Necessary Optimality Conditions for Fractional Action-Like Problems with Intrinsic and Observer Times

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Abstract: We prove higher-order Euler-Lagrange and DuBois-Reymond stationary conditions to fractional action-like variational problems. More general fractional action-like optimal control problems are also considered.

Key–Words: calculus of variations, FALVA problems, higher-order Euler-Lagrange equations, higher-order DuBois-Reymond stationary condition, multi-time control theory.

1 Introduction

The study of fractional problems of the calculus of variations and respective Euler-Lagrange type equations is a subject of strong current research because of its numerous applications: see e.g. [1, 3, 4, 6, 7, 10, 12, 13, 14]. F. Riewe [14] obtained a version of the Euler-Lagrange equations for problems of the calculus of variations with fractional derivatives, that combines both conservative and non-conservative cases. In 2002 O. P. Agrawal proved a formulation for variational problems with right and left fractional derivatives in the Riemann-Liouville sense [1]. Then, these Euler-Lagrange equations were used by D. Baleanu and T. Avkar to investigate problems with Lagrangians which are linear on the velocities [3]. In [12] fractional problems of the calculus of variations with symmetric fractional derivatives are considered and correspondent Euler-Lagrange equations obtained, using both Lagrangian and Hamiltonian formalisms. In all the above mentioned studies, Euler-Lagrange equations depend on left and right fractional derivatives, even when the problem depend only on one type of them. In [13] problems depending on symmetric derivatives are considered for which Euler-Lagrange equations include only the derivatives that appear in the formulation of the problem. In [4, 8] Euler-Lagrange type equations for problems of the calculus of variations which depend on the Riemann-Liouville derivatives of order (α, β), $\alpha > 0$, $\beta > 0$, are studied.

In [5, 16, 17], C. Udriste and his coauthors remark that the standard multi-variable variational calculus has some limitations which the multi-time control theory successfully overcomes. For instance, the classical multi-variable variational calculus cannot be applied directly to create a multi-time maximum principle. In [6, 7] two-time Riemann-Liouville fractional integral functionals, depending on a parameter α but not on fractional-order derivatives of order α, are introduced and respective fractional Euler-Lagrange type equations obtained. In [11], Jumarie uses the variational calculus of fractional order to derive an Hamilton-Jacobi equation, and a Lagrangian variational approach to the optimal control of one-dimensional fractional dynamics with fractional cost function. Here, we extend the Euler-Lagrange equations of [6, 7] by considering two-time fractional action-like variational problems with higher-order derivatives. A DuBois-Reymond stationary condition is also proved for such problems. Finally, we study more general two-time optimal control type problems.

2 Preliminaries

In 2005, El-Nabulsi (cf. [6]) introduced the following Fractional Action-Like VAriational (FALVA) problem.

Problem 1 *Find the stationary values of the integral functional*

$$I[q(\cdot)] = \frac{1}{\Gamma(\alpha)} \int_a^t L\left(\theta, q(\theta), \dot{q}(\theta)\right) (t-\theta)^{\alpha-1} d\theta \qquad (P_1)$$

under given initial conditions $q(a) = q_a$, *where* $\dot{q} = \frac{dq}{d\theta}$, Γ *is the Euler gamma function,* $0 < \alpha \leq 1$, θ *is the intrinsic time,* t *is the observer time,* $t \neq \theta$, *and the*

Lagrangian $L : [a,b]\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ *is a* C^2*-function with respect to all its arguments.*

Along all the work, we denote by $\partial_i L$ the partial derivative of a function L with respect to its i-th argument, $i \in \mathbb{N}$.

Next theorem summarizes the main result of [6].

Theorem 2 *(cf. [6]) If $q(\cdot)$ is a solution of Problem 1, that is, $q(\cdot)$ offers a stationary value to functional (P_1), then $q(\cdot)$ satisfies the following* Euler-Lagrange equations*:*

$$\partial_2 L\left(\theta, q(\theta), \dot{q}(\theta)\right) - \frac{d}{d\theta}\partial_3 L\left(\theta, q(\theta), \dot{q}(t)\right) = \frac{1-\alpha}{t-\theta}\partial_3 L\left(\theta, q(\theta), \dot{q}(\theta)\right) . \quad (1)$$

In this work we begin by generalizing the Euler-Lagrange equations (1) for FALVA problems with higher-order derivatives.

