(t),x_\gamma(t))=L(t,x(t),x_\gamma(t))G_\tau(t)$$

is called *Lagrangian*.

4.1. Proposition. *The multitime Euler-Lagrange PDEs associated to the action (8) are*

$$\frac{\partial L}{\partial x^i}-\frac{d}{dt^\alpha}\frac{\partial L}{\partial x^i_\alpha}=\frac{g''_{\tau^\alpha}(t^\alpha)}{g'_{\tau^\alpha}(t^\alpha)}\frac{\partial L}{\partial x^i_\alpha} \tag{9}$$

$$i=1,...,n;\ \alpha=1,...,m,$$

where the symbol $\dfrac{d}{dt^\alpha}=\dfrac{\partial}{\partial t^\alpha}+x^i_\alpha\dfrac{\partial}{\partial x^i}+x^i_{\alpha\beta}\dfrac{\partial}{\partial x^i_\beta}$ *stands for the total derivative.*

Proof. Since

$$\mathcal{L}(t,x(t),x_\alpha(t))=L(t,x(t),x_\alpha(t))G_\tau(t)$$

and

$$\frac{\partial G_\tau}{\partial t^\alpha}(t)=\frac{g''_{\tau^\alpha}(t)}{g'_{\tau^\alpha}(t)}G_\tau(t),$$

the classical Euler-Lagrange PDEs

$$\frac{\partial \mathcal{L}}{\partial x^i}-\frac{d}{dt^\alpha}\frac{\partial \mathcal{L}}{\partial x^i_\alpha}=0$$

can be written as in Proposition.

Remarks. 1) A particular weight $G_\tau(t)$ can be obtained taking a Riemannian diagonal manifold $(T,h_{\tau^\alpha}(t^\alpha))$ instead the Euclidean manifold $(T,\delta_{\alpha\beta})$. In this case the Lagrangian is $\mathcal{L}=L(t,x(t),x_\gamma(t))\sqrt{\det(h_{\tau^\alpha}(t^\alpha))}$ and the weight is $G_\tau(t)=\sqrt{\det(h_{\tau^\alpha}(t^\alpha))}$.

2) If we have in mind only the Lagrangian density L, then the term $F_i=\dfrac{g''_{\tau^\alpha}(t^\alpha)}{g'_{\tau^\alpha}(t^\alpha)}\dfrac{\partial L}{\partial x^i_\alpha}$ in Euler-Lagrange PDEs (9) stands for the *external forces*.

Examples. 1) If

$$g_{\tau^\alpha}(t^\alpha)=\frac{(\tau^\alpha)^{r_\alpha}-(\tau^\alpha-t^\alpha)^{r_\alpha}}{\Gamma(1+r_\alpha)},\ 0<r_\alpha\leq 1,$$

then $\dfrac{g''_{\tau^\alpha}(t^\alpha)}{g'_{\tau^\alpha}(t^\alpha)}=\dfrac{1-r_\alpha}{\tau^\alpha-t^\alpha}$ and the PDEs (9) are written as *multitime Euler-Lagrange PDEs with fractional forces*

$$\frac{\partial L}{\partial x^i}-\frac{d}{dt^\alpha}\frac{\partial L}{\partial x^i_\alpha}=\frac{1-r_\alpha}{\tau^\alpha-t^\alpha}\frac{\partial L}{\partial x^i_\alpha}.$$

2) If $g_{\tau^\alpha}(t^\alpha)=t^\alpha$, the PDEs (9) are written as the classical multitime Euler-Lagrange PDEs.

3) If $g_{\tau^\alpha}(t^\alpha)=-\dfrac{e^{-\tau^\alpha t^\alpha}}{\tau^\alpha}$, the PDEs (9) are written as Euler-Lagrange PDEs from economics

$$\frac{\partial L}{\partial x^i}-\frac{d}{dt^\alpha}\frac{\partial L}{\partial x^i_\alpha}=-\tau^\alpha\frac{\partial L}{\partial x^i_\alpha};$$

4) If $g_{\tau^\alpha}(t^\alpha)=\dfrac{t^{\tau^\alpha}}{\tau^\alpha}$, the PDEs (9) are written as Euler-Lagrange PDEs from fractal theory of solids

$$\frac{\partial L}{\partial x^i}-\frac{d}{dt^\alpha}\frac{\partial L}{\partial x^i_\alpha}=t^{\tau^\alpha-1}\frac{\partial L}{\partial x^i_\alpha}.$$

In order to introduce the multitime fractional functional like a path independent curvilinear integral, we start with a generic Lagrangian density of energy L and we build the total derivative

$$L_\beta(t,x(t),x_\alpha(t))=\frac{\partial L}{\partial t^\beta}(t,x(t),x_\alpha(t))+$$

$$\frac{\partial L}{\partial x^i}(t,x(t),x_\alpha(t))\frac{\partial x^i}{\partial t^\beta}(t)+\frac{\partial L}{\partial x^i_\lambda}(t,x(t),x_\alpha(t))\frac{\partial x^i_\lambda}{\partial t^\beta}(t).$$

For such type of functions we define the *curvilinear Stieltjes functional*

$$\mathcal{J}_\tau(x(\cdot))=\int_{\Gamma_{0\tau}}L_\beta(t,x(t),x_\alpha(t))dg_{\tau^\beta}(t^\beta), \tag{10}$$

where $\Gamma_{0,\tau}$ is an arbitrary piecewise C^1 curve joining the points 0 and τ in $\Omega_{0\tau}\subset R^m_+$.

4.2 Proposition [20], [22]. *1) If $x^*(\cdot)$ is an extremal of the Lagrangian density of energy L, then $x^*(\cdot)$ is an extremal of dL.*

2) If $x^*(\cdot)$ is an optimum point of the functional $\mathcal{J}_\tau(x(\cdot))$, then $x^*(\cdot)$ is the solution of the multitime Euler-Lagrange PDEs

$$\frac{\partial L_\beta}{\partial x^i} - \frac{d}{dt^\alpha}\frac{\partial L_\beta}{\partial x^i_\alpha} = a_{\beta i} + \frac{g''_{\tau^\alpha}(t^\alpha)}{g'_{\tau^\alpha}(t^\alpha)}\frac{\partial L_\beta}{\partial x^i_\alpha}, \qquad (11)$$

$$a_{\beta i} = const,\ i = 1,..,n;\ \alpha = 1,...,m.$$

Commentary. 1) The fractional multitime action can be represented as multiple integral or as curvilinear integral. For this purpose it is enough to replace the volume element $dt^1...dt^m$ by $dg_{\tau^1}(t^1)...dg_{\tau^m}(t^m)$ or the linear element (dt^β) by $(dg_{\tau^\beta}(t^\beta))$.

