Non-Classical Lagrangian Dynamics and Potential Maps

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Abstract: The basic theory regarding Nonclassical Lagrangian Dynamics and Potential Maps was announced in [7]. Since its mathematical impact is now at large vogue, we reinforce some arguments. Section 1 extends the theory of harmonic and potential maps in the language of differential geometry. Section 2 defines a generalized Lorentz world-force law and shows that any PDE system of order one (in particular, p-flow) generates such a law in a suitable geometrical structure. In other words, the solutions of any PDE system of order one are harmonic or potential maps, i.e., they are solutions of Euler-Lagrange prolongation PDE system of order two built via Riemann-Lagrange structures and a least squares Lagrangian. Section 3 formulates open problems regarding the geometry of potential maps, i.e., they are solutions of Euler-Lagrange prolongation PDE system of order two built via Riemann-Lagrange structures and a least squares Lagrangian. Section 4 shows that the Lorentz-Udriste world-force law is equivalent to certain covariant Hamilton PDEs on \((J^1(T, M), S_1)\). Section 5 describes the maps determining a continuous group of transformations as ultra-potential maps.

Key–Words: ultra-harmonic map, ultra-potential map, Lagrangian, Hamiltonian, Lorentz-Udriste world force law.

1 Harmonic and Potential Maps

All maps throughout the paper are smooth, while manifolds are real, finite-dimensional, Hausdorff, second-countable and connected.

Let \((T, h)\) and \((M, g)\) be semi-Riemann manifolds of dimensions \(p\) and \(n\). Hereafter we shall assume that the manifold \(T\) is oriented. Greek (Latin) letters will be used for indexing the components of geometrical objects attached to the manifold \(T\) (manifold \(M\)). Local coordinates will be written
\[
t = (t^\alpha), \quad \alpha = 1, \ldots, p
\]
\[
x = (x^i), \quad i = 1, \ldots, n,
\]
and the components of the corresponding metric tensor and Christoffel symbols will be denoted by \(h_{\alpha\beta}, g_{ij}, H^\alpha_{\beta\gamma}, G^i_{jk}\). Indices of tensors or distinguished tensors will be rised and lowered in the usual fashion.

Let \(\varphi: T \to M, \varphi(t) = x, x^i = x^i(t^\alpha)\) be a \(C^\infty\) map (parametrized sheet). We set
\[
\begin{aligned}
x_i^\alpha &= \frac{\partial x^i}{\partial t^\alpha}, & x_{\alpha\beta} &= \frac{\partial^2 x_i^\alpha}{\partial t^\alpha \partial t^\beta} - H^\gamma_{\alpha\beta} x^\gamma_i + G^i_{jk} x_j^\alpha x_k^\beta.
\end{aligned}
\]
Then \(x_i^\alpha, x_{\alpha\beta}\) transform like tensors under coordinate transformations \(t \to \tilde{t}, x \to \tilde{x}\). In the sequel \(x_i^\alpha, x_{\alpha\beta}\) will be interpreted like distinguished tensors.

The canonical form of the energy density \(E(\varphi)\) of the map \(\varphi\) is defined by
\[
E_0(\varphi)(t) = \frac{1}{2} h^{\alpha\beta}(t) g_{ij}(x(t)) x_i^\alpha(t) x_j^\beta(t).
\]
For a relatively compact domain \(\Omega \subset T\), we define the energy
\[
E_0(\varphi, \Omega) = \int_{\Omega} E_0(\varphi)(t) dv_h,
\]
where \(dv_h = \sqrt{|h|} dt^1 \wedge \ldots \wedge dt^p\) denotes the volume element induced by the semi-Riemann metric \(h\). A map \(\varphi\) is called ultra-harmonic map if it is a critical point of the energy functional \(E_0\), i.e., an extremal of the Lagrangian
\[
L = E_0(\varphi)(t) \sqrt{|h|},
\]
for all compactly supported variations. The ultra-harmonic map equation is a system of nonlinear ultra-hyperbolic-Laplace PDEs of second order and is expressed in local coordinates as
\[
\tau(\varphi)^i = h^{\alpha\beta} x_i^\alpha = 0.
\]
The vector field \(\tau(\varphi)^i\) defines a section of the pull-back bundle \(\varphi^{-1} TM\) of the tangent bundle \(TM\) of the
2 Lorentz-Udriște World-Force Law

In nonquantum relativity there are three basic laws for particles: the Lorentz World-Force Law and two conservation laws [6]. Now we shall generalize the Lorentz World-Force Law (see also [7]-[18]).