3 Main Results

In §3.1 and §3.2 we study FALVA problems with higher-order derivatives. The results are: Euler-Lagrange equations (Theorem 5) and a DuBois-Reymond stationary condition (Theorem 10) for such problems. Then, on section §3.3, we introduce the two-time optimal control FALVA problem, obtaining more general stationary conditions (Theorems 16 and 19).

3.1 Euler-Lagrange equations for higher-order FALVA problems

We prove Euler-Lagrange equations to higher-order problems of the calculus of variations with fractional integrals of Riemann-Liouville, i.e. to FALVA problems with higher-order derivatives.

Problem 3 *The higher-order FALVA problem consists to find stationary values of an integral functional*

$$I^m[q(\cdot)] = \frac{1}{\Gamma(\alpha)}\int_a^t L\Big(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta)\Big)(t-\theta)^{\alpha-1}d\theta\,, \quad (P_m)$$

$m \geq 1$*, subject to initial conditions*

$$q^{(i)}(a) = q_a^i\,, \;\; i = 0, \ldots, m-1\,, \quad (2)$$

where $q^{(0)}(\theta) = q(\theta)$, $q^{(i)}(\theta)$ is the i-th derivative, $i \geq 1$; Γ is the Euler gamma function; $0 < \alpha \leq 1$; θ is the intrinsic time; t the observer's time, $t \neq \theta$; and the Lagrangian $L : [a,b] \times \mathbb{R}^{n\times(m+1)} \to \mathbb{R}$ is a function of class C^{2m} with respect to all the arguments.

Remark 4 *In the particular case when $m = 1$, functional (P_m) reduces to (P_1) and Problem 3 to 1.*

To establish the Euler-Lagrange stationary condition for Problem (P_m), we follow the standard steps used to derive the necessary conditions in the calculus of variations.

Let us suppose $q(\cdot)$ a solution to Problem 3. The variation $\delta I^m[q(\cdot)]$ of the integral functional (P_m) is given by

$$\frac{1}{\Gamma(\alpha)}\int_a^t \left(\sum_{i=0}^{m}\partial_{i+2}L\cdot\delta q^{(i)}\right)(t-\theta)^{\alpha-1}\,d\theta\,, \quad (3)$$

where $\delta q^{(i)} \in C^{2m}\left([a,b];\mathbb{R}^n\right)$ represents the variation of $q^{(i)}$, $i = 1, \ldots, m$, and satisfy

$$\delta q^{(i)}(a) = 0\,. \quad (4)$$

Having in account conditions (4), repeated integration by parts of each integral containing $\delta q^{(i)}$ in (3) leads to

$$m = 1: \quad \delta I[q(\cdot)] = \frac{1}{\Gamma(\alpha)}\int_a^t \left[\left(\partial_2 L - \frac{d}{d\theta}\partial_3 L\right) - \frac{1-\alpha}{t-\theta}\partial_3 L\right](t-\theta)^{\alpha-1}\cdot\delta q d\theta\,; \quad (5)$$

$$m = 2: \quad \delta I^2[q(\cdot)] = \frac{1}{\Gamma(\alpha)}\int_a^t \Bigg[\left(\partial_2 L - \frac{d}{d\theta}\partial_3 L + \frac{d^2}{d\theta^2}\partial_4 L\right)\left(\frac{1-\alpha}{t-\theta}\left(\partial_3 L - 2\frac{d}{d\theta}\partial_4 L\right) - \frac{(1-\alpha)(2-\alpha)}{(t-\theta)^2}\partial_4 L\right)\Bigg](t-\theta)^{\alpha-1}\cdot\delta q d\theta\,; \quad (6)$$