2) The multitime dynamics with fractional action is suitable for the differential geometry of problems in Continuous Mechanics including fractal theory. Particularly, it describes qualitative properties of m-flows and their associated geometric dynamics [13]-[23].

3) A fractional multi-time action lead to the Euler-Lagrange PDEs with external forces which are proper for the system.

4) Let us point out some criteria to select the functions $g_{\tau^\beta}(t^\beta)$. For example, if t^1 represents the time, then it is suitable to take $g_{\tau^1}(t^1) = \dfrac{(\tau^1)^{r_1} - (\tau^1 - t^1)^{r_1}}{\Gamma(1+r_1)}$; if t^2 represents the dilatation, then $g_{\tau^2}(t^2) = t^2$; if t^3 represents the discounting, then $g_{\tau^3}(t^3) = -\dfrac{e^{-\tau^3 t^3}}{\tau^3}$; if t^4 represents the fractalization, then $g_{\tau^4}(t^4) = \dfrac{(t^4)^{\tau^4}}{\tau^4}$.

5) The results from [13]-[24] can be reformulated for the fractional multi-time actions.

Applications and Examples. We start from examples in continuous mechanics [9], modified in the previous sense.

1) (**Modified sine-Gordon PDE**). The two-time Lagrangian

$$\mathcal{L} : J^1(R^2, R) \to R$$

$$\mathcal{L}(t^1, t^2, x) = (\frac{1}{2}x_1x_2 - \cos x)g'_{\tau^1}(t^1)g'_{\tau^2}(t^2)$$

determines the *modified sine-Gordon PDE*

$$\sin x - x_{12} = \frac{1}{2}\frac{g''_{\tau^1}(t^1)}{g'_{\tau^1}(t^1)}x_1 + \frac{1}{2}\frac{g''_{\tau^2}(t^2)}{g'_{\tau^2}(t^2)}x_2.$$

Taking

$$g_{\tau^1}(t^1) = \frac{(\tau^1)^{r_1} - (\tau^1 - t^1)^{r_1}}{\Gamma(1+r_1)}$$

and $g_{\tau^2}(t^2) = -\dfrac{e^{-\tau^2 t^2}}{\tau^2}$, we find

$$\sin x - x_{12} = -\frac{r_1 - 1}{\tau^1 - t^1}x_1 - \tau^2 x_2.$$

2) (**Degenerate Lagrangian**). The degenerate two-time Lagrangian

$$\mathcal{L} : J^1(R^2, R^3) \to R,$$

$$\mathcal{L}(t^1, t^2, x) = \frac{1}{2}\Big(-(x^2)^2 - (x^3)^2 + x^2x^1_1 + x^3x^1_2 - x^1x^2_1 - x^1x^3_2\Big)\, g'_{\tau^1}(t^1)g'_{\tau^2}(t^2)$$

produces an Euler-Lagrange system of order one

$$-x^2_1 - x^3_2 = \frac{1}{2}\frac{g''_{\tau^1}(t^1)}{g'_{\tau^1}(t^1)}x^2 + \frac{1}{2}\frac{g''_{\tau^2}(t^2)}{g'_{\tau^2}(t^2)}x^3$$

$$-x^2 + x^1_1 = -\frac{1}{2}\frac{g''_{\tau^1}(t^1)}{g'_{\tau^1}(t^1)}x^1$$

$$-x^3 + x^1_2 = -\frac{1}{2}\frac{g''_{\tau^1}(t^1)}{g'_{\tau^1}(t^1)}x^1.$$

3) (**Modified hyperbolic PDE**). The two-time Lagrangian

$$\mathcal{L} : J^1(R^2, R) \to R,$$

$$\mathcal{L}(t^1, t^2, x) = \frac{1}{2}e^{kt^2}\Big((x_1)^2\omega^2 - (x_2)^2 - 2kxx_2 - k^2x^2\Big)\, g'_{\tau^1}(t^1)g'_{\tau^2}(t^2)$$

defines the hyperbolic Euler-Lagrange PDE

$$-x_{11}\omega^2 + x_{22} + kx_2 = \frac{g''_{\tau^1}(t^1)}{g'_{\tau^1}(t^1)}\omega^2 x^1 - \frac{1}{2}\frac{g''_{\tau^2}(t^2)}{g'_{\tau^2}(t^2)}(x_2 + kx).$$

Taking successively

$$g_{\tau^\alpha}(t^\alpha) = t^\alpha,\ g_{\tau^\alpha}(t^\alpha) = \frac{(\tau^\alpha)^{r_\alpha} - (\tau^\alpha - t^\alpha)^{r_\alpha}}{\Gamma(1+r_\alpha)},$$

$$g_{\tau^\alpha}(t^\alpha) = -\frac{e^{-\tau^\alpha t^2}}{\tau^\alpha},$$

we find

$$-x_{11}\omega^2 + x_{22} + kx_2 = 0$$

$$-x_{11}\omega^2 + x_{22} + kx_2 = \frac{1 - r_1}{\tau^1 - t^1}\omega^2 x^1 - \frac{1 - r_2}{\tau^2 - t^2}(x_2 + kx)$$

$$-x_{11}\omega^2 + x_{22} + kx_2 = -\tau^1\omega^2 x^1 + \tau^2(x_2 + kx)$$

respectively.