2.1 Definition. Let \( F_\alpha = (F^i_\alpha) \) and \( U_{\alpha\beta} = (U_{\alpha\beta}^i) \) be \( C^\infty \) distinguished tensors on \( T \times M \), where \( \omega_{jia} = g_{ij} F^h_\alpha h^i_\alpha \) is skew-symmetric with respect to \( j \) and \( i \). Let \( c(\tau, x) \) be a \( C^\infty \) real function on \( T \times M \). A \( C^\infty \) map \( \varphi : T \rightarrow M \) obeys the Lorentz-Udriște World-Force Law with respect to \( F_\alpha, U_{\alpha\beta}, c \) if it is a solution of the ultra-hyperbolic-Poisson PDEs

\[
\tau(\varphi)^i = g^{ij} \frac{\partial \varphi}{\partial x^j} + h^{\alpha\beta} F_\alpha^i \alpha^j + h^{\alpha\beta} U_{\alpha\beta}^i.
\]

Now we show that the solutions of a system of PDEs of order one are ultra-potential maps in a suitable geometrical structure. First we remark that a \( C^\infty \) distinguished tensor field \( X_\alpha^i(t, x) \) on \( T \times M \) defines a family of \( p \)-dimensional sheets as solutions of the PDE system of order one

\[
x_\alpha^i = X_\alpha^i(t, x(t)),
\]

if the complete integrability conditions

\[
\frac{\partial X_\alpha^i}{\partial x^j} + \frac{\partial X_\beta^i}{\partial x^j} X_\beta^j = \frac{\partial X_\alpha^j}{\partial x^i} + \frac{\partial X_\beta^j}{\partial x^i} X_\beta^i
\]

are satisfied.

The distinguished tensor field \( X_\alpha^i \) and semi-Riemann metrics \( h \) and \( g \) determine the potential energy

\[
f : T \times M \rightarrow R, \quad f = \frac{1}{2} h^{\alpha\beta} g_{ij} X_\alpha^i X_\beta^j.
\]

The evolution point \( t = (t^1, \ldots, t^p) \) is called multitime. The distinguished tensor field (family of \( p \)-dimensional sheets) \( X_\alpha^i \) on \( (T \times M, h + g) \) is called:

1) multitimelike, if \( f < 0 \);
2) nonspacelike or causal, if \( f \leq 0 \);
3) null or lightlike, if \( f = 0 \);
4) spacelike, if \( f > 0 \).

Let \( X_\alpha^i \) be a distinguished tensor field of everywhere constant energy. If \( X_\alpha^i \) (the system (8)) has no critical point on \( M \), then upon rescaling, it may be supposed that \( f \in \left\{ -\frac{1}{2}, 0, \frac{1}{2} \right\} \). Generally, \( E \subset M \) is the set of critical points of the distinguished tensor field \( X_\alpha^i \), and this rescaling is possible only on \( T \times (M \setminus E) \).

Using the operator (derivative along a solution of PDEs (8)),

\[
\frac{\delta}{\delta h^\beta} x_\alpha^i = x_{\alpha\beta} = \frac{\partial^2 x^i}{\partial h^{\alpha\beta} \partial h^\gamma} - H_{\alpha\beta}^\gamma x_\gamma^i + C_{jk}^h x^j_h x^k_h,
\]
we obtain the prolongation (system of PDEs of order two)

\[ x_{i\alpha\beta} = D_{\beta}X_{i\alpha} + (\nabla_j X_{i\alpha})x_{j\beta}. \]  

The distinguished tensor field \(X_i\), the metric \(g\), and the connection \(\nabla\) determine the external distinguished tensor field

\[ F_i^\alpha_a = \nabla_j X_i^\alpha_a - g^{ih}g_{kj}\nabla_h X_i^k, \]

which characterizes the helicity of the distinguished tensor field \(X_i\).

