Non-Classical Lagrangian Dynamics and Potential Maps

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Abstract: The basic theory regarding Nonclassical Lagrangian Dynamics and Potential Maps was announced in [7]. Since its mathematical impact is now at large vogue, we reinforce some arguments. Section 1 extends the theory of harmonic and potential maps in the language of differential geometry. Section 2 defines a generalized Lorentz world-force law and shows that any PDE system of order one (in particular, p-flow) generates such a law in a suitable geometrical structure. In other words, the solutions of any PDE system of order one are harmonic or potential maps, i.e., they are solutions of Euler-Lagrange prolongation PDE system of order two built via Riemann-Lagrange structures and a least squares Lagrangian. Section 3 formulates open problems regarding the geometry of semi-Riemann manifolds $(J^1(T,M),S_1)$, $(J^2(T,M),S_2)$. Section 4 shows that the Lorentz-Udriste world-force law is equivalent to certain covariant Hamilton PDEs on $(J^1(T,M),S_1)$. Section 5 describes the maps determining a continuous group of transformations as ultra-potential maps.

Key–Words: ultra-harmonic map, ultra-potential map, Lagrangian, Hamiltonian, Lorentz-Udriste world force law.

1 Harmonic and Potential Maps

All maps throughout the paper are smooth, while manifolds are real, finite-dimensional, Hausdorff, second-countable and connected.

Let (T,h) and (M,g) be semi-Riemann manifolds of dimensions p and n. Hereafter we shall assume that the manifold T is oriented. Greek (Latin) letters will be used for indexing the components of geometrical objects attached to the manifold T (manifold M). Local coordinates will be written

$$t=(t^\alpha),\quad \alpha=1,\dots,p$$
$$x=(x^i),\quad i=1,\dots,n,$$

and the components of the corresponding metric tensor and Christoffel symbols will be denoted by $h_{\alpha\beta}, g_{ij},\ H^\alpha_{\beta\gamma},\ G^i_{jk}$. Indices of tensors or distinguished tensors will be rised and lowered in the usual fashion.

Let $\varphi: T\to M$, $\varphi(t)=x$, $x^i=x^i(t^\alpha)$ be a C^∞ map (parametrized sheet). We set

(1)
$$x^i_\alpha=\frac{\partial x^i}{\partial t^\alpha},\ x^i_{\alpha\beta}=\frac{\partial^2 x^i}{\partial t^\alpha\partial t^\beta}-H^\gamma_{\alpha\beta}x^i_\gamma+G^i_{jk}x^j_\alpha x^k_\beta.$$

Then $x^i_\alpha,\ x^i_{\alpha\beta}$ transform like tensors under coordinate transformations $t\to\bar t$, $x\to\bar x$. In the sequel $x^i_\alpha,\ x^i_{\alpha\beta}$ will be interpreted like distinguished tensors.

The canonical form of the energy density $E(\varphi)$ of the map φ is defined by

$$E_0(\varphi)(t)=\frac{1}{2}h^{\alpha\beta}(t)g_{ij}(x(t))x^i_\alpha(t)x^j_\beta(t).$$

For a relatively compact domain $\Omega\subset T$, we define the energy

$$E_0(\varphi,\Omega)=\int_\Omega E_0(\varphi)(t)dv_h,$$

where $dv_h=\sqrt{|h|}dt^1\wedge\ldots\wedge dt^p$ denotes the volume element induced by the semi-Riemann metric h. A map φ is called *ultra-harmonic map* if it is a critical point of the energy functional E_0, i.e., an extremal of the Lagrangian

$$L=E_0(\varphi)(t)\sqrt{|h|},$$

for all compactly supported variations. The *ultra-harmonic map equation* is a system of nonlinear *ultra-hyperbolic-Laplace PDEs* of second order and is expressed in local coordinates as

$$\tau(\varphi)^i=h^{\alpha\beta}x^i_{\alpha\beta}=0. \qquad (2)$$

The vector field $\tau(\varphi)^i$ defines a section of the pull-back bundle $\varphi^{-1}TM$ of the tangent bundle TM of the manifold M along φ, and is called the *tension field* of φ. If the metric h is a Riemann metric, then we recover the theory of harmonic maps [1].

The product manifold $T \times M$ is coordinated by (t^α, x^i). The first order jet manifold $J^1(T, M)$, i.e., the configuration bundle, is endowed with the adapted coordinates $(t^\alpha, x^i, x^i_\alpha)$. The distinguished tensors fields and other distinguished geometrical objects on $T \times M$ are introduced using the jet bundle $J^1(T, M)$ [4], [5], [15].