5 The multitime perimetric problem of non-renewable resources

Consider a society endowed with a known finite stock S of some non-renewable resources which are essential to the economy, i.e.,

$$\int_{\Gamma_{0\tau}} q_\alpha(t)dt^\alpha = S,$$

where $q(t) = (q_1(t), ..., q_m(t))$ is the vector of quantities of the resources extracted for consumption at multitime t. The objective is to maximize the utility of consumption $u_\alpha(q_\alpha)$, with $u''_\alpha(q_\alpha) < 0 < u'_\alpha(q_\alpha)$, discounted at rate $r = (r_\alpha)$, i.e.,

$$\max \int_{\Gamma_{0\tau}} u_\alpha(q_\alpha(t))e^{-r_\beta t^\beta}dt^\alpha.$$

Define the remaining stock at multitime $t \in \Omega_{0\tau}$ as

$$x(t) = S - \int_{\Gamma_{0t}} q_\alpha(s)ds^\alpha,$$

i.e.,

$$\frac{\partial x}{\partial t^\gamma}(t) = -q_\gamma(t),\ x(0) = S,\ x(\tau) = 0.$$

The objective functional

$$\int_{\Gamma_{0\tau}} L_\alpha(t, q(t), x_\gamma(t), p(t))dt^\alpha,$$

is based on the Lagrangian covector

$$L_\alpha(t, q(t), x_\gamma(t), p(t)) = u_\alpha(q_\alpha(t))e^{-r_\beta t^\beta}$$
$$-p(t)\left(q_\alpha(t) + \frac{\partial x}{\partial t^\alpha}(t)\right).$$

Here we use the multitime Euler-Lagrange PDEs associated to path independent curvilinear integral [20],

$$\frac{\partial L_\alpha}{\partial q_\beta} - \frac{d}{dt^\gamma}\frac{\partial L_\alpha}{\partial(\frac{\partial q_\beta}{\partial t^\gamma})} = (u'_\alpha(q_\alpha)e^{-r_\beta t^\beta} - p)\delta_{\alpha\beta} = a_{\alpha\beta}$$

$$\frac{\partial L_\alpha}{\partial x} - \frac{d}{dt^\gamma}\frac{\partial L_\alpha}{\partial(\frac{\partial x}{\partial t^\gamma})} = \frac{\partial p}{\partial t^\alpha} = b_\alpha.$$

It follows $p(t) = b_\alpha t^\alpha + c,\ u'_\alpha(q_\alpha(t)) = (p(t) + a_{\alpha\alpha})e^{r_\beta t^\beta}$. Consequently, the optimal extraction rate $q^*(t)$ should be such that

$$u'_\alpha(q^*_\alpha(t)) = (p(t) + a_{\alpha\alpha})e^{r_\beta t^\beta},$$

i.e., the marginal utility of consuming non-renewable resource $u'_\alpha(q^*_\alpha)$ should increase exponentially at rate r_α which, in view of the concavity of each $u_\alpha(q_\alpha)$, implies that later generations should consume less than earlier generations.

6 Hamilton PDEs associated to multitime Stieltjes actions

To convert the multitime Euler-Lagrange PDEs of second order to multitime Hamilton PDEs of first order, we accept that the *multi-momentum* system $p^\alpha_i = \frac{\partial L}{\partial x^i_\alpha}(t, x, x_\gamma)$ determine a bijection $x_\alpha \leftrightarrow p^\alpha$. A sufficient condition is that the Lagrangian density of energy L to be regular, i.e.,

$$\det\left(\frac{\partial^2 L}{\partial x^i_\alpha \partial x^j_\beta}\right) \neq 0.$$

In the geometrical theories [1], [3], [4], [13]-[23], the d-tensor field

$$g^{\alpha\beta}_{ij}(t, x(t), x_\gamma(t)) = \frac{\partial^2 L}{\partial x^i_\alpha \partial x^j_\beta}(t, x(t), x_\gamma(t))$$

is used like a vertical metric. A very important case for geometry and field theory is that of Kronecker decomposability

$$g^{\alpha\beta}_{ij}(t, x(t), x_\gamma(t)) = g_{ij}(t, x(t), x_\gamma(t))h^{\alpha\beta}(t).$$

The Lagrangian function L determines the *Hamiltonian function*

$$H(t, x, p) = p^\alpha_i x^i_\alpha(t, x, p) - L(t, x, p).$$

If $x(\cdot)$ is a solution of the multitime Euler-Lagrange PDEs

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt^\alpha}\frac{\partial L}{\partial x^i_\alpha} = 0$$

and define $p(\cdot) = (p^\alpha_i(\cdot))$ as above, then the pair $(x(\cdot), p(\cdot))$ is a solution of the multitime Hamilton PDEs

$$\frac{\partial x^i}{\partial t^\beta}(t) = \frac{\partial H}{\partial p^\beta_i}(t, x(t), p(t))$$

$$\frac{\partial p^\alpha_i}{\partial t^\alpha}(t) = -\frac{\partial H}{\partial x^i}(t, x(t), p(t)).$$

We remark that the classical multitime Euler-Lagrange or Hamilton PDEs are of divergence-type PDEs. We can generalize the Hamilton PDEs introducing two tensor fields:

- the *Hamiltonian tensor field*

$$H^\alpha_\beta(t, x, p) = p^\alpha_i x^i_\beta(t, x, p) - \frac{1}{m}L(t, x, p)\delta^\alpha_\beta,$$

$$H(t, x, p) = H^\alpha_\alpha(t, x, p);$$

- the *moment-energy tensor field* from physics

$$T^\alpha_\beta(t, x, p) = p^\alpha_i x^i_\beta(t, x, p) - L(t, x, p)\delta^\alpha_\beta.$$

The classical Hamilton PDEs can be extended to PDEs that contains the Jacobian matrix of the Legendre transformation.