First we write the PDE system (9) in the equivalent form

\[ x_{i\alpha\beta} = g^{ih}g_{kj}(\nabla_h X_i^k)X_{j\beta} + F_i^h_a x_{j\beta} + D_{\beta}X_i^j. \]

Now we modify this PDE system into

\[ x_{i\alpha\beta} = g^{ih}g_{kj}(\nabla_h X_i^k)X_{j\beta} + F_i^h_a x_{j\beta} + D_{\beta}X_i^j. \]

Of course, the PDE system (10) is still a prolongation of the PDE system (8).

Taking the trace of (10) with respect to \(h^\alpha\beta\) we obtain that any solution of PDE system (8) is also a solution of the ultra-hyperbolic-Poisson PDE system

\[ h^\alpha\beta x_{i\alpha\beta} = g^{ih}h^\alpha\beta g_{kj}(\nabla_h X_i^k)X_{j\beta} + F_i^h_a x_{j\beta} + D_{\beta}X_i^j. \]

\[ + h^\alpha\beta F_i^h_a x_{j\beta} + h^\alpha\beta D_{\beta}X_i^j. \]

2.2 Theorem. The ultra-hyperbolic-Poisson PDE system (11) is an Euler-Lagrange prolongation of the PDEs system (8).

If \(F_i^h_a = 0\), then the PDE system (11) reduces to

\[ h^\alpha\beta x_{i\alpha\beta} = g^{ih}h^\alpha\beta g_{kj}(\nabla_h X_i^k)X_{j\beta} + h^\alpha\beta D_{\beta}X_i^j. \]

The first term in the second hand member of the PDE systems (11) or (12) is \((\text{grad } f)^i\). Consequently, choosing the metrics \(h\) and \(g\) such that \(f \in \{-\frac{1}{2}, 0, \frac{1}{2}\}\), the preceding PDE systems reduce to

\[ h^\alpha\beta x_{i\alpha\beta} = h^\alpha\beta F_i^h_a x_{j\beta} + h^\alpha\beta D_{\beta}X_i^j. \]

2.3 Theorem. 1) The solutions of PDE system (11) are the extremals of the Lagrangian

\[ L = \left(\frac{1}{2} h^\alpha\beta g_{ij} x_{i\alpha} x_{j\beta} + f\right) \sqrt{|h|} \]

2) The solutions of PDE system (12) are the extremals of the Lagrangian

\[ L = \left(\frac{1}{2} h^\alpha\beta g_{ij} x_{i\alpha} x_{j\beta} + f\right) \sqrt{|h|}. \]

3) If the Lagrangians \(L\) are independent of the variable \(t\), then the PDE systems (11) or (12) are conservative, the energy-impulse tensor field being

\[ T^\alpha = x^j_j \frac{\partial L}{\partial x_{\alpha}} - L \delta^\alpha_0. \]

4) Both previous Lagrangians produce the same Hamiltonian

\[ H = \left(\frac{1}{2} h^\alpha\beta g_{ij} x_{i\alpha} x_{j\beta} - f\right) \sqrt{|h|}. \]

Proof. 1) and 2) If we write \(L = E \sqrt{|h|}\), where \(E\) is the energy density, then the Euler-Lagrange equations of extremals

\[ \frac{\partial L}{\partial x^k} - \frac{\partial}{\partial t^\alpha} \frac{\partial L}{\partial x_{\alpha}^k} = 0. \]

can be written

\[ \frac{\partial E}{\partial x^k} - \frac{\partial}{\partial t^\alpha} \frac{\partial E}{\partial x_{\alpha}^k} - H \gamma^\alpha \frac{\partial E}{\partial x_{\alpha}^k} = 0. \]