and, in general,

$$\delta I^m[q(\cdot)] = \frac{1}{\Gamma(\alpha)}\int_a^t \Bigg[\left(\partial_2 L + \sum_{i=1}^{m}(-1)^i\frac{d^i}{d\theta^i}\,\partial_{i+2}L\right) - \frac{1-\alpha}{t-\theta}\sum_{i=1}^{m} i(-1)^{i-1}\frac{d^{i-1}}{d\theta^{i-1}}\,\partial_{i+2}L - \sum_{k=2}^{m}\sum_{i=2}^{k}(-1)^{i-1}\frac{\Gamma(i-\alpha+1)}{(t-\theta)^i\Gamma(1-\alpha)}\binom{k}{k-i}\frac{d^{k-i}}{d\theta^{k-i}}\,\partial_{k+2}L\Bigg]\cdot(t-\theta)^{\alpha-1}\cdot\delta q d\theta\,.$$

The integral functional $I^m[\cdot]$ has, by hypothesis, a stationary value for $q(\cdot)$, so that

$$\delta I^m[q(\cdot)] = 0\,.$$

The fundamental lemma of the calculus of variations asserts that all the coefficients of δq must vanish.

Theorem 5 ***(higher-order Euler-Lagrange equations)*** *If $q(\cdot)$ gives a stationary value to functional (P_m), then $q(\cdot)$ satisfy the* higher-order Euler-Lagrange equations

$$\sum_{i=0}^{m}(-1)^i \frac{d^i}{d\theta^i}\partial_{i+2}L\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta)\right) = F\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(2m-1)}(\theta)\right), \quad (7)$$

where $m \geq 1$ and

$$F\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(2m-1)}(\theta)\right) = \frac{1-\alpha}{t-\theta}\sum_{i=1}^{m} i(-1)^{i-1}\frac{d^{i-1}}{d\theta^{i-1}}\partial_{i+2}L + \sum_{k=2}^{m}\sum_{i=2}^{k}(-1)^{i-1}\frac{\Gamma(i-\alpha+1)}{(t-\theta)^i\Gamma(1-\alpha)}\binom{k}{k-i}\frac{d^{k-i}}{d\theta^{k-i}}\partial_{k+2}L \quad (8)$$

with the partial derivatives of the Lagrangian L evaluated at $\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta)\right)$.

Remark 6 *Function F in* (7) *may be viewed as an external non-conservative friction force acting on the system. If $\alpha = 1$, then $F = 0$ and equation* (7) *is nothing more than the standard Euler-Lagrange equation for the classical problem of the calculus of variations with higher-order derivatives:*

$$\sum_{i=0}^{m}(-1)^i \frac{d^i}{d\theta^i}\partial_{i+2}L\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta)\right) = 0\,.$$

Remark 7 *If $m = 1$, the Euler-Lagrange equations* (7) *coincide with the Euler-Lagrange equations* (1).

Remark 8 *For $m = 2$, the Euler-Lagrange equations* (7) *reduce to*

$$\left(\partial_2 L(\theta, q, \dot{q}, \ddot{q}) - \frac{d}{d\theta}\partial_3 L(\theta, q, \dot{q}, \ddot{q}) + \frac{d^2}{d\theta^2}\partial_4 L(\theta, q, \dot{q}, \ddot{q})\right) = F(\theta, q, \dot{q}, \ddot{q}, \dddot{q}) \quad (9)$$

where

$$\begin{aligned} F(\theta, q, \dot{q}, \ddot{q}, \dddot{q}) &= \frac{1-\alpha}{t-\theta}\Big(\partial_3 L(\theta, q, \dot{q}, \ddot{q}) - 2\frac{d}{d\theta}\partial_4 L(\theta, q, \dot{q}, \ddot{q})\Big) - \frac{\Gamma(3-\alpha)}{(t-\theta)^2\Gamma(1-\alpha)}\binom{2}{0}\partial_4 L(\theta, q, \dot{q}, \ddot{q}) \\ &= \frac{1-\alpha}{t-\theta}\left(\partial_3 L(\theta, q, \dot{q}, \ddot{q}) - 2\frac{d}{d\theta}\partial_4 L(\theta, q, \dot{q}, \ddot{q})\right) - \frac{(1-\alpha)(2-\alpha)}{(t-\theta)^2}\partial_4 L(\theta, q, \dot{q}, \ddot{q})\,. \end{aligned} \quad (10)$$