6.1. Proposition. *Let $x(\cdot)$ be a solution of the multitime Euler-Lagrange PDEs and define $p(\cdot) = (p_i^\alpha(\cdot))$ as above. Then the pair $(x(\cdot), p(\cdot))$ is a solution respectively for the generalized multitime Hamilton PDEs*

$$\delta^\alpha_\gamma \frac{\partial x^i}{\partial t^\beta}(t) = \frac{\partial H^\alpha_\beta}{\partial p^\gamma_i}(t, x(t), p(t)) \qquad (12)$$

$$+\left(\frac{1}{m}\delta^\alpha_\beta p^\lambda_j(t) - \delta^\lambda_\beta p^\alpha_j(t)\right)\frac{\partial x^j_\lambda}{\partial p^\gamma_i}(t, x(t), p(t)),$$

$$\frac{1}{m}\delta^\alpha_\beta \frac{\partial p^\gamma_i}{\partial t^\gamma}(t) = -\frac{\partial H^\alpha_\beta}{\partial x^i}(t, x(t), p(t));$$

$$\delta^\alpha_\gamma \frac{\partial x^i}{\partial t^\beta}(t) = \frac{\partial T^\alpha_\beta}{\partial p^\gamma_i}(t, x(t), p(t)) \qquad (13)$$

$$+\left(\delta^\alpha_\beta p^\lambda_j(t) - \delta^\lambda_\beta p^\alpha_j(t)\right)\frac{\partial x^j_\lambda}{\partial p^\gamma_i}(t, x(t), p(t)),$$

$$\delta^\alpha_\beta \frac{\partial p^\gamma_i}{\partial t^\gamma}(t) = -\frac{\partial T^\alpha_\beta}{\partial x^i}(t, x(t), p(t)).$$

Proof: Let us justify the PDEs (12). We find

$$\frac{\partial}{\partial x^i} H^\alpha_\beta(t, x, p) = -\frac{1}{m}\delta^\alpha_\beta \frac{\partial}{\partial x^i} L(t, x, x_\gamma(t, x, p)).$$

Now $p^\alpha_i(t) = \frac{\partial L}{\partial x^i_\alpha}(t, x(t), x_\gamma(t))$ if and only if $\frac{\partial x^i}{\partial t^\alpha}(t) = x_\alpha(t, x(t), p(t))$. Therefore the Euler-Lagrange PDEs imply the multitime Hamilton PDEs in the second place.

Now we compute the partial derivatives

$$\frac{\partial H^\alpha_\beta}{\partial p^\gamma_i} = \delta^\alpha_\gamma x^i_\beta + p^\alpha_j \frac{\partial x^j_\beta}{\partial p^\gamma_i} - \frac{1}{m}\delta^\alpha_\beta \frac{\partial L}{\partial x^j_\lambda}\frac{\partial x^j_\lambda}{\partial p^\gamma_i},$$

which contains the Jacobian matrix $\left(\frac{\partial x^j_\lambda}{\partial p^\gamma_i}\right)$ of the Legendre transformation. On the other hand, $p^\alpha_i(t) = \frac{\partial L}{\partial x^i_\alpha}(x(t), x_\gamma(t))$, implies $x_\alpha(t) = x_\alpha(x(t), p(t))$. That is why, we get the multitime Hamilton PDEs in the first place.

Remark. After our knowledge, here is the first time when the Jacobian matrix of the Legendre transformation is involved in the Hamilton PDEs.

6.2. Proposition. *The multitime Euler-Lagrange PDEs (9) are equivalent to the multitime Hamilton PDEs*

$$\frac{\partial x^i}{\partial t^\beta}(t) = \frac{\partial H}{\partial p^\beta_i}(t, x(t), p(t))$$

$$\frac{\partial p^\alpha_i}{\partial t^\alpha}(t) = -\frac{\partial H}{\partial x^i}(t, x(t), p(t)) + F_i(t, p(t)), \qquad (14)$$

$$F_i = \frac{g''_{\tau^\alpha}(t)}{g'_{\tau^\alpha}(t)} p^\alpha_i.$$

7 Geometry associated to multitime Euler-Lagrange derivative

Dynamical connection and semi-spray. We use the jet bundle of order one $J^1(T, M)$ and a local chart (t, x, x_α) defined by a local chart $t = (t^\alpha)$, $\alpha = 1, ..., m$, on the manifold T, a local chart $x = (x^i)$, $i = 1, ..., n$, on the manifold M and a local chart for partial velocities $x_\alpha = (x^i_\alpha) = \left(\frac{\partial x^i}{\partial t^\alpha}\right)$. Explicitly, the system of local coordinates is $(t^\alpha, x^i, x^i_\alpha)$. The manifold $J^1(T, M)$ is endowed with the following natural structures:

1) the total derivative operator

$$d_\alpha = \frac{\partial}{\partial t^\alpha} + x^i_\alpha \frac{\partial}{\partial x^i};$$

2) the contact 1-forms $\theta^i = dx^i - x^i_\alpha dt^\alpha$;

3) the total derivative 1-form operator

$$\theta_1 = d_\alpha \otimes dt^\alpha;$$

4) the vector-valued contact form $\theta_2 = \frac{\partial}{\partial x^i} \otimes \theta^i$;

5) the vertical endomorphism field

$$J = J^\alpha \otimes d_\alpha,\ J^\alpha = \frac{\partial}{\partial x^i_\alpha} \otimes \theta^i,$$

where $\{\frac{\partial}{\partial x^i_\alpha}\}$ is a basis of vertical distribution V (vertical vector fields).

A C^∞ vector-valued 1-form H on $J^1(T, M)$ is called *dynamical connection* on $J^1(T, M)$ if it satisfies the conditions

$$\theta_1 \circ H = 0,\ \theta_2 \circ H = \theta_2,\ H_{|V} = -id_{|V}.$$

7.1. Proposition. *The local expression of the dynamical connection H with respect to the chart $(t^\alpha, x^i, x^i_\alpha)$ is*

$$H = (-x^i_\alpha \frac{\partial}{\partial x^i} + H^i_{\alpha\beta}\frac{\partial}{\partial x^i_\beta}) \otimes dt^\alpha$$

$$+(\frac{\partial}{\partial x^i} + H^j_{i\alpha}\frac{\partial}{\partial x^j_\alpha}) \otimes dx^i - \frac{\partial}{\partial x^i_\beta} \otimes dx^i_\beta.$$