We compute

\[ \frac{\partial E}{\partial x^k} = \frac{1}{2} h^\alpha\beta \frac{\partial g_{ij}}{\partial x^k} x_{i\alpha} x_{j\beta} - h^\alpha\beta \frac{\partial g_{ij}}{\partial x^k} x_{i\alpha} x_{j\beta} \]

\[ + \frac{1}{2} h^\alpha\beta g_{ij}(\nabla_h X_i^k)X_{j\beta} - h^\alpha\beta g_{ij} x_{i\alpha} \frac{\partial X_{j\beta}}{\partial x^k} \]

\[ + h^\alpha\beta \frac{\partial x_{i\alpha}}{\partial x^k} X_{j\beta}, \]

\[ \frac{\partial E}{\partial x_{\alpha}^k} = h^\alpha\beta g_{ij} x_{i\alpha} x_{j\beta} - h^\alpha\beta g_{ij} x_{i\alpha} x_{j\beta} \]

\[ - \frac{\partial}{\partial t^\alpha} \frac{\partial E}{\partial x_{\alpha}^k} = - \frac{\partial h^\alpha\beta}{\partial t^\alpha} g_{ij} x_{i\alpha} x_{j\beta} - h^\alpha\beta \frac{\partial g_{ij}}{\partial x^k} x_{i\alpha} x_{j\beta} \]

\[ - h^\alpha\beta g_{ij} \frac{\partial x_{i\alpha}}{\partial t^\alpha} x_{j\beta} + \frac{\partial h^\alpha\beta}{\partial t^\alpha} g_{ij} x_{i\alpha} x_{j\beta} \]

\[ + h^\alpha\beta g_{ij} \frac{\partial x_{i\alpha}}{\partial t^\alpha} x_{j\beta} + h^\alpha\beta g_{ij} \left(\frac{\partial X_{j\beta}}{\partial t^\alpha} + \frac{\partial X_{j\beta}}{\partial x^k} x_{i\alpha}\right). \]

We replace in (13) taking into account the formulas (1), (4) and (6). We find

\[ h^\alpha\beta g_{ij} x_{i\alpha} x_{j\beta} = h^\alpha\beta g_{ij} (\nabla_h X_i^k) X_{j\beta}. \]
\[ h^{\alpha \beta} g_{kj} \left( \nabla_i X^j_\beta \right)_\alpha - h^{\alpha \beta} g_{ij} x^i_\alpha \nabla_k X^j_\beta + h^{\alpha \beta} g_{kj} D_\alpha X^j_\beta. \]

Transvecting by \( g^{hk} \) and using the formula (5), we obtain
\[ h^{\alpha \beta} x^i_\alpha = g^{ik} h^{\alpha \beta} g_{ij} (\nabla_k X^j_\beta) x^i_\alpha + h^{\alpha \beta} F^i_\alpha x^j_\beta + h^{\alpha \beta} D_\alpha X^j_\beta. \]

3) Taking into account the Euler-Lagrange equations, we have
\[
\frac{\partial T^j_\beta}{\partial t^\alpha} = \frac{\partial^2 x^i}{\partial t^\alpha} \frac{\partial L}{\partial x^i} + \frac{\partial^2 L}{\partial x^i \partial t^\alpha} \frac{\partial x^i}{\partial t^\alpha} + \frac{\partial^2 L}{\partial t^\alpha \partial x^i} \frac{\partial x^i}{\partial t^\alpha} + \frac{\partial^2 L}{\partial t^\alpha \partial x^i} \frac{\partial x^i}{\partial t^\alpha} \frac{\partial L}{\partial x^i} - \frac{\partial L}{\partial x^i} \frac{\partial^2 x^i}{\partial t^\alpha \partial t^\beta} \frac{\partial \delta L}{\partial t^\beta} \frac{\partial \delta L}{\partial t^\beta} \frac{\partial \delta L}{\partial t^\beta} = - \frac{\partial L}{\partial t^\alpha} \frac{\partial^2 x^i}{\partial t^\alpha \partial t^\beta} \frac{\partial \delta L}{\partial t^\beta} \frac{\partial \delta L}{\partial t^\beta} \frac{\partial \delta L}{\partial t^\beta}.
\]

Open problem. Determine the general expression of the energy-impulse tensor field as object on \( J^1(T,M) \), and compute its divergence.

4) We use the formula
\[ H = x^i_\alpha \frac{\partial L}{\partial x^i_\alpha} - L. \]

2.4 Corollary. Every PDE generates a Lagrangian of order one via the associated first order PDE system and suitable metrics on the manifold of independent variables and on the manifold of functions. In this sense the solutions of the initial PDE are ultra-potential maps.