Let $X^i_\alpha(t, x)$ be a given C^∞ distinguished tensor field on $T \times M$ and $c(t, x)$ be a given C^∞ real function on $T \times M$. The general energy density $E(\varphi)$ of the map φ is defined by

$$E(\varphi(t)) = \frac{1}{2} h^{\alpha\beta}(t) g_{ij}(x(t)) x^i_\alpha(t) x^j_\beta(t)$$

$$-h^{\alpha\beta}(t) g_{ij}(x(t)) x^i_\alpha(t) X^j_\beta(t, x(t)) + c(t, x).$$

Of course $E(\varphi)$ is a perfect square iff

$$c = \frac{1}{2} h^{\alpha\beta}(t) g_{ij}(x(t)) X^i_\alpha(t, x(t)) X^j_\beta(t, x(t)).$$

Similarly, for a relatively compact domain $\Omega \subset T$, we define the energy

$$E(\varphi; \Omega) = \int_\Omega E(\varphi)(t) dv_h.$$

A map φ is called *ultra-potential map* if it is a critical point of the energy functional E, i.e., an extremal of the Lagrangian

$$L = E(\varphi)(t)\sqrt{|h|},$$

for all compactly supported variations. The *ultra-potential map equation* is a system of nonlinear *ultra-hyperbolic-Poisson PDEs* and is expressed locally by

$$\tau(\varphi)^i = h^{\alpha\beta} x^i_{\alpha\beta} = g^{ij}\frac{\partial c}{\partial x^j} + h^{\alpha\beta}(\nabla_k X^i_\beta$$

$$(3) \qquad -g_{kj} g^{il} \nabla_l X^j_\beta) x^k_\alpha + h^{\alpha\beta} D_\alpha X^i_\beta,$$

where D is the Levy-Civita connection of (T, h) and ∇ is the Levy-Civita connection of (M, g). Explicitly, we have

(4)

$$\nabla_j X^i_\alpha = \frac{\partial X^i_\alpha}{\partial x^j} + G^i_{jk} X^k_\alpha, \quad D_\beta X^i_\alpha = \frac{\partial X^i_\alpha}{\partial t^\beta} - H^\gamma_{\beta\alpha} X^i_\gamma$$

$$(5) \qquad F_j{}^i{}_\alpha = \nabla_j X^i_\alpha - g_{hj} g^{ik} \nabla_k X^h_\alpha,$$

(6)

$$\frac{\partial g_{ij}}{\partial x^k} = G^h_{ki} g_{hj} + G^h_{kj} g_{hi}, \ \frac{\partial h^{\alpha\beta}}{\partial t^\gamma} = -H^\alpha_{\gamma\lambda} h^{\lambda\beta} - H^\beta_{\gamma\lambda} h^{\alpha\lambda}.$$

If the metric h is a Riemann metric, then we recover the theory of potential maps.

2 Lorentz-Udrişte World-Force Law

In nonquantum relativity there are three basic laws for particles: the Lorentz World-Force Law and two conservation laws [6]. Now we shall generalize the Lorentz World-Force Law (see also [7]-[18]).

2.1 Definition. Let $F_\alpha = (F_j{}^i{}_\alpha)$ and $U_{\alpha\beta} = (U^i_{\alpha\beta})$ be C^∞ distinguished tensors on $T \times M$, where $\omega_{ji\alpha} = g_{hi} F_j{}^h{}_\alpha$ is skew-symmetric with respect to j and i. Let $c(t, x)$ be a C^∞ real function on $T \times M$. A C^∞ map $\varphi : T \to M$ obeys the *Lorentz-Udrişte World-Force Law* with respect to F_α, $U_{\alpha\beta}$, c iff it is a solution of the *ultra-hyperbolic-Poisson PDEs*

$$(7) \qquad \tau(\varphi)^i = g^{ij}\frac{\partial c}{\partial x^j} + h^{\alpha\beta} F_j{}^i{}_\alpha x^j_\beta + h^{\alpha\beta} U^i_{\alpha\beta}.$$

Now we show that the solutions of a system of PDEs of order one are ultra-potential maps in a suitable geometrical structure. First we remark that a C^∞ distinguished tensor field $X^i_\alpha(t, x)$ on $T \times M$ defines a family of p-dimensional sheets as solutions of the PDE system of order one

$$(8) \qquad x^i_\alpha = X^i_\alpha(t, x(t)),$$

if the complete integrability conditions

$$\frac{\partial X^i_\alpha}{\partial t^\beta} + \frac{\partial X^i_\alpha}{\partial x^j} X^j_\beta = \frac{\partial X^i_\beta}{\partial t^\alpha} + \frac{\partial X^i_\beta}{\partial x^j} X^j_\alpha$$

are satisfied.

The distinguished tensor field X^i_α and semi-Riemann metrics h and g determine the *potential energy*

$$f : T \times M \to R, \quad f = \frac{1}{2} h^{\alpha\beta} g_{ij} X^i_\alpha X^j_\beta.$$

The evolution point $t = (t^1, ..., t^p)$ is called *multitime*. The distinguished tensor field (family of p-dimensional sheets) X^i_α on $(T \times M, h + g)$ is called:

1) *multitimelike*, if $f < 0$;
2) *nonspacelike or causal*, if $f \leq 0$;
3) *null or lightlike*, if $f = 0$;
4) *spacelike*, if $f > 0$.

Let X^i_α be a distinguished tensor field of everywhere constant energy. If X^i_α (the system (8)) has no critical point on M, then upon rescaling, it may be supposed that $f \in \left\{-\frac{1}{2}, 0, \frac{1}{2}\right\}$. Generally, $\mathcal{E} \subset \mathcal{M}$ is the set of critical points of the distinguished tensor field X^i_α, and this rescaling is possible only on $T \times (M \setminus \mathcal{E})$.

