# Convergence of the collocation methods for singular integro-differential equations in Lebesgue spaces

Iurie Caraus
Katholieke Universiteit Leuven,
Department of Computer Science,
Celestijnenlaan 200A, B-3001 Leuven (Heverlee)
Belgium
Iurie.Caraus@cs.kuleuven.be

Nikos E. Mastorakis,
WSEAS
A.I. Theologou 17-23
15773 Zographou,Athens
Greece
mastor@wseas.org

*Abstract:* In this article, the numerical schemes of collocation methods for an approximate solution of singular integro- differential equations with kernels of Cauchy type are explained. The equations are defined on arbitrary smooth closed contours. The theoretical background of collocations methods in Lebesgue spaces is obtained.

*key–words: singular integro-differential equations, collocation methods.*

## 1 Introduction

Singular integral equations with Cauchy kernels (SIE) and Singular Integro- differential equations with Cauchy kernels (SIDE) are used to model many problems in elasticity theory, aerodynamics, mechanics, thermoelasticity, queuing system analysis, etc. [1]-[9].

The general theory of SIE and SIDE has been widely investigated in the last decades [10]-[12].

It is known that the exact solution for SIDE is possible in some particular cases. That is why it is necessary to elaborate the approximation methods for solving SIDE connected with the corresponding theoretical background. In this article we will study the collocation methods for approximate solution of SIDE.

The problems for an approximate solution of SIDE using collocation methods were studied in [13]-[15]. The SIDE were defined on the unit circle.

However, the case in which the contour of integration is arbitrary smooth closed curve (not a unit circle), has not been throughly investigated. Transition to another contour, different from the standard one, creates many difficulties. Conformal mapping from the arbitrary smooth contour to the unit circle using the Riemann function does not solve the problem on the contrary, it complicates it.

The theoretical background of collocation methods and reduction methods for SIDE in Generalized Hölder spaces is proven in [22]-[25]. In this study the equations were defined on arbitrary smooth closed contours. Numerical examples can be found in [24], [25]. The stability of collocation methods for SIDE was proven in [25].

## 2 The main definitions and notations

Let $\Gamma$ be an arbitrary smooth closed contour bounding a simply connected region $F^+$ of the complex plane and $t = 0 \in F^+$, $F^- = C \setminus \{F^+ \cup \Gamma\}$, $C$ is the complex plane.

Let $z = \psi(w)$ be a Riemann function, mapping conformably and unambiguously the outside of unit circle $\Gamma_0 = \{|w| = 1\}$ on the domain $F^-$, so that $\psi(\infty) = \infty$ , $\psi^{(\prime)}(\infty) = 1$.

We assume that the contour belongs to the class $C(r;\mu)$ (where $r$ is a positive integer and $0 < \mu < 1$ ) if the Riemann function $\psi(w)$ is $r$ times continuously differentiable function on $\{|w|\} > 1$ and $\psi^{(r)}(w) \in H_\mu(\Gamma_0)$, on $\Gamma_0 = \{|w| = 1\}$. [21].

Let $U_n$ be the Lagrange interpolating polynomial operator constructed on the points $\{t_j\}_{j=0}^{2n}$ for any continuous function on $\Gamma$ :

$$\begin{gathered}(U_n g)(t) = \sum_{s=0}^{2n} g(t_s)\cdot l_s(t);\\ l_j(t) = \prod_{k=0,k\neq j}^{2n} \frac{t-t_k}{t_j-t_k}\left(\frac{t_j}{t}\right)^n \equiv \\ \sum_{k=-n}^{n} \Lambda_k^{(j)} t^k, \quad t\in\Gamma, \quad j=0,\dots,2n.\end{gathered} \tag{1}$$

$S$ denotes the singular integral operator (with a kernel of Cauchy type )on $\Gamma$ :

$$(S\varphi)(t) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}d\tau \quad t\in\Gamma.$$

We introduce the Riesz operators $P = \frac{1}{2}(I + S)$ and

$Q = \frac{1}{2}(I - S)$, where $I$ is an identical operator and $S$ is a singular operator.