2.5 Theorem (Lorentz-Udriste World-Force Law).
1) Every solution of the PDE system (12) is a ultra-potential map on the semi-Riemann manifold \((T \times M, h + g)\).

2) Let \( N^i_\alpha \beta = G^i_\alpha \beta - F^i_\alpha \beta \), \( M^i_\alpha \beta = -H^i_\alpha \beta x^i_\alpha \). Every solution of the PDE system (11) is a horizontal ultra-potential map of the semi-Riemann-Lagrangian manifold
\[ (T \times M, h + g, N^i_\alpha \beta, M^i_\alpha \beta). \]

3 Open problems regarding the geometry of jet bundles

If \((t^\alpha, x^i, x^i_\alpha)\) are the coordinates of a point in \( J^1(T,M) \), and \( H^i_\alpha \beta, G^i_\alpha \beta \) are the components of the connection induced by \( h \) and \( g \), respectively, then
\[ \left( \frac{\delta}{\delta t^\alpha} = \frac{\partial}{\partial t^\alpha} + H^i_\alpha \beta x^i_\alpha \frac{\partial}{\partial x^i_\beta}, \right. \]
\[ \left. \left( dt^\beta, dx^i, \delta x^i_\beta = dx^i_\beta - H^i_\beta \lambda dt^\lambda + G^i_\lambda x^i_\lambda dx^\lambda \right) \right) \]
are dual frames on \( J^1(T,M) \), i.e.,
\[ dt^\beta \left( \frac{\delta}{\delta t^\alpha} \right) = \delta^\beta_\alpha, dt^\beta \left( \frac{\delta}{\delta x^i_\alpha} \right) = 0, dt^\beta \left( \frac{\partial}{\partial x^i_\alpha} \right) = 0 \]
\[ dx^i_\beta \left( \frac{\delta}{\delta t^\alpha} \right) = 0, dx^i_\beta \left( \frac{\delta}{\delta x^i_\alpha} \right) = \delta^i_\alpha, dx^i_\beta \left( \frac{\partial}{\partial x^i_\alpha} \right) = \delta^i_\alpha \delta^j_\beta. \]

The induced Sasaki-like metric on \( J^1(T,M) \) is defined by
\[ S_1 = h_{\alpha \beta} dt^\alpha \otimes dt^\beta + g_{ij} dx^i \otimes dx^j + h^{\alpha \beta} g_{ij} \delta x^i_\alpha \otimes \delta x^j_\beta. \]

Problems:
1) The geometry of the semi-Riemann manifold \((J^1(T,M), S_1)\), which is similar to the geometry of the tangent bundle endowed with Sasaki metric, was finalized in our research group [4]-[5], [7]-[18]. As was shown here this geometry permits the interpretation of solutions of PDE systems of order one (8) as potential maps. In this sense the solutions of every PDE of any order are extremals of a least squares Lagrangian of order one.

2) Study the geometry of the dual space of \((J^1(T,M), S_1)\).

3) Find a Sasaki-like \( S_2 \) metric on the jet bundle of order two and develop the geometry of the semi-Riemann manifold \((J^2(T,M), S_2)\). In this manifold, PDEs of Mathematical Physics (of order two) appear like hypersurfaces. Most of them are in fact algebraic hypersurfaces.

Hint. The tangent vectors
\[ D_\alpha = \frac{\partial}{\partial x^i_\alpha}, D_1 = \frac{\partial}{\partial x^i}, D^i_\alpha = \frac{\partial}{\partial x^i_\alpha}, \]
\[ D^{i\alpha} = \frac{\partial}{\partial x^i_\alpha}, D^{i\alpha}_\beta (\alpha < \beta) = \frac{\partial}{\partial x^i_\beta}, \]
determine a natural basis
\[ (D_\alpha, D_1, D^i_\alpha, D^{i\alpha}_\beta (\alpha < \beta)), \]
whose dual basis is
\[ (dt^\alpha, dx^i_\alpha, dx^i_\beta, dx^i_\beta_\alpha (\alpha < \beta)). \]