7.2. Proposition. *1) The rank of the matrix associated to the dynamical connection is* $(m+1)n$.

2) The dynamical connection defines an $f(3,-1)$*-structure.*

As any $f(3,-1)$-structure, the dynamical connection determines the projectors

$$\ell = H \circ H = H^2,\ m = -H^2 + I$$

having the following properties:

$$\ell^2 = \ell,\ m^2 = m,\ \ell \circ m = m \circ \ell = 0,\ \ell + m = I$$

$$\ell(\frac{\partial}{\partial t^\alpha}) = -x^i_\alpha \frac{\partial}{\partial x^i} - (x^i_\alpha H^j_{i\beta} + H^j_{\alpha\beta})\frac{\partial}{\partial x^j_\beta}$$

$$\ell(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i},\ \ell(\frac{\partial}{\partial x^i_\alpha}) = \frac{\partial}{\partial x^i_\alpha}$$

$$m(\frac{\partial}{\partial t^\alpha}) = \frac{\partial}{\partial t^\alpha} + x^i_\alpha \frac{\partial}{\partial x^i} + (x^i_\alpha H^j_{i\beta} + H^j_{\alpha\beta})\frac{\partial}{\partial x^j_\beta}$$

$$m(\frac{\partial}{\partial x^i}) = 0,\ m(\frac{\partial}{\partial x^i_\alpha}) = 0.$$

A set of C^∞ vector fields $\Gamma_\alpha,\ \alpha = 1, ..., m$, on $J^1(T, M)$ is called semi-spray (*multitime-dependent second order vector field or field of second order PDEs*) if it satisfies the conditions

$$dt^\alpha(\Gamma_\beta) = \delta^\alpha_\beta,\ \theta^i(\Gamma_\beta) = 0,\ i = 1, ..., n.$$

Locally,

$$\Gamma_\alpha = \frac{\partial}{\partial t^\alpha} + x^i_\alpha \frac{\partial}{\partial x^i} + F^i_{\alpha\beta}\frac{\partial}{\partial x^i_\beta},$$

$$\Lambda^i_{\alpha\beta} \in C^\infty(J^1(T, M)).$$

This semi-spray can be used to study "multitime-dependent mechanics" on $J^1(T, M)$.

7.3. Proposition. *The vector-valued* 1*-form* $H = (m-1)\theta_2 - \mathcal{L}_{\Gamma_\alpha} J^\alpha$ *is a dynamical connection on* $J^1(T, M)$*, where* $\mathcal{L}_{\Gamma_\alpha}$ *is the Lie derivative with respect to the vector field* Γ_α.

7.4. Proposition. *A quasilinear second order PDEs system of the type*

$$A^{\alpha\beta}_{ij}(t, x(t), x_\gamma(t))x^j_{\alpha\beta}(t) + B_i(t, x(t), x_\gamma(t)) = 0,$$

where

$$A^{\alpha\beta}_{ij} = A^{\beta\alpha}_{ji} = A^{\alpha\beta}_{ji},\ \textit{det}\left(A^{\alpha\beta}_{ij}\right) \neq 0,$$

$$^\alpha_i - \textit{rows},\ ^\beta_j - \textit{columns},$$

extends to a semi-spray

$$\Gamma_\alpha = \frac{\partial}{\partial t^\alpha} + x^i_\alpha \frac{\partial}{\partial x^i} + F^i_{\alpha\beta}\frac{\partial}{\partial x^i_\beta}, \tag{15}$$

where

$$F^i_{\alpha\beta} = A^{ij}_{\alpha\gamma}\left(\phi^\gamma_{j\beta} - \frac{1}{m}\delta^\gamma_\beta B_j\right)$$

and

$$\phi^\beta_{j\beta} = 0,\ A^{ij}_{\alpha\gamma}\phi^\gamma_{j\beta} = A^{ij}_{\beta\gamma}\phi^\gamma_{j\alpha}.$$

Proof. To prove this statement we use two ingredients: (1) an anti-trace PDEs system

$$A^{\gamma\alpha}_{ij}(t, x(t), x_\sigma(t))x^j_{\alpha\beta}(t) + \frac{1}{m}\delta^\gamma_\beta B_i(t, x(t), x_\sigma(t))$$
$$= \phi^\gamma_{i\beta}(t, x(t), x_\sigma(t)),$$

where $\phi^\gamma_{i\beta}$ are arbitrary C^∞ functions satisfying $\phi^\beta_{i\beta} = 0$, (2) the inverse $(A^{ij}_{\alpha\beta})$ of the matrix $(A^{\alpha\beta}_{ij})$, i.e., $A^{\alpha\beta}_{ij}A^{ik}_{\alpha\gamma} = \delta^\beta_\gamma\delta^k_j$. In fact, the anti-trace PDE system is equivalent to the semi-spray

$$x^i_{\alpha\beta} = A^{ij}_{\alpha\gamma}\left(\phi^\gamma_{j\beta} - \frac{1}{m}\delta^\gamma_\beta B_j\right)$$

if $A^{ij}_{\alpha\gamma}\phi^\gamma_{j\beta} = A^{ij}_{\beta\gamma}\phi^\gamma_{j\alpha}$.

To simplify, we accept $\phi^\gamma_{j\beta} = 0$. Then the components of the dynamical connection determined by the previous PDE system are

$$\begin{aligned} H^j_{i\alpha} &= \frac{1}{m}A^{jk}_{\beta\alpha}\left(\frac{\partial A^{\delta\epsilon}_{kh}}{\partial x^i_\beta}A^{hl}_{\delta\epsilon}B_l - \frac{\partial B_k}{\partial x^i_\beta}\right) \\ H^j_{\gamma\beta} &= \frac{1}{m}A^{jk}_{\delta\beta}\left(\frac{\partial A^{\delta\epsilon}_{kh}}{\partial x^i_\gamma}A^{hl}_{\delta\epsilon}B_l - \frac{\partial B_k}{\partial x^i_\gamma}\right)x^i_\gamma. \end{aligned} \tag{16}$$

