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A Note on Banach Principle for *JW*-algebras

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Abstract: - In the sequel we establish the Banach Principle for semifinite *JW*-algebras without direct summand of type I_2, which extends the recent results of Chilin and Litvinov on the Banach Principle for semifinite von Neumann algebras to the case of *JW*-algebras.

Key-Words: - von Neumann algebras, Jordan operator algebras, *JW*-algebras, Banach Principle, *-algebra of τ-measurable operators affiliated to a semifinite von Neumann algebra, Jordan algebra of τ-measurable operators affiliated to a semifinite *JW*-algebra.

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1 Introduction

Let (Ω, Σ, μ) be a probability space. Denote by $\mathcal{L} = \mathcal{L}(\Omega, \mu)$ the set of all (classes of) complex-valued measurable functions on Ω. Let τ_μ be the measure topology on $\mathcal{L}$. The classical Banach Principle (see for example [13]) can be stated as follows:

Classical Banach Principle. *Let* $(X \|\cdot\|)$ *be a Banach space, and let* $a_n : (X \|\cdot\|) \to (\mathcal{L}, \tau_\mu)$ *be a sequence of continuous linear maps. Consider the following properties:*

(I) the sequence $\{a_n(x)\}$ *converges almost everywhere (a.e.) for every* $x \in X$ *;*

(II) $\hat{a}(x)(\omega) = \sup_n |a_n(x)(\omega)| < \infty$ *a.e. for every* $x \in X$ *;*

(III) (II) holds, and the maximal operator $\hat{a} : (X \|\cdot\|) \to (\mathcal{L}, \tau_\mu)$ *is continuous at* 0*;*

(IV) the set $\{x \in X : \{a_n(x)\}$ *converges a.e.*$\}$ *is closed in* X .

Then the implications $(I) \Rightarrow (II) \Rightarrow (III) \Rightarrow (IV)$ *are always true. If in addition, there exists a dense subset* $D \subset X$ *, such that the sequence* $\{a_n(x)\}$ *converges a.e. for every* $x \in D$*, then all four conditions* $(I) - (IV)$ *above are equivalent.*

The Banach Principle above was often applied in the case $X = (L^p, \|\cdot\|_p)$, where $1 \le p < \infty$. However, in the case $p = \infty$ the uniform topology on L^∞ appears to be too strong for the classical Banach Principle to be effective in L^∞. For example, one can notice that continuous functions are not uniformly dense in L^∞.

Bellow and Jones [9], using the fact that the unit ball $L_1^\infty = \{x \in L^\infty : \|x\|_\infty \le 1\}$ is complete in τ_μ, suggested to consider the measure topology on L^∞ by replacing $(X \|\cdot\|)$ by (L_1^∞, τ_μ). Since L_1^∞ is not a linear space, geometrical complications occur, which, however, were successfully resolved in [9].

Non-commutative versions of Banach Principle for measurable operators affiliated to a semifinite von Neumann algebra were established in [14] and [18]. These results were extended to the case of semifinite *JBW*-algebras in [16] and [17], following ideas introduced in [1], [2], [4], [5], [10]. A non-commutative version of the Banach Principle for L^∞ was proposed by Chilin and Litvinov in [12]. The present notes are devoted to a presentation of an extension of the results in [12] to the case of JW-algebras without direct summand of the type I_2.

2 Preliminaries

Let *M* be a semifinite von Neumann algebra of bounded operators acting on a complex Hilbert space *H* ([11]), and let *B(H)* be the algebra of all bounded operators on *H*. A densely defined closed operator x on H is called *affiliated* to *M* if $y'z = zy'$, with $z \in M$ ([19], [21]). Denote by *P*(*M*) the complete lattice of projections in *M*. Let τ be a faithful normal semifinite trace on *M*. Denote by $e^{\perp} = \mathbf{1} - e$ the orthogonal complemented projection for the projection $e \in P(M)$. An operator *x* affiliated to M is called τ*-measurable* if $\forall \varepsilon > 0$, $\exists e \in P(M)$ with $\tau(e^{\perp}) \le \varepsilon$ such that *eH* belongs to the domain of the operator *x*. Let $L(M,\tau)$ be the set of all τ - measurable operators affiliated to *M*.