From [20],[23] we have the following formulae:

$$(Px)^{(r)}(t) = (Px^{(r)})(t), \quad (Qx)^{(r)}(t) = (Qx^{(r)})(t) \tag{2}$$

and the relations

$$(t^{k+q})^{(r)} = \frac{(k+q)!}{(k+q-r)!} t^{k+q-r},\ k = 0, \dots, n;$$

$$(t^{-k})^{(r)} = (-1)^r \frac{(k+r-1)!}{(k-1)!} t^{-k-r},\ k = 1, \dots, n; \tag{3}$$

**Problem formulation.** In the complex space $L_p(\Gamma)(1 < p < \infty)$ with norm

$$||g||_p = \left( \frac{1}{l} \int_\Gamma |g|^p |d\tau| \right)^{\frac{1}{p}},$$

where $l$ is the length of $\Gamma$, we consider the singular integro-differential equation (SIDE)

$$(Mx \equiv) \sum_{r=0}^{\nu} [\tilde{A}_r(t) x^{(r)}(t) + \tilde{B}_r(t) \frac{1}{\pi i} \int_\Gamma \frac{x^{(r)}(\tau)}{\tau - t} d\tau + \frac{1}{2\pi i} \int_\Gamma K_r(t,\tau) \cdot x^{(r)}(\tau) d\tau] = f(t), \quad t \in \Gamma, \tag{4}$$

where $\tilde{A}_r(t)$, $\tilde{B}_r(t)$, $f(t)$ and $K_r(t,\tau)$, $(r = 0, \dots, \nu)$ are known functions on $\Gamma$; $x^{(0)}(t) = x(t)$ is an unknown function; $x^{(r)}(t) = \frac{d^r x(t)}{dt^r}$ $(r = 1, \dots, \nu)$ and $\nu$ is a positive integer.

We search for a solution of (4) in the class of functions, satisfying the conditions

$$\frac{1}{2\pi i} \int_\Gamma x(\tau) \tau^{-k-1} d\tau = 0, \quad k = 0, \dots, \nu - 1. \tag{5}$$

Using the Riesz operators we rewrite (4) in the following convenient form

$$(Mx \equiv) \sum_{r=0}^{\nu} \Big( A_r(t)(Px^{(r)})(t) + B_r(t)(Qx^{(r)})(t) + \frac{1}{2\pi i} \int_\Gamma K_r(t,\tau) x^{(r)}(\tau) d\tau \Big) = f(t), \quad t \in \Gamma, \tag{6}$$

where $A_r(t) = \tilde{A}_r(t) + \tilde{B}_r(t)$, $B_r(t) = \tilde{A}_r(t) - \tilde{B}_r(t)$, $r = 0, \dots, \nu$.
Equation (6) with conditions (5) will be denoted as "problem (6)-(5)."

We define the spaces $\overset{\circ}{W}{}_p^{(\nu)}(\Gamma) = \{g;\ \exists\, g^{(r)} \in C(\Gamma),\ r = \overline{0, \nu - 1},\ g^{(\nu)} \in L_p(\Gamma)\}$. The functions from $\overset{\circ}{W}{}_p^{(\nu)}(\Gamma)$ satisfy conditions (5). The norm in $\overset{\circ}{W}{}_p^{(\nu)}(\Gamma)$ is given by formula

$$||g||_{p,\nu} = ||g^{(\nu)}||_{L_p}.$$

$L_{p,\nu}$ denote the image of space $L_p$ under the mapping $P + t^{-\nu} Q$ with the same norm as in $L_p$. The following lemmas are used to prove convergence theorems.

**Lemma 1** *The differential operator* $\hat{D}^\nu : \overset{\circ}{W}{}_p^{(\nu)} \to L_{p,\nu}$, $(\hat{D}^\nu g)(t) = g^{(\nu)}(t)$ *is continuously invertible one and its inverse operator* $\hat{D}^{-\nu} : L_{p,\nu} \to \overset{\circ}{W}{}_p^{(\nu)}$ *is determined by the equality*

$$(\hat{D}^{-\nu} g)(t) = (\hat{N}^+ g)(t) + (\hat{N}^- g)(t),$$

$$(\hat{N}^+ g)(t) = \frac{(-1)^\nu}{2\pi i(\nu - 1)!} \int_\Gamma (Pg)(\tau)(\tau - t)^{\nu - 1} \ln(1 - \frac{t}{\tau}) d\tau,$$

$$(\hat{N}^- g)(t) = \frac{(-1)^{\nu-1}}{2\pi i(\nu - 1)!} \int_\Gamma (Qg)(\tau)(\tau - t)^{\nu - 1} \ln(1 - \frac{\tau}{t}) d\tau.$$

The proof can be found in [18].