Proof: Theorem 5 is proved by induction. For $m = 1$ and $m = 2$, the Euler-Lagrange equations (1) and (9)-(10) are obtained applying the fundamental lemma of the calculus of variations respectively to (5) and (6). From the induction hypothesis,

$$\sum_{i=0}^{j}(-1)^i \frac{d^i}{d\theta^i}\partial_{i+2}L\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta)\right) = F\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(2j-1)}(\theta)\right), \; m = j > 2. \quad (11)$$

We need to prove that equations (7)–(8) hold for $m = j + 1$. For simplicity, let us focus our attention on the variation of $q^{(j+1)}(\theta)$. From hypotheses (variation of $q^{(j+1)}(\theta)$ up to order $m = j$), and having in mind that $C_i^j + C_{i+1}^j = C_{i+1}^{j+1}$ and $m\Gamma(m) = \Gamma(m+1)$, we obtain equations (7) for $m = j+1$ using integration by parts followed by the application of the fundamental lemma of the calculus of variations. □

It is convenient to introduce the following quantity (cf. [15]):

$$\psi^j = \sum_{i=0}^{m-j}(-1)^i \frac{d^i}{d\theta^i}\partial_{i+j+2}L\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta)\right), \quad (12)$$

$j = 1, \ldots, m$. This notation is useful for our purposes because of the following property:

$$\frac{d}{d\theta}\psi^j = \partial_{j+1}L\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta)\right) - \psi^{j-1}, \quad (13)$$

$j = 1, \ldots, m$.

Remark 9 *Equation* (7) *can be written in the following form:*

$$\partial_2 L\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta)\right) - \frac{d}{d\theta}\psi^1 = F\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(2m-1)}(\theta)\right). \quad (14)$$

3.2 DuBois-Reymond condition for higher-order FALVA problems

We now prove a DuBois-Reymond condition for FALVA problems. The result seems to be new even for $m = 1$ (Corollary 12).

Theorem 10 ***(higher-order DuBois-Reymond condition)*** *A necessary condition for $q(\cdot)$ to be a solution to Problem 3 is given by the following higher-order*

DuBois-Reymond condition:

$$\frac{d}{d\theta}\left\{L\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta)\right) - \sum_{j=1}^{m} \psi^j \cdot q^{(j)}(\theta)\right\}$$
$$= \partial_1 L\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta)\right)$$
$$+ F\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(2m-1)}(\theta)\right) \cdot \dot{q}(\theta)\,, \quad (15)$$

where F and ψ^j are defined by (8) *and* (12)*, respectively.*

Remark 11 *If $\alpha = 1$, then $F = 0$ and condition* (15) *is reduced to the classical higher-order DuBois-Reymond condition (see e.g. [15]):*

$$\partial_1 L\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta)\right)$$
$$= \frac{d}{d\theta}\left\{L\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta)\right) - \sum_{j=1}^{m} \psi^j \cdot q^{(j)}(\theta)\right\}$$

Proof: The total derivative of

$$L\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta)\right) - \sum_{j=1}^{m} \psi^j \cdot q^{(j)}(\theta)$$

with respect to θ is:

$$\frac{d}{d\theta}\left\{L\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta)\right) - \sum_{j=1}^{m} \psi^j \cdot q^{(j)}(\theta)\right\}$$
$$= \frac{\partial L}{\partial \theta}\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta)\right)$$
$$+ \sum_{j=0}^{m} \frac{\partial L}{\partial q^{(j)}}\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta)\right) \cdot q^{(j+1)}(\theta)$$
$$- \sum_{j=1}^{m}\left(\dot{\psi}^j \cdot q^{(j)}(\theta) + \psi^j \cdot q^{(j+1)}(\theta)\right)\,. \quad (16)$$