Set $V(\varepsilon,\delta) = \{x \in L(M,\tau) : \| xe \| < \delta \ \ for \ \ some \ \ e \in P(M) \ \ with \ \ \tau(e^{\perp}) < \varepsilon\}$, for arbitrary $\varepsilon > 0$ and $\delta > 0$, where $\|\cdot\|$ stands for the operator norm on *B(H)*. The topology t_τ defined on $L(M,\tau)$ by the family $\{V(\varepsilon,\delta) : \varepsilon > 0, \delta > 0\}$ of neighborhoods of zero is called the *measure topology* ([19], [21]).

Theorem 1. *$(L(M,\tau), t_\tau)$ is a complete metrizable topological *- algebra.*

Proof. *See [19], [21] for details.* □

Proposition 1. *For any $d > 0$, the sets $M_d = \{x \in M : \| x \| \le d\}$, $M_d^h = \{x \in M_d : x = x^*\}$ are t_τ-complete.*

Proof. *See [12] for details.* □

A sequence $\{y_n\} \subset L(M,\tau)$ is said to converge *almost uniformly* (a.u.) to $y \in L(M,\tau)$ if $\forall \varepsilon > 0$, $\exists e \in P(M)$ with $\tau(e^{\perp}) < \varepsilon$ such that $\| (y - y_n)e \| \to 0$.

Proposition 2. *For $\{y_n\} \subset L(M,\tau)$ the conditions*

(i) $\{y_n\}$ converges a.u. in $L(M,\tau)$;

(ii) $\forall \varepsilon > 0$, $\exists e \in P(M)$ with $\tau(e^{\perp}) < \varepsilon$ such that $\| (y_m - y_n)e \| \to 0$ as $m,n \to \infty$;

are equivalent.

Proof. *See [12] for details.* □

The following theorem is a non-commutative version of Riesz theorem ([13]).

Theorem 2. *If $\{y_n\} \subset L(M,\tau)$ and $y = t_\tau - \lim_{n\to\infty} y_n$, then $y = a.u. - \lim_{k\to\infty} y_{n_k}$ for some subsequence $\{y_{n_k}\} \subset \{y_n\}$.*

Proof. *See [21] and [14] for details.* □

Let *A* be a semifinite *JW*-subalgebra of $B(H)_{SA}$ without a direct summand of type I_2 (see [15] and [20] for definitions), *P*(*A*) be the complete lattice of projections in *A*, and τ be a faithful normal semifinite trace on A. Let *M* = *M*(*A*) be the von Neumann enveloping algebra of the Jordan algebra *A*. Then τ can be uniquely extended to a faithful normal semifinite trace on *M*, for which we will use the same symbol τ (see [3], [6] and [8]). A self adjoint operator $x \in L(M,\tau)$ is called affiliated to a *JW*-algebra *A*, if all its spectral projections belong to *A* (see [3], [6], [7] and [8]). An operator *x* affiliated to A is called τ*-measurable* if $\forall \varepsilon > 0$, $\exists e \in P(A)$ with $\tau(e^{\perp}) \le \varepsilon$ such that *eH* belongs to the domain of the operator *x*. Let $L(A,\tau)$ be the set of all τ - measurable operators affiliated to *A*.

Proposition 3. *An operator $x \in L(M,\tau)_{SA}$ is affiliated to A iff $x \in L(A,\tau)$.*

Proof. *Follows from arguments in [20].* □

Theorem 3. *$(L(A,\tau), t_\tau)$ is a complete topological Jordan subalgebra of $(L(M,\tau), t_\tau)_{SA}$.*

Proof. *A direct consequence of Theorem 1 and arguments in [20].* □

A sequence $\{y_n\} \subset L(A,\tau)$ is said to converge *bilaterally with square almost uniformly* (b.s.a.u.) to $y \in L(A,\tau)$ if $\forall \varepsilon > 0$, $\exists e \in P(A)$ with $\tau(e^{\perp}) < \varepsilon$ such that $\| e(y - y_n)^2 e \| \to 0$.