**Lemma 2** *The operator* $\hat{B} : \overset{\circ}{W}{}_p^{(\nu)} \to L_p, \hat{B} = (P + t^\nu Q)\hat{D}^\nu$ *is continuously invertible operator and*

$$\hat{B}^{-1} = \hat{D}^{-\nu}(P + t^{-\nu} Q).$$

The proof can be found in [18].

**Auxiliary result** We will formulate one result from [17], establishing the equivalence (in terms of solvability) of problem (6)-(5) and the singular integral equation (SIE). We will use this result to prove Theorem 4 .

Using the integral representations

$$\left. \begin{aligned} \frac{d^\nu (Px)(t)}{dt^\nu} &= \frac{1}{2\pi i} \int_\Gamma \frac{\zeta(\tau)}{\tau - t} d\tau, \quad t \in F^+ \\ \frac{d^\nu (Qx)(t)}{dt^\nu} &= \frac{t^{-\nu}}{2\pi i} \int_\Gamma \frac{\zeta(\tau)}{\tau - t} d\tau, \quad t \in F^-. \end{aligned} \right\} \tag{7}$$

we reduce problem (6)-(5) to equivalent (in terms of solvability) singular integral equation (SIE) for unknown function $\zeta(t)$.

$$(\Theta\zeta \equiv) C(t)\zeta(t) + \frac{D(t)}{\pi i}\int\limits_{\Gamma}\frac{\zeta(\tau)}{\tau - t}d\tau$$

$$+\frac{1}{2\pi i}\int\limits_{\Gamma} h(t,\tau)\zeta(\tau)d\tau = f(t), \quad t \in \Gamma, \quad (8)$$

where

$$C(t) = \frac{1}{2}[A_\nu(t) + t^{-\nu}B_\nu(t)],$$

$$D(t) = \frac{1}{2}[A_\nu(t) - t^{-\nu}B_\nu(t)] \quad (9)$$

and $h(t,\tau)$ is a continuous Hölder function. The obvious formula for $h(t,\tau)$ can be found [17, (4.7)].

Note that the right-hand sides in SIDE (4) and SIE (8) coincide by virtue of condition (5). The equivalence of the existence of the solutions of SIE (8) and problem (6)-(5) is formulated in the following lemma.

**Lemma 3** *The system of SIE (8) and problem (6)-(5) are equivalent in terms of solvability. That is, for each solution $\zeta(t)$ of (8) there is a solution $x(t)$ of problem (6)-(5), determined by formulae*

$$(Px)(t) = \frac{(-1)^\nu}{2\pi i(\nu-1)!}\int\limits_{\Gamma}\zeta(\tau)[(\tau - t)^{\nu-1}\ln\left(1 - \frac{t}{\tau}\right)$$

$$+\sum_{k=1}^{\nu-1}\tilde{\alpha}_k\tau^{\nu-k-1}t^k]d\tau, \quad (10)$$

$$(Qx)(t) = \frac{(-1)^\nu}{2\pi i(\nu-1)!}\int\limits_{\Gamma}\zeta(\tau)\tau^{-\nu}[(\tau - t)^{\nu-1}\ln\left(1 - \frac{\tau}{t}\right) +$$

$$\sum_{k=1}^{\nu-2}\tilde{\beta}_k\tau^{\nu-k-1}t^k]d\tau,$$

*( $\tilde{\alpha}_k, k = 1,\ldots,\nu-1$, and $\tilde{\beta}_k, k = 0,\ldots,\nu-2$, are real numbers), and vice verso for each solution $x(t)$ of problem (6)-(5) there is a solution $\zeta(t)$*

$$\zeta(t) = \frac{d^\nu(Px)(t)}{dt^\nu} + t^\nu\frac{d^\nu(Qx)(t)}{dt^\nu},$$

*to the SIE (8).*

*Furthermore, for linear-independent solutions $\zeta(t)$ of (8) there are corresponding linear-independent solutions of problem (6)-(5) and vice verso.*

In formulae (10) $\ln\left(1 - \frac{t}{\tau}\right)$ and $\ln\left(1 - \frac{\tau}{t}\right)$ there are branches that vanish at the points $t = 0$ and $t = \infty$, respectively.