Particularly, let us consider a PDE of the form

$$A^{\alpha\beta}(t, x(t), x_\gamma(t))x_{\alpha\beta}(t) + B(t, x(t), x_\gamma(t)) = 0,$$

where $(A^{\alpha\beta})$ is a nondegenerate matrix with the inverse $A_{\alpha\beta}$. Then the associated anti-trace PDE system is

$$A^{\alpha\gamma}(t, x(t), x_\gamma(t))x_{\alpha\beta}(t) + \frac{1}{m}B(t, x(t), x_\gamma(t)) = 0$$

or

$$x_{\alpha\beta} + \frac{1}{m}A_{\alpha\beta}B = 0.$$

That is why, our initial PDE system extends to the semi-spray $F_{\alpha\beta} = -\frac{1}{m}A_{\alpha\beta}B$ and the components of the associated dynamical connection are

$$H_\alpha = \frac{1}{m}A_{\alpha\beta}\left(\frac{\partial A^{\mu\lambda}}{\partial x_\beta}A_{\mu\lambda}B - \frac{\partial B}{\partial x_\beta}\right)$$

$$H_{\alpha\beta} = \frac{1}{m} A_{\gamma\beta} \left(\frac{\partial A^{\mu\lambda}}{\partial x_\gamma} A_{\mu\lambda} B - \frac{\partial B}{\partial x_\gamma} \right) x_\alpha.$$

Example. Let us take the PDE

$$\omega^2 x_{11} - x_{22} - k x_2 + h_1 \omega^2 x_1 - h_2(x_2 + kx) = 0,$$

where ω, k are constants and $h_i = h_i(t^1, t^2)$, $i = 1, 2$. In this case

$$A^{11} = \omega^2, A^{12} = A^{21} = 0, A^{22} = -1$$

$$A_{11} = \frac{1}{\omega^2}, A_{12} = A_{21} = 0, A_{22} = -1.$$

It appears the semi-spray

$$F_{11} = -\frac{1}{2\omega^2}(h_1 \omega^2 x_1 - (k + h_2) x_2 - k h_2 x),$$

$$F_{12} = F_{21} = 0, F_{22} = -\omega^2 F_{11}$$

and the dynamical connection

$$H_1 = -\frac{h_1}{2}, H_2 = -\frac{k + h_2}{2}$$

$$H_{11} = -\frac{h_1 x_1}{2}, H_{12} = H_{21} = 0, H_{22} = -\frac{(k + h_2) x_2}{2}.$$

Now, let

$$\mathcal{L}(t, x(t), x_\alpha(t)) = L(t, x(t), x_\alpha(t)) G_\tau(t)$$

be a multitime Lagrangian. Since the d-tensor field $g^{\alpha\beta}_{ij} = \dfrac{\partial^2 L}{\partial x^i_\alpha \partial x^j_\beta}$ is the dominant coefficient for a geometrical theory, we writte the Euler-Lagrange PDEs of $\mathcal{L}$ in the form

$$\frac{d}{dt^\alpha} \frac{\partial L}{\partial x^i_\alpha} - \frac{\partial L}{\partial x^i} + \frac{g''_{\tau^\alpha}(t^\alpha)}{g'_{\tau^\alpha}(t^\alpha)} \frac{\partial L}{\partial x^i_\alpha} =$$

$$\frac{\partial^2 L}{\partial x^i_\alpha \partial x^j_\beta} x^j_{\alpha\beta} + \frac{\partial^2 L}{\partial x^i_\alpha \partial x^j} x^j_\alpha$$

$$+ \frac{\partial^2 L}{\partial x^i_\alpha \partial t^\alpha} - \frac{\partial L}{\partial x^i} + \frac{g''_{\tau^\alpha}(t^\alpha)}{g'_{\tau^\alpha}(t^\alpha)} \frac{\partial L}{\partial x^i_\alpha} = 0.$$

This system can be identified directly to $g^{\alpha\beta}_{ij} x^j_{\alpha\beta} + B_i = 0$ and we can apply the previous theory. But, to show that the previous way is not unique, we prefer another extension as the anti-trace PDEs system

$$\frac{d}{dt^\gamma} \frac{\partial L}{\partial x^i_\alpha} - \frac{1}{m} \delta^\alpha_\gamma \left(\frac{\partial L}{\partial x^i} - \frac{g''_{\tau^\sigma}(t^\sigma)}{g'_{\tau^\sigma}(t^\sigma)} \frac{\partial L}{\partial x^i_\sigma} \right) =$$

$$\frac{\partial^2 L}{\partial x^i_\alpha \partial x^j_\beta} x^j_{\beta\gamma} + \frac{\partial^2 L}{\partial x^i_\alpha \partial x^j} x^j_\gamma + \frac{\partial^2 L}{\partial x^i_\alpha \partial t^\gamma}$$

$$- \frac{1}{m} \delta^\alpha_\gamma \left(\frac{\partial L}{\partial x^i} - \frac{g''_{\tau^\sigma}(t^\sigma)}{g'_{\tau^\sigma}(t^\sigma)} \frac{\partial L}{\partial x^i_\sigma} \right) = 0.$$

If the Lagrangian density of energy L is nondegenerate, then the matrix $(g^{\alpha\beta}_{ij})$ has an inverse $(g^{ij}_{\alpha\beta})$. Therefore a semi-spray associated to the Euler-Lagrange PDEs is characterized by the functions

$$F^i_{\alpha\beta} = g^{ij}_{\alpha\epsilon} \left(\frac{1}{m} \delta^\epsilon_\beta \left(\frac{\partial L}{\partial x^j} - \frac{g''_{\tau^\gamma}(t^\gamma)}{g'_{\tau^\gamma}(t^\gamma)} \frac{\partial L}{\partial x^j_\gamma} \right) \right.$$

$$\left. - \frac{\partial^2 L}{\partial x^j_\epsilon \partial x^k} x^k_\beta - \frac{\partial^2 L}{\partial x^j_\epsilon \partial t^\beta} \right).$$

Automatically, the formulas (15) produce the components of the associated *dynamical connection*.