From (13) it follows that (16) is equivalent to

$$\frac{d}{d\theta}\left\{L\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta)\right) - \sum_{j=1}^{m} \psi^j \cdot q^{(j)}(\theta)\right\}$$
$$= \frac{\partial L}{\partial \theta}\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta)\right)$$
$$+ \sum_{j=0}^{m} \frac{\partial L}{\partial q^{(j)}}\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta)\right) \cdot q^{(j+1)}(\theta)$$
$$- \sum_{j=1}^{m}\Big[\Big(\frac{\partial L}{\partial q^{(j-1)}}\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta)\right)$$
$$- \psi^{j-1}\Big) \cdot q^{(j)}(\theta) + \psi^j \cdot q^{(j+1)}(\theta)\Big]\,. \quad (17)$$

We now simplify the last term on the right-hand side of (17):

$$\sum_{j=1}^{m}\left[\left(\frac{\partial L}{\partial q^{(j-1)}} - \psi^{j-1}\right) \cdot q^{(j)}(\theta) + \psi^j \cdot q^{(j+1)}(\theta)\right]$$
$$= \sum_{j=0}^{m-1}\Big[\frac{\partial L}{\partial q^{(j)}} \cdot q^{(j+1)}(\theta) - \psi^j \cdot q^{(j+1)}(\theta)$$
$$+ \psi^{j+1} \cdot q^{(j+2)}(\theta)\Big]$$
$$= \sum_{j=0}^{m-1}\left[\frac{\partial L}{\partial q^{(j)}} \cdot q^{(j+1)}(\theta)\right] - \psi^0 \cdot \dot{q}(\theta) + \psi^m \cdot q^{(m+1)}(\theta)\,, \quad (18)$$

where the partial derivatives of the Lagrangian L are evaluated at $\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta)\right)$. Substituting (18) into (17), and using the higher-order Euler-Lagrange equations (7), we obtain the intended result, that is,

$$\frac{d}{d\theta}\left\{L\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta)\right) - \sum_{j=1}^{m} \psi^j \cdot q^{(j)}(\theta)\right\}$$
$$= \frac{\partial L}{\partial \theta}\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta)\right)$$
$$+ \frac{\partial L}{\partial q^{(m)}}\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta)\right) \cdot q^{(m+1)}(\theta)$$
$$+ \psi^0 \cdot \dot{q}(\theta) - \psi^m \cdot q^{(m+1)}(\theta)$$
$$= \partial_1 L\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta)\right)$$
$$+ F\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(2m-1)}(\theta)\right) \cdot \dot{q}(\theta)\,,$$

since, by definition,

$$\psi^m = \frac{\partial L}{\partial q^{(m)}}\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta)\right)$$

and

$$\psi^0 = \sum_{i=0}^{m}(-1)^i \frac{d^i}{d\theta^i}\partial_{i+2} L\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta)\right)\,.$$

□

Corollary 12 ***(DuBois-Reymond condition)*** *If $q(\cdot)$ is a solution of Problem 1, then the following (first-order) DuBois-Reymond condition holds:*

$$\frac{d}{d\theta}\left\{L\left(\theta, q(\theta), \dot{q}(\theta)\right) - \partial_3 L\left(\theta, q(\theta), \dot{q}(\theta)\right) \cdot \dot{q}(\theta)\right\}$$
$$= \partial_1 L\left(\theta, q(\theta), \dot{q}(\theta)\right) + \frac{1-\alpha}{t-\theta}\partial_3 L\left(\theta, q(\theta), \dot{q}(\theta)\right) \cdot \dot{q}(\theta)\,. \quad (19)$$

Proof: For $m = 1$, condition (15) is reduced to

$$\frac{d}{d\theta}\left\{L\left(\theta, q(\theta), \dot{q}(\theta)\right) - \psi^1 \cdot \dot{q}(\theta)\right\} = \partial_1 L\left(\theta, q(\theta), \dot{q}(\theta)\right) + F\left(\theta, q(\theta), \dot{q}(\theta)\right) \cdot \dot{q}(\theta)\,. \tag{20}$$