# 3 Numerical schemes of the collocation methods

The numerical schemes of collocation methods were obtained.

Now we need to find the approximate solution of problem (6)-(5) in polynomial form

$$x_n(t) = \sum_{k=0}^{n}\xi_k^{(n)}t^{k+\nu} + \sum_{k=-n}^{-1}\xi_k^{(n)}t^k, \quad t \in \Gamma, \quad (11)$$

where unknown coefficients $\xi_k^{(n)} = \xi_k$, $(k = -n,\ldots,n)$; obviously $x_n(t)$, derived from formula (11), satisfies conditions (5).

Let $R_n(t) = Mx_n(t) - f(t)$ be the residual of SIDE. The collocation methods consist in setting it equal to zero at chosen points $t_j$, $j = 0,\ldots,2n$ on $\Gamma$ and thus obtaining system linear algebraic equations for the unknown coefficients $\xi_k$, which will be determined by solving it.

$$R_n(t_j) = 0, j = 0,\ldots,2n. \quad (12)$$

Using formulae (2), (3) from (12), we obtain the following system of linear algebraical equations (SLAE) for collocation methods:

$$\sum_{r=0}^{\nu}\{A_r(t_j)\sum_{k=0}^{n}\frac{(k+\nu)!}{(k+\nu-r)!}t^{k+\nu-r}\xi_k$$

$$+B_r(t_j)\sum_{k=1}^{n}(-1)^r\frac{(k+r-1)!}{(k-1)!}$$

$$\cdot t_j^{-k-r}\cdot\xi_{-k} + \frac{1}{2\pi i}\cdot\sum_{k=0}^{n}\frac{(k+\nu)!}{(k+\nu-r)!}$$

$$\int\limits_{\Gamma}K_r(t_j,\tau)\tau^{k+\nu-r}d\tau\cdot\xi_k \quad (13)$$

$$+\sum_{k=1}^{n}(-1)^r\frac{(k+r-1)!}{(k-1)!}\cdot\frac{1}{2\pi i}$$

$$\int\limits_{\Gamma}K_r(t_j,\tau)\tau^{-k-r}d\tau\cdot\xi_{-k}\} = f(t_j),$$

where $j = 0,\ldots 2n$ are distinct points on $\Gamma$ and $A_r(t) = \tilde{A}_r(t) + \tilde{B}_r(t)$, $B_r(t) = \tilde{A}_r(t) - \tilde{B}_r(t)$, $r = 0,\ldots,\nu$.

## 4 Convergence theorem of collocation methods

The theoretical backgrounds of collocation methods methods are given in the following theorems.

**Theorem 4** *Let the following conditions be satisfied: be*

*contour* $\Gamma \in C(2,\mu), \quad 0<\mu<1;$

*the functions* $A_r(t)$ *and* $B_r(t)$ *belong to the space* $H_\alpha(\Gamma), 0<\alpha<1, \quad r=0,\ldots,\nu;$

$A_\nu(t)\neq 0\ B_\nu(t)\neq 0, \quad t\in\Gamma;$

*the index of function* $t^\nu B_\nu^{-1}(t)A_\nu(t)$ *is equal to zero;*

*the function* $K_r(t,\tau)\in H_\beta(\Gamma)\ \ 0<\beta\leq 1$ *(for both variables) and function* $f(t)\in C(\Gamma)$*;*

*operator* $M : \overset{\circ}{W}{}_p^{(\nu)} \to L_p(\Gamma)$ *is linear and invertible;*

*points* $t_j$ $(j=0,\ldots,2n)$ *form a system of Fe[redacted]ér knots on* $\Gamma$ *[16]:*

$$t_j=\psi\left[\exp\left(\frac{2\pi i}{2n+1}(j-n)\right)\right], \quad j=0,\ldots,2n, \quad i^2=-1. \tag{14}$$