Poincaré-Cartan form. Let Γ_α, $\alpha = 1, ..., m$, be a semi-spray on $J^1(T, M)$. The semi-spray is called *compatible* to a Lagrangian

$$\mathcal{L}(t, x(t), x_\alpha(t)) = L(t, x(t), x_\alpha(t)) G_\tau(t)$$

if it satisfies the multitime PDEs

$$\Gamma_\alpha \left(\frac{\partial L}{\partial x^i_\alpha} \right) - \frac{\partial L}{\partial x^i} + \frac{g''_{\tau^\alpha}(t^\alpha)}{g'_{\tau^\alpha}(t^\alpha)} \frac{\partial L}{\partial x^i_\alpha} = 0. \qquad (17)$$

If the semi-spray is given by the formulas (15), then the condition (17) leads to the PDEs

$$g^{\alpha\beta}_{ij} F^j_{\alpha\beta} + B_i = 0,$$

where

$$F^j_{\alpha\beta} = F^j_{\beta\alpha},\ g^{\alpha\beta}_{ij} = \frac{\partial^2 L}{\partial x^i_\alpha \partial x^j_\beta},$$

$$B_i = \frac{\partial^2 L}{\partial x^i_\epsilon \partial x^k} x^k_\epsilon + \frac{\partial^2 L}{\partial t^\epsilon \partial x^j_\epsilon} - \frac{\partial L}{\partial x^i} + \frac{g''_{\tau^\gamma}(t^\gamma)}{g'_{\tau^\gamma}(t^\gamma)} \frac{\partial L}{\partial x^i_\gamma}.$$

An arbitrary dynamical connection H on $J^1(T, M)$ determines the dual bases

$$\Gamma_\alpha = \frac{\partial}{\partial t^\alpha} + x^i_\alpha \frac{\partial}{\partial x^i} + F^i_{\alpha\beta} \frac{\partial}{\partial x^i_\beta}$$

$$H_i = \frac{\partial}{\partial x^i} + \frac{1}{2} H^j_{i\alpha} \frac{\partial}{\partial x^j_\alpha},\ V^\alpha_i = \frac{\partial}{\partial x^i_\alpha}$$

$$dt^\alpha,\ \theta^i = dx^i - x^i_\alpha dt^\alpha,\ \psi^i_\alpha = dx^i_\alpha - \frac{1}{2} H^i_{j\alpha} \theta^j - F^i_{\alpha\beta} dt^\beta.$$

Let $\omega = dt^1 \wedge ... \wedge dt^m$ and

$$\omega_\alpha = (-1)^m dt^1 \wedge ... \wedge \hat{dt}^\alpha \wedge ... \wedge dt^m.$$

Then the m-form $\theta^1 = \mathcal{L}\omega + \dfrac{\partial \mathcal{L}}{\partial x^i_\alpha} \theta^i \wedge \omega_\alpha$ is called the *Poincaré-Cartan form*.

7.5. Proposition. *The* $(m+1)$*-form* $\Omega^1 = d\theta^1$ *can be written*

$$\Omega^1 = (g^{\alpha\beta}_{ij}\psi^i_\beta \wedge \theta^j - \frac{1}{2}J^{1\alpha}_{ij}\theta^i \wedge \theta^j)\omega_\alpha + \mathcal{E}^1_i(\mathcal{L})\theta^i \wedge \omega,$$

where

$$\mathcal{E}^1_i(\mathcal{L}) = \frac{\partial \mathcal{L}}{\partial x^i} - \Gamma_\alpha \frac{\partial \mathcal{L}}{\partial x^i_\alpha},\ J^{1\alpha}_{ij} = H_i(\frac{\partial \mathcal{L}}{\partial x^j_\alpha}) - H_j(\frac{\partial \mathcal{L}}{\partial x^i_\alpha}).$$

7.6 Proposition. *If the dynamical connection* H *is associated to a semi-spray which is compatible to* $\mathcal{L}$*, i.e.,* $\mathcal{E}^1_i(\mathcal{L}) = 0$*, then*

$$\Omega^1 = (g^{\alpha\beta}_{ij}\psi^i_\beta \wedge \theta^j - \frac{1}{2}J^{1\alpha}_{ij}\theta^i \wedge \theta^j)\omega_\alpha$$

$$J^{1\alpha}_{ij} = \frac{m-1}{2m}\left(\frac{\partial B^1_j}{\partial x^i_\alpha} - \frac{\partial B^1_i}{\partial x^j_\alpha}\right) + \frac{1}{2}\left(\frac{\partial \phi^\alpha_{i\beta}}{\partial x^j_\beta} - \frac{\partial \phi^\alpha_{j\beta}}{\partial x^i_\beta}\right),$$

where

$$B^1_j = \frac{\partial^2 \mathcal{L}}{\partial x^i \partial x^j_\alpha}x^i_\alpha + \frac{\partial^2 \mathcal{L}}{\partial t^\alpha \partial x^j_\alpha} - \frac{\partial \mathcal{L}}{\partial x^j}.$$

If $m = 1$, then $\Omega^1 = g_{ij}\psi^i \wedge \theta^j$, $g_{ij} = \dfrac{\partial \mathcal{L}}{\partial \dot{x}^i \partial \dot{x}^j}$, $\dot{x}^i = \dfrac{dx^i}{dt}$. If $n = 1$, then $\Omega^1 = h^{\alpha\beta}\psi_\alpha \wedge \theta \wedge \omega_\beta$, $h^{\alpha\beta} = \dfrac{\partial \mathcal{L}}{\partial x_\alpha \partial x_\beta}$.

The previous theory refers to classical Riemann actions. Its reformulation for Stieltjes actions is obvious. For example, the Poincaré-Cartan m-form can be written $\theta = L\tilde{\omega} + \dfrac{\partial L}{\partial x^i_\alpha}\theta^i \wedge \tilde{\omega}_\alpha$, where

$$\tilde{\omega} = dg_{\tau^1}(t^1) \wedge ... \wedge dg_{\tau^m}(t^m),$$

$$\tilde{\omega}_\alpha = (-1)^m dg_{\tau^1}(t^1) \wedge ... \wedge \hat{d}g_{\tau^\alpha}(t^\alpha) \wedge ... \wedge dg_{\tau^m}(t^m).$$

Since $\tilde{\omega} = G_\tau(t)\omega,\ \tilde{\omega}_\alpha = \dfrac{g''_{\tau^\alpha}(t^\alpha)}{g'_{\tau^\alpha}(t^\alpha)}\omega_\alpha$, we can writte

$$\theta = G_\tau(t)L\omega + \frac{\partial L}{\partial x^i_\alpha}\frac{g''_{\tau^\alpha}(t^\alpha)}{g'_{\tau^\alpha}(t^\alpha)}\theta^i \wedge \omega_\alpha.$$