Having in mind (8) and (12), we obtain that

$$\psi^1 = \partial_3 L\left(\theta, q(\theta), \dot{q}(\theta)\right)\,, \tag{21}$$

$$F\left(\theta, q(\theta), \dot{q}(\theta)\right) = \frac{1-\alpha}{t-\theta}\partial_3 L\left(\theta, q(\theta), \dot{q}(\theta)\right)\,. \tag{22}$$

One finds the intended equality (19) by substituting the quantities (21) and (22) into (20). □

3.3 Stationary conditions for optimal control FALVA problems

Fractional optimal control problems have been studied in [2, 8, 9]. Here we obtain stationary conditions for two-time FALVA problems of optimal control. We begin by defining the problem.

Problem 13 *The two-time optimal control FALVA problem consists in finding the stationary values of the integral functional*

$$I[q(\cdot), u(\cdot)] = \frac{1}{\Gamma(\alpha)}\int_a^t L\left(\theta, q(\theta), u(\theta)\right)(t-\theta)^{\alpha-1}d\theta\,, \tag{23}$$

when subject to the control system

$$\dot{q}(\theta) = \varphi\left(\theta, q(\theta), u(\theta)\right) \tag{24}$$

and the initial condition $q(a) = q_a$. The Lagrangian $L : [a,b] \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}$ and the velocity vector $\varphi : [a,b] \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$ are assumed to be C^1 functions with respect to all their arguments. In accordance with the calculus of variations, we suppose that the control functions $u(\cdot)$ take values on an open set of $\mathbb{R}^r$.

Remark 14 *Problem 1 is a particular case of Problem 13 where $\varphi(\theta, q, u) = u$. FALVA problems of the calculus of variations with higher-order derivatives are also easily written in the optimal control form* (23)-(24). *For example, the integral functional of the second-order FALVA problem of the calculus of variations,*

$$I^2[q(\cdot)] = \frac{1}{\Gamma(\alpha)}\int_a^t L\left(\theta, q(\theta), \dot{q}(\theta), \ddot{q}(\theta)\right)(t-\theta)^{\alpha-1}d\theta\,,$$

is equivalent to

$$\frac{1}{\Gamma(\alpha)}\int_a^t L\left(\theta, q^0(\theta), q^1(\theta), u(\theta)\right)(t-\theta)^{\alpha-1}d\theta\,,$$

$$\begin{cases}\dot{q}^0(\theta) = q^1(\theta)\,,\\ \dot{q}^1(\theta) = u(\theta)\,.\end{cases}$$

We now adopt the Hamiltonian formalism. We reduce (23)-(24) to the form (P_1) by considering the augmented functional:

$$J[q(\cdot), u(\cdot), p(\cdot)] = \frac{1}{\Gamma(\alpha)}\int_a^t \left[\mathcal{H}\left(\theta, q(\theta), u(\theta), p(\theta)\right) - p(\theta)\cdot\dot{q}(\theta)\right]d\theta\,, \tag{25}$$

where the Hamiltonian $\mathcal{H}$ is defined by

$$\mathcal{H}\left(\theta, q, u, p\right) = L\left(\theta, q, u\right)(t-\theta)^{\alpha-1} + p\cdot\varphi\left(\theta, q, u\right)\,. \tag{26}$$

Definition 15 ***(Process)*** *A pair $(q(\cdot), u(\cdot))$ that satisfies the control system $\dot{q}(\theta) = \varphi\left(\theta, q(\theta), u(\theta)\right)$ and the initial condition $q(a) = q_a$ of Problem 13 is said to be a* process.

Next theorem gives the weak Pontryagin maximum principle for Problem 13.