*Then, beginning with numbers* $n\geq N_1,$ *the SLAE (13) of collocation methods have the unique solution* $\xi_k$ $(k=-n,\ldots,n)$*. The an approximate solution* $x_n(t)$*, constructed by formula (11), converge in the norm of space* $\overset{\circ}{W}{}_p^{(\nu)}(\Gamma)$ *as* $n\to\infty$ *to the exact solution* $x(t)$ *of problem (6)-(5). Furthermore, the following formula holds:*

$$||x-x_n||_{p,\nu}\leq O\left(\frac{1}{n^\alpha}\right)+O\left(\omega(f;\frac{1}{n})\right)+O\left(\omega^t(h;\frac{1}{n})\right). \tag{15}$$

**Proof of Theorem 4**

Using conditions (12)

$$R_n(t_j)=0,\ j=0,\ldots,2n \tag{16}$$

we obtain that (13) is equivalent to the operator equation

$$U_nMU_nx_n=U_nf, \tag{17}$$

where $M$ is the operator defined in (6).

We show that if $n\geq N_1$ large enough, then the operator $U_nMU_n$ is invertible. The operator maps from the subspace $\overset{\circ}{X}_n=\left\{t^\nu\sum_{k=0}^{n}\xi_kt^k+\sum_{k=-n}^{-1}\xi_kt^k\right\}$ (the norm defined as in $\overset{\circ}{W}{}_p^{(\nu)}(\Gamma)$) to the subspace $X_n=\sum_{k=-n}^{n}\bar{r}_kt^k$, (the norm defined as in $L_p(\Gamma)$ ).

Similarly, using formulae (7), we represent the functions $\frac{d^\nu(Px_n)(t)}{dt^\nu}$ and $\frac{d^\nu(Qx_n)(t)}{dt^\nu}$ by Cauchy type integrals with the same density $\zeta_n(t)$ :

$$\left.\begin{aligned}\frac{d^\nu(Px_n)(t)}{dt^\nu}&=\frac{1}{2\pi i}\int_\Gamma\frac{\zeta_n(\tau)}{\tau-t}d\tau, \quad t\in F^+\\ \frac{d^\nu(Qx_n)(t)}{dt^\nu}&=\frac{t^{-\nu}}{2\pi i}\int_\Gamma\frac{\zeta_n(\tau)}{\tau-t}d\tau, \quad t\in F^-.\end{aligned}\right\} \tag{18}$$

By $\Upsilon_n$ we denote the polynomial class of the form $\sum_{k=-n}^{n}\gamma_kt^k$, where $\gamma_k$ are arbitrary complex numbers.

Using formulae (2) and relations (3) we obtain from (18) that

$$\zeta_n(t)=\sum_{k=0}^{n}\frac{(k+\nu)!}{k!}t^k\xi_k+(-1)^\nu\sum_{k=1}^{n}\frac{(k+\nu-1)!}{(k-1)!}t^{-k}\xi_{-k}$$

and so $\zeta_n(t)\in\Upsilon_n$, for $t\in\Gamma$.

Using (17), (18) as well as problem (6)-(5) can be reduced to an equivalent equation (in terms of solvability)

$$U_n\Theta U_n\zeta_n=U_nf, \tag{19}$$

treated as an equation in the subspace $X_n$.

Obviously, (19) is the equation of collocation methods for SIE (8). The collocation methods were considered in [21] for SIE, where sufficient conditions for the solvability and convergence of these methods were obtained. From (18) and $\zeta_n(t)\in\Upsilon_n$ we conclude that if $\zeta_n(t)$ is the solution of equation (19), then $y_n(t)$ is the discrete solution of the system $U_nMU_nx_n=U_nf$ and vice verso.

We can determine $y_n(t)$ from the following relations

$$(Py_n)(t)=\frac{(-1)^\nu}{2\pi i(\nu-1)!}\int_\Gamma\zeta_n(\tau)\left\{(\tau-t)^{\nu-1}\ln\left(1-\frac{t}{\tau}\right)+\sum_{k=1}^{\nu-1}\tilde{\alpha}_k\tau^{\nu-k-1}t^k\right\}d\tau; \tag{20}$$

$$(Qy_n)(t)=\frac{(-1)^\nu}{2\pi i(\nu-1)!}\int_\Gamma\zeta_n(\tau)\tau^{-\nu}\left\{(\tau-t)^{\nu-1}\ln\left(1-\frac{\tau}{t}\right)+\sum_{k=1}^{\nu-2}\tilde{\beta}_k\tau^{\nu-k-1}t^k\right\}d\tau.$$

As mentioned above, function $y_n(t)$ is determined through $\zeta_n(t)$ from (20) uniquely.