7.7 Proposition. *The* $(m+1)$*-form* $\Omega = d\theta$ *can be written*

$$\Omega = \left(g^{\alpha\beta}_{ij}\psi^i_\beta \wedge \theta^j - \frac{1}{2}J^\alpha_{ij}\theta^i \wedge \theta^j\right)\frac{g''_{\tau^\alpha}(t^\alpha)}{g'_{\tau^\alpha}(t^\alpha)}\omega_\alpha$$

$$+\mathcal{E}_i(L)G_\tau(t)\theta^i \wedge \omega,$$

where

$$\mathcal{E}_i(L) = \Gamma_\alpha\left(\frac{\partial L}{\partial x^i_\alpha}\right) - \frac{\partial L}{\partial x^i} + \frac{g''_{\tau^\alpha}(t^\alpha)}{g'_{\tau^\alpha}(t^\alpha)}\frac{\partial L}{\partial x^i_\alpha}$$

$$J^\alpha_{ij} = H_i\left(\frac{\partial L}{\partial x^j_\alpha}\right) - H_j\left(\frac{\partial L}{\partial x^i_\alpha}\right).$$

7.8 Proposition. *If the dynamical connection* H *is associated to a semi-spray which is compatible to* L*, i.e.,* $\mathcal{E}_i(L) = 0$*, then*

$$\Omega^1 = (g^{\alpha\beta}_{ij}\psi^i_\beta \wedge \theta^j - \frac{1}{2}J^\alpha_{ij}\theta^i \wedge \theta^j)G_\tau(t)\omega_\alpha$$

$$J^\alpha_{ij} = \frac{m-1}{2m}\left(\frac{\partial B_j}{\partial x^i_\alpha} - \frac{\partial B_i}{\partial x^j_\alpha}\right) + \frac{1}{2}\left(\frac{\partial \phi^\alpha_{i\beta}}{\partial x^j_\beta} - \frac{\partial \phi^\alpha_{j\beta}}{\partial x^i_\beta}\right),$$

where

$$B_j = \frac{\partial^2 L}{\partial x^i \partial x^j_\alpha}x^i_\alpha + \frac{\partial^2 L}{\partial t^\alpha \partial x^j_\alpha} - \frac{\partial L}{\partial x^j} + \frac{g''_{\tau^\alpha}(t^\alpha)}{g'_{\tau^\alpha}(t^\alpha)}\frac{\partial L}{\partial x^i_\alpha}.$$

8 Multitime Hamilton-Poisson systems on jet bundle

If (T, h) and (R, g) are Riemannian manifolds, we shall use the adapted dual bases

$$\left(\frac{\delta}{\delta t^\alpha} = \frac{\partial}{\partial t^\alpha} + H^\gamma_{\alpha\beta}x^i_\gamma\frac{\partial}{\partial x^i_\beta},\right.$$

$$\left.\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G^h_{ik}x^k_\alpha\frac{\partial}{\partial x^h_\alpha},\ \frac{\partial}{\partial x^i_\alpha}\right)$$

$$(dt^\beta,\ dx^j,\ \delta x^j_\beta = dx^j_\beta - H^\gamma_{\beta\lambda}x^j_\gamma dt^\lambda + G^j_{hk}x^h_\beta dx^k)$$

as frames on the jet bundle $J^1(T, M)$. Then the induced Riemann Sasaki-like metric on $J^1(T, M)$ is

$$S = h_{\alpha\beta}dt^\alpha \otimes dt^\beta + g_{ij}dx^i \otimes dx^j + h^{\alpha\beta}g_{ij}\delta x^i_\alpha \otimes \delta x^j_\beta.$$

We first notice that, on the Riemannian manifold $(J^1(T, M), S)$ there exists a globally defined 1-form d-tensor

$$\omega = g_{ij}x^j_\alpha dx^i \otimes dt^\alpha.$$

Its exterior differential

$$\Omega = d\omega = (-g_{ij}dx^i \wedge \delta x^j_\alpha) \otimes dt^\alpha$$

is also globally defined 2-form d-tensor, and has the components

$$(\Omega_{\alpha\, AB}) = \begin{pmatrix} 0 & -g_{ij}\delta^\beta_\alpha \\ g_{ij}\delta^\beta_\alpha & 0 \end{pmatrix}$$

in the adapted frame. Of course we can find a suitable geometry produced by ω and Ω on $J^1(T, M)$.

The section $t^\alpha = c^\alpha$, $\alpha = 1, \ldots, p$, is an $(1+p)n$-dimensional Riemann submanifold of $J^1(T, M)$ which can be identified with the Riemann manifold $({}^p\mathcal{T}(M),\ g + h^{-1} \otimes g)$, where h has constant components, and ${}^p\mathcal{T}(M) = \bigcup\limits_{x\in M} (\mathcal{T}_x M)^p$. The closed 2-forms $\Omega_\alpha = -g_{ij} dx^i \wedge \delta x^j_\alpha$, and the metric $g + h^{-1} \otimes g$ produce an almost p-Kählerian structure on ${}^p\mathcal{T}(M)$ in the sense of Grassi [16].

A theory of Hamilton-Poisson systems on $J^1(T, M)$ can be obtained in the following way. Let $L_1,\ L_2$ be two real C^∞ functions on $J^1(T, M)$, i.e., two Lagrangians. The maps

$$\{L_1,\ L_2\}_\alpha = g^{ij} h_{\alpha\beta} \left(\frac{\delta L_1}{\delta x^i} \frac{\partial L_2}{\partial x^j_\beta} - \frac{\partial L_1}{\partial x^i_\beta} \frac{\delta L_2}{\delta x^j} \right),$$

$$\alpha = 1, \ldots, m$$

define a Poisson structure on the jet bundle $J^1(T, M)$ via the 1-form Poisson bracket $\{L_1,\ L_2\} = \{L_1,\ L_2\}_\alpha dt^\alpha$. Also the maps $\{L_1,\ L_2\}_\alpha$ define a p-Poisson structure on $({}^p\mathcal{T}(M), g + h^{-1} \otimes g)$ compatible with the almost p-Kählerian structure $\Omega_\alpha = -g_{ij} dx^i \wedge \delta x^j_\alpha$.

A similar theory can be introduced on the dual jet bundle $J^1(T, M)^*$ of local coordinates $(t^\alpha, x^i, p^\alpha_i)$.

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