Using the operator (derivative along a solution of PDEs (8)),

$$\frac{\delta}{\partial t^\beta} x^i_\alpha = x^i_{\alpha\beta} = \frac{\partial^2 x^i}{\partial t^\alpha \partial t^\beta} - H^\gamma_{\alpha\beta} x^i_\gamma + G^i_{jk} x^j_\alpha x^k_\beta,$$

we obtain the prolongation (system of PDEs of order two)

(9) $$x^i_{\alpha\beta} = D_\beta X^i_\alpha + (\nabla_j X^i_\alpha) x^j_\beta.$$

The distinguished tensor field X^i_α, the metric g, and the connection ∇ determine the *external distinguished tensor field*

$$F_j{}^i{}_\alpha = \nabla_j X^i_\alpha - g^{ih} g_{kj} \nabla_h X^k_\alpha,$$

which characterizes the *helicity* of the distinguished tensor field X^i_α.

First we write the PDE system (9) in the equivalent form

$$x^i_{\alpha\beta} = g^{ih} g_{kj} (\nabla_h X^k_\alpha) x^j_\beta + F_j{}^i{}_\alpha x^j_\beta + D_\beta X^i_\alpha.$$

Now we modify this PDE system into

(10) $$x^i_{\alpha\beta} = g^{ih} g_{kj} (\nabla_h X^k_\alpha) X^j_\beta + F^i_{j\alpha} x^j_\beta + D_\beta X^i_\alpha.$$

Of course, the PDE system (10) is still a prolongation of the PDE system (8).

Taking the trace of (10) with respect to $h^{\alpha\beta}$ we obtain that any solution of PDE system (8) is also a solution of the ultra-hyperbolic-Poisson PDE system

$$h^{\alpha\beta} x^i_{\alpha\beta} = g^{ih} h^{\alpha\beta} g_{kj} (\nabla_h X^k_\alpha) X^j_\beta$$

(11) $$+ h^{\alpha\beta} F_j{}^i{}_\alpha x^j_\beta + h^{\alpha\beta} D_\beta X^i_\alpha.$$

2.2 Theorem. *The ultra-hyperbolic-Poisson PDE system (11) is an Euler-Lagrange prolongation of the PDEs system (8).*

If $F_j{}^i{}_\alpha = 0$, then the PDE system (11) reduces to

(12) $$h^{\alpha\beta} x^i_{\alpha\beta} = g^{ih} h^{\alpha\beta} g_{kj} (\nabla_h X^k_\alpha) X^j_\beta + h^{\alpha\beta} D_\beta X^i_\alpha.$$

The first term in the second hand member of the PDE systems (11) or (12) is $(grad\ f)^i$. Consequently, choosing the metrics h and g such that $f \in \left\{-\frac{1}{2}, 0, \frac{1}{2}\right\}$, then the preceding PDE systems reduce to

(11′) $$h^{\alpha\beta} x^i_{\alpha\beta} = h^{\alpha\beta} F_j{}^i{}_\alpha x^j_\beta + h^{\alpha\beta} D_\beta X^i_\alpha$$

(12′) $$h^{\alpha\beta} x^i_{\alpha\beta} = h^{\alpha\beta} D_\beta X^i_\alpha.$$

2.3 Theorem. 1) *The solutions of PDE system (11) are the extremals of the Lagrangian*

$$\begin{aligned} L &= \frac{1}{2} h^{\alpha\beta} g_{ij} (x^i_\alpha - X^i_\alpha)(x^j_\beta - X^j_\beta)\sqrt{|h|} = \\ &= \left(\frac{1}{2} h^{\alpha\beta} g_{ij} x^i_\alpha x^j_\beta - h^{\alpha\beta} g_{ij} x^i_\alpha X^j_\beta + f\right)\sqrt{|h|}. \end{aligned}$$

2) *The solutions of PDE system (12) are the extremals of the Lagrangian*

$$L = \left(\frac{1}{2} h^{\alpha\beta} g_{ij} x^i_\alpha x^j_\beta + f\right)\sqrt{|h|}.$$

3) *If the Lagrangians L are independent of the variable t, then the PDE systems (11) or (12) are conservative, the energy-impulse tensor field being*

$$T^\alpha{}_\beta = x^i_\beta \frac{\partial L}{\partial x^i_\alpha} - L\delta^\alpha_\beta.$$

4) *Both previous Lagrangians produce the same Hamiltonian*

$$H = \left(\frac{1}{2} h^{\alpha\beta} g_{ij} x^i_\alpha x^j_\beta - f\right)\sqrt{|h|}.$$

Proof. 1) and 2) If we write $L = E\sqrt{|h|}$, where E is the energy density, then the Euler-Lagrange equations of extremals

$$\frac{\partial L}{\partial x^k} - \frac{\partial}{\partial t^\alpha}\frac{\partial L}{\partial x^k_\alpha} = 0$$

can be written

(13) $$\frac{\partial E}{\partial x^k} - \frac{\partial}{\partial t^\alpha}\frac{\partial E}{\partial x^k_\alpha} - H^\gamma_{\gamma\alpha}\frac{\partial E}{\partial x^k_\alpha} = 0.$$