Theorem 16 *If $(q(\cdot), u(\cdot))$ is a stationary process for Problem 13, then there exists a vectorial function $p(\cdot) \in C^1([a,b];\mathbb{R}^n)$ such that for all θ the tuple $(q(\cdot), u(\cdot), p(\cdot))$ satisfy the following conditions:*

- *the Hamiltonian system*

$$\begin{cases}\dot{q}(\theta) = \partial_4\mathcal{H}(\theta, q(\theta), u(\theta), p(\theta))\,,\\ \dot{p}(\theta) = -\partial_2\mathcal{H}(\theta, q(\theta), u(\theta), p(\theta))\,;\end{cases} \tag{27}$$

- *the stationary condition*

$$\partial_3\mathcal{H}(\theta, q(\theta), u(\theta), p(\theta)) = 0\,; \tag{28}$$

where $\mathcal{H}$ is given by (26).

Proof: We begin by remarking that the first equation in the Hamiltonian system, $\dot{q} = \partial_4\mathcal{H}$, is nothing more than the control system (24). We write the augmented functional (25) in the following form:

$$\frac{1}{\Gamma(\alpha)}\int_a^t \left[\frac{\mathcal{H} - p(\theta)\cdot\dot{q}(\theta)}{(t-\theta)^{\alpha-1}}\right](t-\theta)^{\alpha-1}d\theta\,, \tag{29}$$

where $\mathcal{H}$ is evaluated at $(\theta, q(\theta), u(\theta), p(\theta))$. Intended conditions are obtained by applying the stationary condition (1) to (29):

$$\begin{cases} \frac{d}{d\theta}\frac{\partial}{\partial \dot{q}}\left[\frac{\mathcal{H}-p\cdot\dot{q}}{(t-\theta)^{\alpha-1}}\right] = \frac{\partial}{\partial q}\left[\frac{\mathcal{H}-p\cdot\dot{q}}{(t-\theta)^{\alpha-1}}\right] - \frac{1-\alpha}{t-\theta}\frac{\partial}{\partial \dot{q}}\left[\frac{\mathcal{H}-p\cdot\dot{q}}{(t-\theta)^{\alpha-1}}\right] \\ \frac{d}{d\theta}\frac{\partial}{\partial \dot{u}}\left[\frac{\mathcal{H}-p\cdot\dot{q}}{(t-\theta)^{\alpha-1}}\right] = \frac{\partial}{\partial u}\left[\frac{\mathcal{H}-p\cdot\dot{q}}{(t-\theta)^{\alpha-1}}\right] - \frac{1-\alpha}{t-\theta}\frac{\partial}{\partial \dot{u}}\left[\frac{\mathcal{H}-p\cdot\dot{q}}{(t-\theta)^{\alpha-1}}\right] \end{cases}$$

$$\Leftrightarrow \begin{cases} -\dot{p} = \partial_2\mathcal{H} \\ 0 = \partial_3\mathcal{H} \end{cases}$$

□

Remark 17 *For FALVA problems of the calculus of variations, Theorem 16 takes the form of Theorem 5.*

Definition 18 ***(Pontryagin FALVA extremal)*** *We call any tuple $(q(\cdot), u(\cdot), p(\cdot))$ satisfying Theorem 16 a* Pontryagin FALVA extremal.

Next theorem generalizes the DuBois-Reymond condition (15) to Problem 13.

Theorem 19 *The following property holds along the Pontryagin FALVA extremals:*

$$\frac{d\mathcal{H}}{d\theta}(\theta, q(\theta), u(\theta), p(\theta)) = \partial_1\mathcal{H}(\theta, q(\theta), u(\theta), p(\theta)) \, . \tag{30}$$

Proof: Equality (30) is a simple consequence of Theorem 16. □

Remark 20 *In the classical framework, i.e. for $\alpha = 1$, the Hamiltonian $\mathcal{H}$ does not depend explicitly on θ when the Lagrangian L and the velocity vector φ are autonomous. In that case, it follows from* (30) *that the Hamiltonian $\mathcal{H}$ (interpreted as energy in mechanics) is conserved. In the FALVA setting, i.e. for $\alpha \neq 1$, this is no longer true: equality* (30) *holds but we have no conservation of energy since, by definition (cf.* (26)*), the Hamiltonian $\mathcal{H}$ is never autonomous ($\mathcal{H}$ always depend explicitly on θ for $\alpha \neq 1$, thus $\partial_1\mathcal{H} \neq 0$).*

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