These basis are not suitable for the geometry of \((J^2(T,M)\) since they induce complicated formulas. We suggest to take frames of the form
\[ \delta = \frac{\partial}{\partial t^\alpha} + A^i_\alpha \frac{\partial}{\partial x^i} + A^i_\alpha \frac{\partial}{\partial x^i_\beta} + A^i_\alpha (i_\beta) \frac{\partial}{\partial x^i_\beta}. \]
\[
\frac{\delta}{\delta x^i} = A^i_\alpha \frac{\partial}{\partial t^\alpha} + \frac{\partial}{\partial x^i} + A_i^j (\beta_j) \frac{\partial}{\partial x^j} + A^i_\gamma (\beta_\gamma) \frac{\partial}{\partial x^\gamma}
\]
\[
\frac{\delta}{\delta x^j} = A^j_\alpha \frac{\partial}{\partial t^\alpha} + A^j_\gamma (\beta_\gamma) \frac{\partial}{\partial x^\gamma}
\]
\[
\frac{\delta}{\delta x^{\alpha\beta}} = A^{\alpha\beta}_\gamma \frac{\partial}{\partial t^\gamma} + A^{\alpha\beta}_\gamma (\beta_\gamma) \frac{\partial}{\partial x^\gamma}
\]
\[
+ A^{\alpha\beta}_\gamma (\beta_\gamma) \frac{\partial}{\partial x^{\alpha\beta}}
\]
dual to coframe of the form
\[
\delta t^\beta = dt^\beta + B^\beta_j dx^j + B^\beta_\gamma dx^\gamma + B^\beta_k dx^k
\]
\[
\delta x^i = B^i_\beta dt^\beta + dx^i + B^i_\gamma dx^\gamma + B^i_k dx^k
\]
\[
\delta x^{\alpha\beta} = B^{\alpha\beta}_\gamma dt^\gamma + B^{\alpha\beta}_k dx^k + dx^{\alpha\beta} + B^{\alpha\beta}_l dx^l + B^{\alpha\beta}_m dx^m
\]
In this context, we can define the Sasaki like metric
\[
S_2 = h_{\alpha\beta} dt^\alpha \otimes dt^\beta + g_{ij} dx^i \otimes dx^j + h_{\alpha\beta} g_{ij} dx^i \otimes dx^j
\]
\[
+ h_{\alpha\beta} g_{ij} dx^i \otimes dx^j + h_{\alpha\beta} g_{ij} dx^i \otimes dx^j
\]
4) Study the geometry of the dual space of \((J^2(T, M), S_2)\).

4 Covariant Hamilton Field Theory (Covariant Hamilton PDEs)

Let us show that the PDE systems (11) and (12) induce Hamilton PDE systems on the manifold \(J^1(T, M)\). The results are similar to those in the papers [2], [3].

Let \((T, h)\) be a semi-Riemann manifold with \(p\) dimensions, and \((M, g)\) be a semi-Riemann manifold with \(n\) dimensions. Then \((J^1(T, M), h + g + h^{-1} \circ g)\) is a semi-Riemann manifold with \(p + n + pn\) dimensions.

We denote by \(X_\alpha^i\) a \(C^\infty\) distinguished tensor field on \(T \times M\), and by \(\omega_{ij\alpha}\) the distinguished 2-form associated to the distinguished tensor field

\[
F_{ji}^\alpha = \nabla_j X_i^\alpha - g^{ij} g_{kj} \nabla_h X_k^\alpha
\]
via the metric \(g\), i.e., \(\omega = \frac{1}{2} g \circ F\). Of course \(X_\alpha^i\), \(F_{ji}^\alpha\) are distinguished globally defined objects on \(J^1(T, M)\).

Recall that on a symplectic manifold \((Q, \Omega)\) of even dimension \(q\), the Hamiltonian vector field \(X_f\) of a function \(f \in F(Q)\) is defined by

\[
X_f \Omega = df,
\]
and the Poisson bracket of \(f_1, f_2\) is defined by

\[
\{f_1, f_2\} = \Omega(X_{f_1}, X_{f_2}).
\]

The polysymplectic analogue of a function is a \(q\)-form called momentum observable. The Hamiltonian vector field \(X_{f_i}\) of such a momentum observable \(f_i\) is defined by

\[
X_{f_i} \Omega = df_i,
\]
where \(\Omega\) is the canonical \((q + 2)\)-form on the appropriate dual of \(J^1(T, M)\). Since \(\Omega\) is nondegenerate, this uniquely defines \(X_{f_i}\). The Poisson bracket of two such \(n\)-forms \(f_1, f_2\) is the \(n\)-form defined by

\[
\{f_1, f_2\} = X_{f_1} \Omega(X_{f_2} \Omega).
\]

Of course \(\{f_1, f_2\}\) is, up to the addition of exact terms, another momentum observable.