We compute

$$\begin{aligned} \frac{\partial E}{\partial x^k} &= \frac{1}{2} h^{\alpha\beta}\frac{\partial g_{ij}}{\partial x^k} x^i_\alpha x^j_\beta - h^{\alpha\beta}\frac{\partial g_{ij}}{\partial x^k} x^i_\alpha X^j_\beta \\ &+ \frac{1}{2} h^{\alpha\beta}\frac{\partial g_{ij}}{\partial x^k} X^i_\alpha X^j_\beta - h^{\alpha\beta} g_{ij} x^i_\alpha \frac{\partial X^j_\beta}{\partial x^k} \\ &+ h^{\alpha\beta} g_{ij}\frac{\partial X^i_\alpha}{\partial x^k} X^j_\beta, \end{aligned}$$

$$\frac{\partial E}{\partial x^k_\alpha} = h^{\alpha\beta} g_{kj} x^j_\beta - h^{\alpha\beta} g_{kj} X^j_\beta,$$

$$\begin{aligned} -\frac{\partial}{\partial t^\alpha}\frac{\partial E}{\partial x^k_\alpha} &= -\frac{\partial h^{\alpha\beta}}{\partial t^\alpha} g_{kj} x^j_\beta - h^{\alpha\beta}\frac{\partial g_{kj}}{\partial x^l} x^l_\alpha x^j_\beta \\ &- h^{\alpha\beta} g_{kj}\frac{\partial^2 x^j}{\partial t^\alpha \partial t^\beta} + \frac{\partial h^{\alpha\beta}}{\partial t^\alpha} g_{kj} X^j_\beta \\ &+ h^{\alpha\beta}\frac{\partial g_{kj}}{\partial x^l} x^l_\alpha X^j_\beta + h^{\alpha\beta} g_{kj}\left(\frac{\partial X^j_\beta}{\partial t^\alpha} + \frac{\partial X^j_\beta}{\partial x^l} x^l_\alpha\right). \end{aligned}$$

We replace in (13) taking into account the formulas (1), (4) and (6). We find

$$h^{\alpha\beta} g_{kj} x^j_{\alpha\beta} = h^{\alpha\beta} g_{ij} (\nabla_k X^i_\alpha) X^j_\beta$$

$$+h^{\alpha\beta}g_{kj}(\nabla_l X^j_\beta)x^l_\alpha - h^{\alpha\beta}g_{ij}x^i_\alpha\nabla_k X^j_\beta + h^{\alpha\beta}g_{kj}D_\alpha X^j_\beta.$$

Transvecting by g^{hk} and using the formula (5), we obtain

$$h^{\alpha\beta}x^i_{\alpha\beta} = g^{ik}h^{\alpha\beta}g_{lj}(\nabla_k X^l_\alpha)X^j_\beta + h^{\alpha\beta}F_j{}^i{}_\alpha x^j_\beta$$

$$+h^{\alpha\beta}D_\alpha X^i_\beta.$$

3) Taking into account the Euler-Lagrange equations, we have

$$\frac{\partial T^\alpha_\beta}{\partial t^\alpha} = \frac{\partial^2 x^i}{\partial t^\alpha \partial t^\beta}\frac{\partial L}{\partial x^i_\alpha} + x^i_\beta\frac{\partial^2 L}{\partial t^\alpha \partial x^i_\alpha} + x^i_\beta\frac{\partial^2 L}{\partial x^j \partial x^i_\alpha}x^j_\alpha +$$

$$+x^i_\beta\frac{\partial^2 L}{\partial x^i_\alpha \partial x^j_\gamma}\frac{\partial^2 x^j}{\partial t^\gamma \partial t^\alpha} - \frac{\partial L}{\partial t^\alpha}\delta^\alpha_\beta - \frac{\partial L}{\partial x^j}x^j_\beta$$

$$-\frac{\partial L}{\partial x^j_\gamma}\frac{\partial^2 x^j}{\partial t^\gamma \partial t^\alpha}\delta^\alpha_\beta = -\frac{\partial L}{\partial t^\beta}.$$

Open problem. Determine the general expression of the energy-impulse tensor field as object on $J^1(T,M)$, and compute its divergence.

4) We use the formula

$$H = x^i_\alpha\frac{\partial L}{\partial x^i_\alpha} - L.$$

2.4 Corollary. *Every PDE generates a Lagrangian of order one via the associated first order PDE system and suitable metrics on the manifold of independent variables and on the manifold of functions. In this sense the solutions of the initial PDE are ultra-potential maps.*

2.5 Theorem (Lorentz-Udrişte World-Force Law).

1) *Every solution of the PDE system (12) is a ultra-potential map on the semi-Riemann manifold* $(T\times M, h+g)$.