Proposition 4. *For $\{y_n\} \subset L(A,\tau) \subset L(M,\tau)_{SA}$ the conditions:*

(i) $\{y_n\}$ converges a.u. in $L(M,\tau)$;

(ii) $\forall \varepsilon > 0$, $\exists e \in P(M)$ with $\tau(e^{\perp}) < \varepsilon$ such that $\| (y_m - y_n)e \| \to 0$ as $m,n \to \infty$;

(iii) $\{y_n\}$ converges b.s.a.u. in $L(A,\tau)$;

(iv) $\forall \varepsilon > 0$, $\exists e \in P(A)$ with $\tau(e^{\perp}) < \varepsilon$ such that $\| e(y_m - y_n)^2 e \| \to 0$ as $m,n \to \infty$;

are equivalent.

Proof. *From* $\| e(y_m - y_n)^2 e \| =$

$$= \| e(y_m - y_n)(y_m - y_n)e \| =$$

$$=\|((y_m-y_n)e)^*(y_m-y_n)e\|\le$$
$$\le\|((y_m-y_n)e)^*\|\cdot\|(y_m-y_n)e\|=$$
$=\|(y_m-y_n)e\|^2$, *so we can see that b.s.a.u. fundamentalness of a sequence in a reversible JW-algebra ([15], [8]) is equivalent to a.u. fundamentalness of the same sequence in its von Neumann enveloping algebra M = M(A). Thus the statement follows from Proposition 2 above.* □

The Riesz theorem 2 above will take the following form.

Theorem 4. *If* $\{y_n\}\subset L(A,\tau)$ *and* $y=t_\tau-\lim_{n\to\infty}y_n$, *then* $y=b.s.a.u.-\lim_{k\to\infty}y_{n_k}$ *for some subsequence* $\{y_{n_k}\}\subset\{y_n\}$.

Proof. *Directly follows from Proposition 4 and Theorem 2 above.* □

3 Bilateral with square uniform equicontinuity for sequences of maps into *L(A,τ)*

Let E be an arbitrary set. If $a_n:E\to L(A,\tau)$, $x\in E$, and $b\in A$ such that $\{b(a_n(x))^2b\}\subset A$. Denote $S(\{a_n^2\},x,b)=\sup_n\|b(a_n(x))^2b\|$.
The following Lemma is valid.

Lemma 1. *Let (X, +) be a semigroup, and* $a_n:X\to L(A,\tau)$ *be a sequence of additive maps. Assume that* $\overline{x}\in X$ *is such that* $\forall\varepsilon>0$, $\exists\{x_k\}\subset X$, *and* $p\in P(A)$ *with* $\tau(p^\perp)<\varepsilon$ *, such that :*
(i) $\{a_n(\overline{x}+x_k)\}$ *converges b.s.a.u. as* $n\to\infty$, *for every* $k\in\boldsymbol{N}$ *;*
(ii) $S(\{a_n^2\},x_k,p)\to0$, *as* $k\to\infty$.
Then the sequence $\{a_n(\overline{x})\}$ *converges b.s.a.u. in* $L(A,\tau)$.

Proof. *Follows from [12] and Proposition 4.* □

Let (*X*, *t*) be a topological space, and $a_n:X\to L(A,\tau)$ and $x_0\in X$ be such that $a_n(x_0)=0$ for $n\in\boldsymbol{N}$. A sequence $\{a_n\}$ is called *bilaterally with square equicontinuous* at x_0 if $\forall\varepsilon,\delta>0$, $\exists$ a neighborhood *U* of x_0 in (*X*, *t*) such that $a_nU\subset V(\varepsilon,\delta)\cap L(A,\tau)$, $n\in\boldsymbol{N}$, i.e. $\forall x\in U$ and $\forall n\in\boldsymbol{N}$ one can find a projection $e=e(x,n)\in P(A)$ with $\tau(e^\perp)<\varepsilon$, satisfying $\|e(a_n(x))^2e\|<\delta$.

Let now $x_0\in E\subset X$. A sequence $\{a_n\}$ is called *bilaterally with square uniformly equicontinuous* at x_0 on *E*, if $\forall\varepsilon,\delta>0$, $\exists$ a neighborhood *U* of x_0 in (*X*, *t*) such that $\forall x\in E\cap U$, $\exists e=e(x)\in P(A)$ with $\tau(e^\perp)<\varepsilon$, satisfying $S(\{a_n^2\},x,e)<\delta$.