It follows that if equation (19) has a unique solution $\zeta_n(t)$ in subspaces $X_n$, then the following relation is true

$$y_n(t) = x_n(t). \quad (21)$$

We will show that for (8) all the conditions of Theorem 1 for the collocation methods from [21] are satisfied.

From condition 3 of theorem 4 and formula (9), we obtain condition 3 of the theorem 1 from [21].

From equality $[C(t) - D(t)]^{-1}[C(t) + D(t)] = t^{\nu} B_{\nu}^{-1}(t) A_{\nu}(t)$ and the condition 4 of the theorem 4, we conclude that the indices of the functions $[C(t) - D(t)]^{-1}[C(t) + D(t)]$ are equal to zero. We have the condition 4 from the Theorem 1. [21]. From conditions 3),4),6), lemma 2, the invertibility of operator $\Upsilon : L_p(\Gamma) \rightarrow L_p(\Gamma)$ follows. The other conditions of theorem 4 coincide with the conditions of Theorem 1[21].

Conditions 1)-7) in theorem 4 provide the validity of all the conditions of theorem 1 in [21]. Therefore, beginning with the numbers $n \geq N_1$, the equation (19) is uniquely solvable.

The an approximate solution $\zeta_n(t) \in \Upsilon_n$ of (19) converge to the exact solution $\zeta(t)$ of SIE (8) in the norm of the space $L_p(\Gamma)$ as $n \to \infty$. Therefore, operator equation $U_n M U_n x_n = U_n f$ and SLAE (13) have unique solutions for $n \geq N_1$.

We know from Theorem 1 in [21] that the following relation holds:

$$||\zeta - \zeta_n||_{L_p} \leq O\left(\frac{1}{n^{\alpha}}\right) + O(\omega(f; \frac{1}{n})) + O\left(\omega^t(h; \frac{1}{n})\right). \quad (22)$$

From (7) and (18) we obtain

$$(Px)^{(\nu)}(t) = (P\zeta)(t) \quad \text{and} \quad (Qx)^{(\nu)}(t) = t^{-\nu} \cdot (Q\zeta)(t).$$

We therefore have that

$$(Px_n)^{(\nu)}(t) = (P\zeta_n)(t),$$
$$(Qx_n)^{(\nu)} = t^{-\nu}(Q\zeta_n)(t).$$

We proceed to get an error estimate

$$\begin{aligned} ||x - x_n||_{p,\nu} &= ||x^{(\nu)} - x_n^{(\nu)}||_{L_p} \\ &\leq ||P(\zeta - \zeta_n)||_{L_p} + ||t^{-\nu} Q(\zeta - \zeta_n)||_{L_p} \\ &\leq ||P|| \cdot ||\zeta - \zeta_n||_{L_p} + ||t^{-\nu}||_{L_p} \cdot ||Q|| \cdot ||\zeta - \zeta_n||_{L_p} \\ &\leq (||P|| + ||t^{-\nu}|| \cdot ||Q||)||\zeta - \zeta_n|| \\ &\leq (||P|| + c_1||Q||)||\zeta - \zeta_n||. \end{aligned} \quad (23)$$

Here we have used the inequality

$$||t^{-\nu}||_{L_p} = \left(\frac{1}{l}\int_{\Gamma} |t^{-\nu}|^p dt\right)^{\frac{1}{p}} = \left(\frac{1}{l}\int_{\Gamma} |t^{-p\nu}| dt\right)^{\frac{1}{p}}$$

$$\leq \left(\frac{1}{l} \cdot \frac{1}{\min\limits_{t\in\Gamma} |t|^{p\cdot\nu}} \cdot l\right)^{\frac{1}{p}}$$

$$= \left(\frac{1}{\min\limits_{t\in\Gamma} |t|^{p\cdot\nu}}\right)^{\frac{1}{p}} = c_1.$$

From the previous relation and from (23), using (22), we obtain (15).

Theorem 4 is then proven.

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