2) *Let* $N(^i_\alpha)_j = G^i_{jk}x^k_\alpha - F_j{}^i{}_\alpha,\ \ M(^i_\alpha)_\beta = -H^\gamma_{\alpha\beta}x^i_\gamma$. *Every solution of the PDE system (11) is a horizontal ultra-potential map of the semi-Riemann-Lagrange manifold*

$$\left(T\times M,\ h+g,\ N(^i_\alpha)_j,\ M(^i_\alpha)_\beta\right).$$

3 Open problems regarding the geometry of jet bundles

If $(t^\alpha, x^i, x^i_\alpha)$ are the coordinates of a point in $J^1(T,M)$, and $H^\alpha_{\beta\gamma}$, G^i_{jk} are the components of the connection induced by h and g, respectively, then

$$\left(\frac{\delta}{\delta t^\alpha} = \frac{\partial}{\partial t^\alpha} + H^\gamma_{\alpha\beta}x^i_\gamma\frac{\partial}{\partial x^i_\beta}\ ,\right.$$

$$\left.\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G^h_{ik}x^k_\alpha\frac{\partial}{\partial x^h_\alpha},\ \frac{\partial}{\partial x^i_\alpha}\right),$$

$$\left(dt^\beta, dx^j,\ \delta x^j_\beta = dx^j_\beta - H^\gamma_{\beta\lambda}x^j_\gamma dt^\lambda + G^j_{hk}x^h_\beta dx^k\right)$$

are dual frames on $J^1(T,M)$, i.e.,

$$dt^\beta\left(\frac{\delta}{\delta t^\alpha}\right) = \delta^\beta_\alpha, dt^\beta\left(\frac{\delta}{\delta x^i}\right) = 0, dt^\beta\left(\frac{\partial}{\partial x^i_\alpha}\right) = 0$$

$$dx^j\left(\frac{\delta}{\delta t^\alpha}\right) = 0, dx^j\left(\frac{\delta}{\delta x^i}\right) = \delta^j_i, dx^j\left(\frac{\partial}{\partial x^i_\alpha}\right) = 0$$

$$\delta x^j_\beta\left(\frac{\delta}{\delta t^\alpha}\right) = 0, \delta x^j_\beta\left(\frac{\delta}{\delta x^i}\right) = 0, \delta x^j_\beta\left(\frac{\partial}{\partial x^i_\alpha}\right) = \delta^j_i\delta^\alpha_\beta.$$

The induced Sasaki-like metric on $J^1(T,M)$ is defined by

$$S_1 = h_{\alpha\beta}dt^\alpha\otimes dt^\beta + g_{ij}dx^i\otimes dx^j + h^{\alpha\beta}g_{ij}\delta x^i_\alpha\otimes\delta x^j_\beta.$$

Problems:

1) The geometry of the semi-Riemann manifold $(J^1(T,M), S_1)$, which is similar to the geometry of the tangent bundle endowed with Sasaki metric, was finalized in our research group [4]-[5], [7]-[18]. As was shown here this geometry permits the interpretation of solutions of PDE systems of order one (8) as potential maps. In this sense the solutions of every PDE of any order are extremals of a least squares Lagrangian of order one.

2) Study the geometry of the dual space of $(J^1(T,M), S_1)$.

3) Find a Sasaki-like S_2 metric on the jet bundle of order two and develop the geometry of the semi-Riemann manifold $(J^2(T,M), S_2)$. In this manifold, the PDEs of Mathematical Physics (of order two) appear like hypersurfaces. Most of them are in fact algebraic hypersurfaces.

Hint. The tangent vectors

$$D_\alpha = \frac{\partial}{\partial t^\alpha}, D_i = \frac{\partial}{\partial x^i}, D^\alpha_i = \frac{\partial}{\partial x^i_\alpha},$$

$$D^{\alpha\alpha}_i = \frac{\partial}{\partial x^i_{\alpha\alpha}}, D^{\alpha\beta}_i(\alpha<\beta) = \frac{\partial}{\partial x^i_{\alpha\beta}}$$

determine a natural basis

$$(D_\alpha, D_i, D^\alpha_i, D^{\alpha\alpha}_i, D^{\alpha\beta}_i(\alpha<\beta)),$$

whose dual basis is

$$(dt^\alpha, dx^i, dx^i_\alpha, dx^i_{\alpha\beta}(\alpha<\beta)).$$

These basis are not suitable for the geometry of $(J^2(T,M)$ since they induce complicated formulas. We suggest to take frames of the form

$$\frac{\delta}{\delta t^\alpha} = \frac{\partial}{\partial t^\alpha} + A_\alpha{}^i\frac{\partial}{\partial x^i} + A_\alpha(^i_\beta)\frac{\partial}{\partial x^i_\beta} + A_\alpha(^i_{\beta\gamma})\frac{\partial}{\partial x^i_{\beta\gamma}}$$

$$\frac{\delta}{\delta x^i} = A_i^\alpha \frac{\partial}{\partial t^\alpha} + \frac{\partial}{\partial x^i} + A_i(^j_\beta)\frac{\partial}{\partial x^i_\beta} + A_i(^j_{\beta\gamma})\frac{\partial}{\partial x^j_{\beta\gamma}}$$

$$\frac{\delta}{\delta x^i_\alpha} = A(^i_\alpha)^\beta \frac{\partial}{\partial t^\beta} + A(^i_\alpha)^j\frac{\partial}{\partial x^j} + \frac{\partial}{\partial x^i_\alpha} + A(^i_\alpha)(^j_{\beta\gamma})\frac{\partial}{\partial x^j_{\beta\gamma}}$$

$$\frac{\delta}{\delta x^i_{\alpha\beta}} = A(^i_{\alpha\beta})^\gamma \frac{\partial}{\partial t^\gamma} + A(^i_{\alpha\beta})^j\frac{\partial}{\partial x^j} + \frac{\partial}{\partial x^i_\alpha}$$

$$+A(^i_{\alpha\beta})(^j_\gamma)\frac{\partial}{\partial x^j_\gamma} + \frac{\partial}{\partial x^i_{\alpha\beta}}$$

dual to coframe of the form

$$\delta t^\beta = dt^\beta + B_j{}^\beta dx^j + B(^\gamma_j)^\beta dx^j_\gamma + B(^{\gamma\delta}_j)^\beta dx^j_{\gamma\delta}$$

$$\delta x^j = B_\gamma{}^j dt^\gamma + dx^j + B(^\gamma_k)^j dx^k_\gamma + B(^{\gamma\delta}_k)^j dx^k_{\gamma\delta}$$

$$\delta x^j_\beta = B_\gamma(^j_\beta)dt^\gamma + B_k(^j_\beta)dx^k + dx^j_\gamma + B(^{\gamma\delta}_k)^j_\beta dx^k_{\gamma\delta}$$

$$\delta x^j_{\beta\gamma} = B_\delta(^j_{\beta\gamma})dt^\delta + B_k(^j_{\beta\gamma})dx^k + B(^\delta_k)(^j_{\beta\gamma})dx^k_\delta + dx^j_{\beta\gamma}$$

In this context, we can define the Sasaki like metric

$$S_2 = h_{\alpha\beta}\delta t^\alpha \otimes \delta t^\beta + g_{ij}\delta x^i \otimes \delta x^j + h^{\alpha\beta}g_{ij}\delta x^i_\alpha \otimes \delta x^j_\beta$$

$$+h^{\alpha\gamma}h^{\beta\lambda}g_{ij}\delta x^i_{\alpha\beta}\otimes \delta x^j_{\gamma\lambda}.$$

4) Study the geometry of the dual space of $(J^2(T,M), S_2)$.

4 Covariant Hamilton Field Theory (Covariant Hamilton PDEs)

Let us show that the PDE systems (11) and (12) induce Hamilton PDE systems on the manifold $J^1(T,M)$. The results are similar to those in the papers [2], [3].

Let (T,h) be a semi-Riemann manifold with p dimensions, and (M,g) be a semi-Riemann manifold with n dimensions. Then $(J^1(T,M), h+g+h^{-1}*g)$ is a semi-Riemann manifold with $p+n+pn$ dimensions.

We denote by X^i_α a C^∞ distinguished tensor field on $T\times M$, and by $\omega_{ij\alpha}$ the distinguished 2-form associated to the distinguished tensor field

$$F_j{}^i{}_\alpha = \nabla_j X^i_\alpha - g^{ih}g_{kj}\nabla_h X^k_\alpha$$

via the metric g, i.e., $\omega = \frac{1}{2} g\circ F$. Of course X^i_α, $F_j{}^i{}_\alpha$ are distinguished globally defined objects on $J^1(T,M)$.

Recall that on a symplectic manifold (Q,Ω) of even dimension q, the Hamiltonian vector field X_{f_1} of a function $f_1 \in \mathcal{F}(\mathcal{Q})$ is defined by

$$X_{f_1}\rfloor\Omega = df_1,$$

and the Poisson bracket of f_1, f_2 is defined by

$$\{f_1,f_2\} = \Omega(X_{f_1}, X_{f_2}).$$

The polysymplectic analogue of a function is a q-form called *momentum observable*. The Hamiltonian vector field X_{f_1} of such a momentum observable f_1 is defined by

$$X_{f_1}\rfloor\Omega = df_1,$$

where Ω is the canonical $(q+2)$-form on the appropriate dual of $J^1(T,M)$. Since Ω is nondegenerate, this uniquely defines X_{f_1}. The Poisson bracket of two such n-forms f_1, f_2 is the n-form defined by

$$\{f_1,f_2\} = X_{f_1}\rfloor(X_{f_2}\rfloor\Omega).$$

Of course $\{f_1,f_2\}$ is, up to the addition of exact terms, another momentum observable.

4.1 Theorem. *The ultra-hyperbolic PDE system*

$$h^{\alpha\beta}x^i_{\alpha\beta} = g^{ih}h^{\alpha\beta}g_{jk}X^j_\beta\nabla_h X^k_\alpha$$

transfers in $J^1(T,M)$ as a covariant Hamilton PDE system with respect to the Hamiltonian (momentum observable)

$$H = \left(\frac{1}{2}h^{\alpha\beta}g_{ij}x^i_\alpha x^j_\beta - f\right)dv_h$$

and the non-degenerate distinguished polysymplectic $(p+2)$-form

$$\Omega = \Omega_\alpha\otimes dt^\alpha, \quad \Omega_\alpha = g_{ij}dx^i\wedge\delta x^j_\alpha\wedge dv_h.$$

Proof. Let

$$\theta = \theta_\alpha\otimes dt^\alpha, \quad \theta_\alpha = g_{ij}x^i_\alpha dx^j\wedge dv_h$$

be the distinguished Liouville $(p+1)$-form on $J^1(T,M)$. It follows

$$\Omega_\alpha = -d\theta_\alpha.$$

We denote by

$$X_H = X^\beta_H\frac{\delta}{\delta t^\beta}, \quad X^\beta_H = u^{\beta l}\frac{\delta}{\delta x^l} + \frac{\delta u^{\beta l}}{\partial t^\alpha}\frac{\partial}{\partial x^l_\alpha}$$

the distinguished Hamiltonian object of the observable H. Imposing

$$X^\alpha_H\rfloor\Omega_\alpha = dH,$$

where

$$dH = (h^{\alpha\beta}g_{ij}x^j_\beta\delta x^i_\alpha - h^{\alpha\beta}g_{ij}X^j_\beta\nabla_k X^i_\alpha dx^k)\wedge dv_h,$$

we find

$$g_{ij}u^{\alpha i}\delta x^j_\alpha - g_{ij}\frac{\delta u^{\alpha j}}{\partial t^\alpha}dx^i = h^{\alpha\beta}g_{ij}x^j_\beta\delta x^i_\alpha$$

$$-h^{\alpha\beta}g_{ij}X^j_\beta\nabla_k X^i_\alpha dx^k \text{ modulo } dv_h.$$

Consequently, it appears the Hamilton PDE system

$$\begin{cases} u^{\alpha i} = h^{\alpha\beta}x^i_\beta \\ \dfrac{\delta u^{\alpha i}}{\partial t^\alpha} = g^{hi}h^{\alpha\beta}g_{jk}X^j_\beta(\nabla_h X^k_\alpha) \end{cases}$$

(up to the addition of terms which are cancelled by the exterior multiplication with dv_h).

4.2 Theorem. *The ultra-hyperbolic PDE system*

$$h^{\alpha\beta}x^i_{\alpha\beta} = g^{ih}h^{\alpha\beta}g_{kj}(\nabla_h X^k_\alpha)X^j_\beta + h^{\alpha\beta}F_j{}^i{}_\alpha x^j_\beta$$

$$+h^{\alpha\beta}D_\beta X^i_\alpha$$

transfers in $J^1(T,M)$ as a covariant Hamilton PDE system with respect to the Hamiltonian (momentum observable)

$$H = \left(\frac{1}{2}h^{\alpha\beta}g_{ij}x^i_\alpha x^j_\beta - f\right)dv_h$$

and the non-degenerate distinguished polysymplectic $(p+2)$-form

$$\Omega = \Omega_\alpha \otimes dt^\alpha,$$

$$\Omega_\alpha = (g_{ij}dx^i \wedge \delta x^j_\alpha + \omega_{ij\alpha}dx^i \wedge dx^j$$

$$+g_{ij}(D_\beta X^i_\alpha)dt^\beta \wedge dx^j) \wedge dv_h.$$

Proof. Let

$$\theta = \theta_\alpha \otimes dt^\alpha, \quad \theta_\alpha = (g_{ij}x^i_\alpha dx^j - g_{ij}X^i_\alpha dx^j) \wedge dv_h$$

be the distinguished Liouville $(p+1)$-form on $J^1(T,M)$. It follows

$$\Omega_\alpha = -d\theta_\alpha$$

(of course the term containing dt^β disappears by exterior multiplication with dv_h). We denote by

$$X_H = X^\beta_H\frac{\delta}{\delta t^\beta}, \quad X^\beta_H = h^{\beta\gamma}\frac{\delta}{\delta t^\gamma} + u^{\beta l}\frac{\delta}{\delta x^l} + \frac{\delta u^{\beta l}}{\partial t^\alpha}\frac{\partial}{\partial x^l_\alpha}$$

the distinguished Hamiltonian object of the observable H. Imposing

$$X^\alpha_H \rfloor \Omega_\alpha = dH,$$

where

$$dH = (h^{\alpha\beta}g_{ij}x^j_\beta\delta x^i_\alpha - h^{\alpha\beta}g_{ij}X^j_\beta(\nabla_k X^i_\alpha)dx^k) \wedge dv_h,$$

we find

$$(g_{ij}u^{\alpha i}\delta x^j_\alpha - g_{ij}\frac{\delta u^{\alpha j}}{\partial t^\alpha}dx^i + 2\omega_{ij\alpha}u^{\alpha i}dx^j$$

$$+h^{\alpha\beta}g_{ij}(D_\beta X^i_\alpha)dx^j) \wedge dv_h = dH.$$

Consequently, it appears the Hamilton PDE system

$$u^{\alpha i} = h^{\alpha\beta}x^i_\beta, \ \frac{\delta u^{\alpha i}}{\partial t^\alpha} = g^{hi}h^{\alpha\beta}g_{jk}X^j_\beta(\nabla_h X^k_\alpha)$$

$$+2g^{hi}\omega_{jh\alpha}u^{\alpha j} + h^{\alpha\beta}D_\beta X^i_\alpha$$

(up to the addition of terms which are cancelled by the exterior multiplication with dv_h).