4.1 Theorem. The ultra-hyperbolic PDE system

\[
h^{\alpha\beta} x_{ij}^\alpha = g^{ij} h^{\alpha\beta} j_{jk} X_j^\alpha \nabla_h X_k^\alpha
\]

transfers in \(J^1(T, M)\) as a covariant Hamilton PDE system with respect to the Hamiltonian (momentum observable)

\[
H = \left(\frac{1}{2} h^{\alpha\beta} g_{ij} x_i^\alpha x_j^\beta - f\right) dv_h
\]

and the non-degenerate distinguished polysymplectic \((p + 2)\)-form

\[
\Omega = \Omega_\alpha \otimes dt^\alpha, \quad \Omega_\alpha = g_{ij} dx^i \wedge dx^j \wedge dv_h.
\]

Proof. Let

\[
\theta = \theta_\alpha \otimes dt^\alpha, \quad \theta_\alpha = g_{ij} x_i^\alpha dx^j \wedge dv_h
\]

be the distinguished Liouville \((p + 1)\)-form on \(J^1(T, M)\). It follows

\[
\Omega_\alpha = -d\theta_\alpha.
\]

We denote by

\[
X_H = X_H^\alpha \frac{\delta}{\delta t^\alpha}, \quad X_H^\alpha = u^{ij} \frac{\delta}{\delta x^i} + \frac{\delta u^{\beta\gamma}}{\delta t^\alpha} \frac{\partial}{\partial x^\alpha}
\]

the distinguished Hamiltonian object of the observable \(H\). Imposing

\[
X_H^\alpha \Omega_\alpha = dH,
\]

where

\[
dH = (h^{\alpha\beta} g_{ij} x_i^\alpha x_j^\beta - h^{\alpha\beta} g_{ij} X_j^\alpha \nabla_k X_i^k) \wedge dv_h,
\]
we find
\[ g_{ij}u^\alpha \delta x^j_\alpha - g_{ij} \frac{\partial u^\alpha}{\partial x^j_\alpha} dx^j = h^{\alpha \beta} g_{ij} x^j_\beta \delta x^i_\alpha \]
\[ - h^{\alpha \beta} g_{ij} X^j_\beta \nabla_k X^i_\alpha dx^k \text{ modulo } dv_h. \]
Consequently, it appears the Hamilton PDE system
\[
\begin{cases}
    u^\alpha = h^{\alpha \beta} x^j_\beta \\
    \frac{\partial u^\alpha}{\partial t^\alpha} = g^{hi} h^{\alpha \beta} g_{jk} X^j_\beta (\nabla_h X^k_\alpha)
\end{cases}
\]
(up to the addition of terms which are cancelled by the exterior multiplication with \(dv_h\)).

4.2 Theorem. The ultra-hyperbolic PDE system
\[ h^{\alpha \beta} x^i_\alpha x^j_\beta = g^{hi} h^{\alpha \beta} g_{jk} (\nabla_h X^k_\alpha) X^j_\beta + h^{\alpha \beta} F^i_\alpha x^j_\beta \]
\[ + h^{\alpha \beta} D^i_\beta X^j_\alpha \]
transfers in \( J^1(T, M) \) as a covariant Hamilton PDE system with respect to the Hamiltonian (momentum observable)
\[ H = \left( \frac{1}{2} h^{\alpha \beta} g_{ij} x^i_\alpha x^j_\beta - f \right) dv_h \]
and the non-degenerate distinguished polysymplectic \((p + 2)\)-form
\[ \Omega = \Omega_\alpha \otimes dt^\alpha, \]
\[ \Omega_\alpha = (g_{ij} dx^i \wedge dx^j + \omega_{ij} dx^i \wedge dx^j) + g_{ij} (D^i_\beta X^j_\alpha) dt^\alpha \wedge dx^j \wedge dv_h. \]