Proposition 5. *Let the sequence* $\{a_n\}$ *and* $x_0\in E\subset X$ *be as above. Then,*
(i) $\{a_n\}$ *is equicontinuous at* x_0 *on E into* $L(M,\tau)$ *iff it is bilaterally with square equicontinuous at* x_0 *on E into* $L(A,\tau)$ *;*
(ii) $\{a_n\}$ *is uniformly equicontinuous ([12]) at* x_0 *on E into* $L(M,\tau)$ *iff it is bilaterally with square uniformly equicontinuous at* x_0 *on E into* $L(A,\tau)$.

Proof. *Directly follows from Proposition 4 and arguments in [12].* □

Theorem 1 and theorem 3 established that that $(L(M,\tau),t_\tau)$ is a complete metrizable topological *-algebra, and $(L(A,\tau),t_\tau)$ is a complete metrizable topological Jordan subalgebra of $(L(M,\tau),t_\tau)_{SA}$. In [12] it has been established that for any $d>0$, the sets

$M_d=\{x\in M:\|x\|\le d\}$, and

$M_d^h=\{x\in M_d:x=x^*\}$ are t_τ-complete. It is easy to see that the set $A_d=M_d^h\cap A$ is t_τ-complete too.

Lemma 2. *Let* $d>0$. *If* $a_n:A\to L(A,\tau)$ *be a sequence of additive maps. Then it is bilaterally with square uniformly equicontinuous at* 0 *on* A_d *iff it is uniformly equicontinuous at* 0 *on* M_d *(where in the second condition we mean that all maps are extended by linearity to the sequence of additive maps* $M\to L(M,\tau)$*).*

Proof. *Directly follows from Proposition 5, and arguments in [12] and [8].* □

Lemma 3. *Let a sequence* $a_n:A\to L(A,\tau)$ *of additive maps be bilaterally with square uniformly equicontinuous at* 0 *on* A_d *for some* $0<d\in\boldsymbol{R}$. *Then* $\{a_n\}$ *is as well bilaterally with square*

uniformly equicontinuous at 0 *on* A_s *for every* $0 < s \in \boldsymbol{R}$.

Proof. *Directly follows from Lemma 2 and arguments in [12] and [8].* □

4 Main results

Let $0 \in E \subset A$. For a sequence $a_n : (A, t_\tau) \to L(A, \tau)$, consider the following conditions:

- Bilateral with square almost uniform convergence of $\{a_n(x)\}$ for every $x \in E$ (BSCNV (*E*));
- Bilateral with square uniform equicontinuity at 0 on *E* (BSCNT (*E*));
- Closedness in (E, t_τ) of the set $C(E) = \{x \in E : \{a_n(x)\}$ *converges b.s.a.u.*$\}$ (BSCLS (*E*)).

In this section we will discuss relationships among the conditions (BSCNV (A_1)), (BSCNT (A_1)), and (BSCLS (A_1)).

Theorem 5. *Let* $a_n : A \to L(A, \tau)$ *be a (BSCNV (* A_1 *)) sequence of positive* t_τ *-continuous linear maps with* $a_n(1) \le 1$, $n \in \boldsymbol{N}$. *Then the sequence* $\{a_n\}$ *is also (BSCNT (* A_1 *)).*

Proof. *Directly follows from arguments in [12] and the previous section.* □

Theorem 6. *A (BSCNT (* A_1 *)) sequence of additive maps* $a_n : A \to L(A, \tau)$ *is as well (BSCLS (* A_1 *)).*

Proof. *Directly follows from arguments in [12] and the results of the previous section.* □

Theorem 7. *Let* $a_n : A \to L(A, \tau)$ *be a sequence of positive* t_τ *-continuous linear maps such that* $a_n(1) \le 1$, $n \in \boldsymbol{N}$. *If a sequence* $\{a_n\}$ *is (BSCNV (D)) with D being* t_τ *-dense in* A_1, *the conditions (BSCNV (* A_1 *)), (BSCNV (* A_1 *)), and (BSCLS (* A_1 *)) are equivalent.*

Proof. *Directly follows from [12] and the results of the previous section.* □

5 Conclusion

Results of the present notes extend the results of [12] to the case of *JW*-algebras without direct summand of type I_2. In a new manuscript under preparation we extend these results to the case of bilateral almost uniform convergence on semifinite von Neumann algebras and semifinite *JBW*-algebras without direct summand of type I_2. This results can be further extended to obtain Stochastic Banach Principle, and then apply it to obtain some new Ergodic type theorems for Jordan algebras.

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