5 Application to continuous groups of transformations

The C^∞ vector fields ξ_α on the manifold M and the 1-forms A^α on the manifold T satisfying

$$[\xi_\alpha, \xi_\beta] = C^\gamma_{\alpha\beta}\xi_\gamma, \quad C^\gamma{}_{\alpha\beta} = \text{constants},$$

$$\frac{\partial A^\alpha_\beta}{\partial t^\gamma} - \frac{\partial A^\alpha_\gamma}{\partial t^\beta} = C^\alpha_{\lambda\delta}A^\lambda_\beta A^\delta_\gamma$$

determine a continuous group of transformations via the PDEs

$$x^i_\alpha = \xi^i_\beta(x(t))A^\beta_\alpha(t).$$

Conversely, if $x^i = x^i(t^\alpha, y^j)$ are solutions of a completely integrable system of PDEs of the preceding form, where the $A's$ and $\xi's$ satisfy the conditions stated above, such that for values t^α_0 of $t's$ the determinant of the $A's$ is not zero and

$$x^i(t^\alpha_0, y^j) = y^j,$$

then $x^i = x^i(t^\alpha, y^j)$ define a continuous group of transformations.

Using a semi-Riemann metric h on the manifold T, a semi-Riemann metric g on the manifold M, then the maps determining a continuous group of transformations appear like extremals (ultra-potential maps) of the Lagrangian

$$L = \frac{1}{2}h^{\alpha\beta}g_{ij}(x^i_\alpha - \xi^i_\lambda A^\lambda_\alpha)(x^j_\beta - \xi^j_\mu A^\mu_\beta)\sqrt{|h|}.$$

Open problem. Find the geometrical meaning of ultra-potential maps which are not transformations in the given group.

References:

[1] J. Eells, L. Lemaire, Harmonic maps of Riemannian manifolds, *Amer. J. Math.*, 86(1964), 109-160.

[2] G. Giachetta, L. Mangiarotti, Covariant Hamiltonian field theory, *http://xxx.lanl.gov/hep-th/9904062*.

[3] M. J. Gotay, J. Isenberg, J. E. Marsden, R. Montgomery, J. Sniatycki, P. B. Yasskin, Momentum maps and Classical relativistic fields, Part.I: Covariant field theory, *http://xxx.lanl.gov/physical/9801019*.

[4] M. Neagu, Riemann-Lagrange Geometry on 1-Jet Spaces, *Matrix Rom*, Bucharest, 2005.

[5] M. Neagu, C. Udrişte, A. Oana, Multi-time sprays and h-traceless maps, *Balkan Journal of Geometry and Its Applications*, Vol. 10, No. 2 (2005), 76-92.

[6] R. K. Sachs and H. Wu, General Relativity for Mathematicians, *Springer-Verlag*, New-York, 1977.

[7] C. Udrişte, Nonclassical Lagrangian Dynamics and Potential Maps, *Conference in Mathematics in Honour of Professor Radu Roşca on the occasion of his Ninetieth Birthday*, Katholieke University Brussel, Katholieke University Leuwen, Belgium, December 11-16, 1999; http://xxx.lanl.gov/math.DS/0007060, (2000).

[8] C. Udrişte, Geometric Dynamics, *Southeast Asian Bulletin of Mathematics*, Springer-Verlag, 24, 1 (2000), 313-322.

[9] C. Udrişte, Geometric Dynamics, *Kluwer Academic Publishers*, 2000.

[10] C. Udrişte, Dynamics induced by second-order objects, *BSG Proceedings 4, Global Analysis, Differential Geometry, Lie Algebras*, Editor Grigorios Tsagas, pp.161-168, Geometry Balkan Press, 2000.

[11] C. Udrişte, Muti-time Dynamics induced by 1-forms and metrics, *BSG Proceedings 5, Global Analysis, Differential Geometry, Lie Algebras*, Editor Grigorios Tsagas, pp.169-178, Geometry Balkan Press, 2000.

[12] C. Udrişte, Solutions of DEs and PDEs as potential maps using first order Lagrangians, *Balkan Journal of Geometry and Its Applications*, Vol.6, No.1 (2001), 93-108.

[13] C. Udrişte, From integral manifolds and metrics to potential maps, *Atti dell'Academia Peloritana dei Pericolanti, Classe I di Scienze Fis. Mat. et Nat.*, 81-82, C1A0401008 (2003-2004), 1-16.

[14] C. Udrişte, Geodesic motion in a gyroscopic field of forces, *Tensor, N. S.*, 66, 3 (2005), 215-228.

[15] C. Udrişte, M. Neagu, Geometrical interpretation of solutions of certain PDEs, *Balkan Journal of Geometry and Its Applications*, 4,1 (1999), 138-145.

[16] C. Udrişte, M. Ferrara, D. Opriş, Economic Geometric Dynamics, *Monographs and Textbooks 6, Geometry Balkan Press*, Bucharest, 2004.

[17] C. Udrişte, Tools of geometric dynamics, *Buletinul Institutului de Geodinamică, Academia Română*, 14, 4 (2003), 1-26; *Proceedings of the XVIII Workshop on Hadronic Mechanics, honoring the 70-th birthday of Prof. R. M. Santilli, the originator of hadronic mechanics*, University of Karlstad, Sweden, June 20-22, 2005; *Eds. Valer Dvoeglazov, Tepper L. Gill, Peter Rowland, Erick Trell, Horst E. Wilhelm, Hadronic Press, International Academic Publishers*, December 2006, ISBN 1-5748.

[18] C. Udrişte, A. Udrişte, From flows and metrics to dynamics and winds, *Bull. Cal. Math. Soc.*, 98, 5 (2006), 389-394.