Proof. Let
\[ \theta = \theta_\alpha \otimes dt^\alpha, \quad \theta_\alpha = (g_{ij} x^i_\alpha dx^j - g_{ij} X^i_\alpha dx^j) \wedge dv_h \]
be the distinguished Liouville \((p + 1)\)-form on \( J^1(T, M) \). It follows
\[ \Omega_\alpha = -d\theta_\alpha \]
(of course the term containing \(dt^j\) disappears by exterior multiplication with \(dv_h\)). We denote by
\[ X_H = X^\beta_\alpha \frac{\delta}{\delta t^\alpha}, \quad X^\beta_\alpha = h^{\beta \gamma} \frac{\delta}{\delta t^\gamma} + u^\gamma \frac{\delta}{\delta x^\gamma} + \omega_{ij} \frac{\partial}{\partial t^\alpha} \frac{\partial}{\partial x^i_\alpha} \]
the distinguished Hamiltonian object of the observable \(H\). Imposing
\[ X^\alpha_\alpha | \Omega_\alpha = dH, \]
where
\[ dH = (h^{\alpha \beta} g_{ij} x^j_\beta \delta x^i_\alpha - h^{\alpha \beta} g_{ij} x^j_\beta (\nabla_h X^i_\alpha) dx^k) \wedge dv_h, \]
we find
\[ (g_{ij} u^\alpha \delta x^j_\alpha - g_{ij} \frac{\partial u^\alpha}{\partial x^j_\alpha} dx^j + 2 \omega_{ij} u^\alpha dx^j \]
\[ + h^{\alpha \beta} g_{ij} (D^j_\alpha X^i_\beta) dx^j) \wedge dv_h = dH. \]
Consequently, it appears the Hamilton PDE system
\[ u^\alpha = h^{\alpha \beta} x^i_\beta, \quad \frac{\partial u^\alpha}{\partial t^\alpha} = g^{hi} h^{\alpha \beta} g_{jk} X^j_\beta (\nabla_h X^k_\alpha) \]
\[ + 2 g^{hi} \omega_{ij} u^\alpha dx^j + h^{\alpha \beta} D^i_\beta X^j_\alpha \]
(up to the addition of terms which are cancelled by the exterior multiplication with \(dv_h\)).

5 Application to continuous groups of transformations
The \( C^\infty \) vector fields \( \xi_\alpha \) on the manifold \( M \) and the 1-forms \( A^\alpha \) on the manifold \( T \) satisfying
\[ [\xi_\alpha, \xi_\beta] = C^\gamma_{\alpha \beta} \xi_\gamma, \quad C^\gamma_{\alpha \beta} = \text{ constants}, \]
\[ \frac{\partial A^\alpha_\beta}{\partial t^\gamma} - \frac{\partial A^\alpha_\gamma}{\partial t^\beta} = C^\alpha_\lambda A^\lambda_\beta A^\delta_\gamma \]
determine a continuous group of transformations via the PDEs
\[ x^i_\alpha = \xi^i_\beta (x(t)) A^i_\alpha (t). \]

Conversely, if \( x^i = x^i(t^\alpha, y^j) \) are solutions of a completely integrable system of PDEs of the preceding form, where the \( A^i_\alpha \)’s and \( \xi_\gamma \)’s satisfy the conditions stated above, such that for values \( t^\alpha_0 \) of \( t^\alpha \) the determinant of the \( A^i_\alpha \)’s is not zero and
\[ x^i(t^\alpha_0, y^j) = y^j, \]
then \( x^i = x^i(t^\alpha, y^j) \) define a continuous group of transformations.

Using a semi-Riemann metric \( h \) on the manifold \( T \), a semi-Riemann metric \( g \) on the manifold \( M \), then the maps determining a continuous group of transformations appear like extremals (ultra-potential maps) of the Lagrangian
\[ L = \frac{1}{2} h^{\alpha \beta} g_{ij} (x^i_\alpha - \xi^i_\lambda A^i_\lambda)(x^j_\beta - \xi^j_\mu A^j_\mu) \sqrt{|h|}. \]

Open problem. Find the geometrical meaning of ultra-potential maps which are not transformations in the given group.
References:


