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The electromagnetic radiation problem in an arbitrary gravitational background vacuum

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Abstract: Electromagnetism in an arbitrary gravitational background vacuum is formulated in terms of Clifford analysis over a pseudo-Riemannian space with signature (1,3). The full electromagnetic radiation problem is solved for given smooth compact support electric and magnetic monopole charge-current density source fields. We show that it is possible to attack this problem by analytical means in four dimensional form and without invoking electromagnetic potentials. Our approach reveals that the solution for the full electromagnetic field can be expressed in terms of a fundamental solution of the Laplace-de Rham scalar wave equation, so that calculating Green's dyadic fields is superfluous. In the absence of gravity our method reproduces (i) Jefimenko's equations and (ii) an expression for the particular solution of the wave equation satisfied by the electromagnetic field. Expression (ii) is simpler than Jefimenko's result and has the additional advantage that its evaluation avoids integrating over singularities.

Key–Words: Electromagnetism, Gravity, Clifford analysis, Exterior differential forms, Curved space-time.

1 Introduction

Calculating the generation and propagation of electromagnetic waves on a curved background vacuum is a complicated problem. Such problems are usually solved in the weak-gravity and slow-motion limit by a method of successive approximations or using multipole expansions, e.g., [1], [4], [7], [16].

The aim of this paper is to show that the solution of the full electromagnetic radiation problems in a general curved space-time background can be obtained by analytical means in terms of a fundamental solution of the underlying *scalar* wave equation. We do this by modelling electromagnetism in terms of modern mathematical language and concepts. It then becomes apparent that the here considered problem, which would take a most cumbersome form when stated in the classical formulation, now allows a short and elegant solution.

We first identify and review the mathematical structures that are required to formulate electromagnetism in a vacuum containing a given arbitrary background gravity field. The latter is assumed not to be influenced by the electromagnetic radiation. We then present a new analytical solution method to calculate the electromagnetic field produced by smooth compact support electric and magnetic monopole sources in curved space-time. We show that it is possible to solve this general electromagnetic radiation problem in four-dimensional form, without assuming the customary time harmonic regime, without making the detour of invoking electromagnetic potentials and without the need for a Green's dyadic field. The expression obtained for the solution shows that it is sufficient to calculate a causal fundamental solution of the scalar wave equation in curved space-time, in order to solve the full electromagnetic radiation problem.

The solution of this problem has important applications in astronomy, for instance related to the observation of electromagnetic radiation coming from ionized in-spiraling matter near neutron stars and black holes. If the observed radiation could be linked to the orbital motion of the matter around the compact object, then this would open an enormous potential for testing effects of the General Theory of Relativity in the strong field limit, as well as for the direct measurement of the mass of the compact object. The here presented solution is useful in this context, as it provides a simple and direct way to model the radiation produced by the current density of a given moving plasma distribution.

Our result is a direct consequence of the mathematical model that we use to represent electromagnetism. The model for electromagnetism in the absence of gravity, still widely in use today in applied physics and engineering, is a virtually unchanged ver-

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sion that goes back to O. W. Heaviside, [17], [29], [27], [19], who simplified two earlier models by J. C. Maxwell, [23], [24]. Heaviside's equations are widely considered to be the correct model for (classical) electromagnetism, because they predict numerical values for the magnitudes of the field components that are in agreement with experimental values. However, Heaviside's model does not correctly represent the geometrical content of the electromagnetic field, nor all its physical invariances. Modern physical insight requires that a good mathematical model not only predicts correct magnitudes, but also correctly models the geometrical content and physical invariances of the physical phenomenon.

Moreover, the mathematical formulation used by Heaviside is not only very outdated, but also obscured all this time the intrinsic simplicity and beauty of this physical phenomenon. Responsible for this state of affairs is a vector algebra, independently created by J. W. Gibbs and Heaviside in the period 1881–1884 and used by Heaviside to build his model. This vector algebra is however quite inappropriate for describing electromagnetism and far better alternatives exist, as we will see further on.

Gibbs was influenced by H. G. Grassmann's work on graded (or exterior) algebras, while Heaviside extracted his version of vector calculus from W. R. Hamilton's quaternion algebra by splitting quaternions in a scalar and 3-component vectorial part, [2]. Slightly earlier in the period 1876–1878, W. K. Clifford introduced his eponymous algebras, [3], [10], [28], which were the result of his desire to combine earlier work published by Grassmann in 1844 on graded algebras with the discovery of the quaternions by Hamilton in 1843. Clifford called his algebras geometrical algebras, because they make it possible to formulate geometrical relationships between the geometrical objects living in a linear space, [10], [15]. Any linear space together with a (quadratic) inner product of signature (p, q) (usually identified with $R^{p,q}$) can be equipped with a Clifford algebra. All Clifford algebras are associative. Familiar examples of Clifford algebras are the complex algebra ($Cl\left(R^{0,1}\right)$), the quaternion algebra ($Cl\left(R^{0,2}\right)$) and the Pauli algebra ($Cl\left(R^{3,0}\right)$). The latter is the most appropriate and natural one to use to express geometrical ideas related to three-dimensional Euclidean space (usually identified with $R^{3,0}$). It was the need for a more appropriate algebra to describe rotations in three dimensions, that led W. E. Pauli to reinvent this Clifford algebra when he derived his (non-relativistic) equation for the electron with spin. Also, in 1928 P.A.M. Dirac reinvented the Clifford algebra $Cl\left(R^{1,3}\right)$ in deriving his eponymous equation for a relativistic spin-$1/2$ particle.

The classical Gibbs–Heaviside vector algebra falls short compared to the geometrical richness of the Pauli algebra. For instance, classical vector calculus lacks the capability to express operations such as the union or intersection of linear subspaces. Nor does it accommodate a way to represent reflections, and in particular rotations, in a coordinate-free way. It explicitly depends on a right-handed Cartesian reference frame, which makes it cumbersome to use with other coordinate systems. As a result, equations expressed in classical vector calculus are not form invariant under a change of basis. These properties make classical vector calculus a very inappropriate mathematical language to model electromagnetism in a background gravitational field. Finally, it is a non-associative algebra and only defined for three dimensions, while our universe is manifestly four-dimensional and its physical laws are generally accepted to be independent of any preferred reference system.

Nature's laws (for gravity, electromagnetism, gauge fields, etc.) are more and more understood as expressing geometrical relationships between geometrical quantities. It thus makes sense to use a more appropriate number system that is up to this task. Once one is willing to give attention to these requirements, by changing to a mathematical model that is also correct in this broader sense, fascinating new progress becomes possible. We use here the Clifford algebra $Cl\left(R^{1,3}\right)$ and the thereupon based Clifford analysis over a pseudo-Riemannian space with signature $(1, 3)$ (the standard reference on Clifford analysis over Euclidean spaces is [5]). This allows us to model the above mentioned electromagnetic radiation problem by a single and simple equation. This equation is equivalent to Heaviside's equations in the narrow sense that both models produce the same field component magnitudes (in the absence of gravity).

We use natural units in our model for electromagnetism. This has the advantage that all superfluous unit conversion factors disappear from the equation, which so acquires its most simple form. Natural units are not unique, but all such unit systems are equivalent in the sense that they all result in the same equation. The use of a natural unit system reveals that there are no fundamental physical constants associated with electromagnetism. A physical constant is regarded as being fundamental iff it is a dimensionless quantity and different from 0 and 1. A convenient natural unit system for electromagnetism (and gravity) is obtained by defining the *dimensionless* constants $c \triangleq 1$ (c: "speed of light"), $8\pi G \triangleq 1$ (G: "gravitational constant") and $4\pi\varepsilon_0 \triangleq 1$ (ε_0: "permittivity of the vacuum").

2 Mathematical preliminaries

A finite subset of consecutive integers will be denoted by $\mathbf{Z}_{[i_1,i_2]} \triangleq \{i \in \mathbf{Z} : i_1 \leq i \leq i_2\}$. Let M designate a real, connected, non-compact, oriented, smooth (i.e., C^∞) differential, (paracompact and Hausdorff) manifold, [6].

2.1 Contravariant tensor fields on M

2.1.1 Contravariant tensors at a point

At any point $x \in M$, we consider the *tangent space at* x, T_xM, which is a linear space over $\mathbf{R}$ of some dimension $n \in \mathbf{Z}_+$ and whose elements are called contravariant (tangent) vectors at x.

Denote further by $\wedge^k T_xM$, with $0 \leq k \leq n$, the linear space over $\mathbf{R}$, of totally antisymmetric contravariant tensors of order k at x, having dimension $\binom{n}{k}$. Elements of $\wedge^k T_xM$ are usually called in the Clifford algebra literature *(contravariant) k-vectors* and the order k of a k-vector is there called its grade. In particular, contravariant 0-vectors are by definition identified with the base field $\mathbf{R}$ and contravariant 1-vectors are identified with elements of the tangent space, i.e., $\wedge^1 T_xM \cong T_xM$.

With respect to the natural (or coordinate) basis $B_x \triangleq \left\{\partial_\mu, \forall\mu \in \mathbf{Z}_{[1,n]}\right\}$ for T_xM, induced by a choice of local coordinates $\left\{x^\mu, \forall\mu \in \mathbf{Z}_{[1,n]}\right\}$ on M, any contravariant vector u has the representative $u = u^\mu \partial_\mu$ with components $\{u^\mu\}$. We use throughout the implicit Einstein summation convention over pairs of corresponding covariant and contravariant indices. Any basis B_x for T_xM induces a basis $\wedge^k B_x \triangleq \left\{\partial_{\mu_1} \wedge \ldots \wedge \partial_{\mu_k}, \forall\mu_1 < \ldots < \mu_k \in \mathbf{Z}_{[1,n]}\right\}$ for $\wedge^k T_xM$, $\forall k \in \mathbf{Z}_{[2,n]}$. Any k-vector $t \in \wedge^k T_xM$ has, with respect to $\wedge^k B_x$, the representative $t = t^{\mu_1 \cdots \mu_k}\left(\partial_{\mu_1} \wedge \ldots \wedge \partial_{\mu_k}\right)$ with strict components $\left\{t^{\mu_1 \cdots \mu_k}, \forall\mu_1 < \ldots < \mu_k \in \mathbf{Z}_{[1,n]}\right\}$. An equivalent expression for t is the expansion

$$t = \frac{1}{k!} t^{\mu_1 \cdots \mu_k}\left(\partial_{\mu_1} \wedge \ldots \wedge \partial_{\mu_k}\right), \tag{1}$$

in terms of non-strict (i.e., unordered) indices $\mu_1, \ldots, \mu_k$.

2.1.2 Contravariant inner product

We now assume that our manifold M admits a bilinear (generalized) *contravariant inner product* $\cdot : T_xM \times T_xM \to \mathbf{R}$, defined by a symmetric, 2-covariant, non-degenerate (i.e., of maximal rank), indefinite, inner product (so called "metric") smooth tensor field g such that $(u, v) \mapsto u \cdot v = g_x(u, v)$, with g_x the tensor obtained by evaluating g at x. This makes the structure (M, g) a smooth, pseudo-Riemannian manifold. Since we assumed that our manifold M is paracompact and non-compact, it always admits a hyperbolic structure, [6, p. 293].

With respect to natural bases, $u \cdot v = g_{\mu\nu}(x)\, u^\mu v^\nu$. In particular, $\partial_\mu \cdot \partial_\nu = g_{\mu\nu}(x)$. Then, the image of a contravariant vector, with representative $u = u^\mu \partial_\mu$, under the canonical isomorphism from $T_xM \to T_x^*M$ such that $u \mapsto u^*$ (see further), has the representative $u^* = u^*_\mu dx^\mu$, with $u^*_\mu = g_{\mu\nu}(x)\, u^\nu$.

The commutative inner product of any pair of contravariant k-vectors, with respect to a basis for $\wedge^k T_xM$, is defined by

$$\begin{aligned}(a, b) \quad \mapsto \quad & a \cdot b = \frac{1}{k!} a^{\mu_1 \cdots \mu_k} b^{\nu_1 \cdots \nu_k} \\ & g_{\mu_1\nu_1}(x) \ldots g_{\mu_k\nu_k}(x).\end{aligned} \tag{2}$$

2.1.3 Contravariant tensor fields

The manifold M together with the set of linear spaces $\wedge^k T_xM$, $\forall x \in M$, can be given the structure of a linear bundle, denoted $\wedge^k TM$ and called the *k-th exterior power of the tangent bundle of* M. Any section of $\wedge^k TM$ is called a totally antisymmetric contravariant tensor field of order (or grade) k on M, or in short a *contravariant k-vector field*. We will denote the set of contravariant k-vector fields by $\Gamma\left(\wedge^k TM\right)$.

In particular, contravariant 0-vector fields are by definition identified with scalar functions from $M \to \mathbf{R}$, called scalar fields and contravariant 1-vector fields are identified with sections of the tangent bundle, i.e., contravariant vector fields.

The manifold M together with the set of bases for $\wedge^k T_xM$, $\forall x \in M$, can also be given the structure of a linear bundle, called the *frame bundle for* $\wedge^k TM$. Any section of this frame bundle is called a contravariant frame field of order (or grade) k on M, denoted $\wedge^k B$, or in short a *(moving) contravariant k-frame*. Any contravariant k-vector field has a representative with respect to any contravariant k-frame.

2.1.4 Signature of M

At any $x \in M$, we can always choose local coordinates on M such that the tensor field g at that point, g_x, becomes $g_x \triangleq [g_{\mu\nu}(x)] = [\eta_{\mu\nu}]$ with η the following diagonal tensor with components in matrix form given by

$$[\eta_{\mu\nu}] \triangleq diag\left[\underbrace{+1, +1, ..., +1}_{p \text{ times}}, \underbrace{-1, -1, ..., -1}_{q \text{ times}}\right], \tag{3}$$

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and where $n = p + q$. The couple (p, q) is called the *signature* of the pseudo-Riemannian manifold M (and is independent of x).

When $p > 0$ and $q > 0$, g_x is indefinite and M is said to be a pseudo-Riemannian manifold with p time dimensions and q space dimensions. If $q = 0$, g_x is positive definite and M is called a Riemannian manifold. If a pseudo-Riemannian manifold has zero curvature (i.e., is flat), a global coordinate system can be found on M such that the tensor field g takes the constant diagonal form (3) everywhere. Flat pseudo-Riemannian manifolds for which $p = 1$ are called Lorentzian (or hyperbolic) manifolds and the particular case $p = 1$ and $q = 3$ is called Minkowski space.

In practice, g will represent a gravitational field present on M and/or will be induced by a particular choice of local coordinates, used to chart M in the vicinity of x. We will refer hereafter to a general pseudo-Riemannian manifold with signature $(1, 3)$ as curved time-space (we reserve the term curved space-time to a pseudo-Riemannian manifold with signature $(3, 1)$).

2.2 Covariant tensor fields on M

For a fixed contravariant vector $u \in T_xM$, the map $g_x(u, .) : T_xM \to \mathbf{R}$ such that $v \mapsto g_x(u, v)$, defines a canonical isomorphism between the tangent space T_xM at x, and its dual T_x^*M, $\forall x \in M$. This canonical isomorphism is then the map from $T_xM \to T_x^*M$ such that $u \mapsto u^* \triangleq g_x(u, .)$, so u^* is the covariant vector corresponding to the contravariant vector u under this isomorphism. This enables us to define a bilinear binary function $\langle ., . \rangle_x : T_x^*M \times T_xM \to \mathbf{R}$ such that $(u^*, v) \mapsto \langle u^*, v \rangle_x \triangleq g_x(u, v)$. The canonical isomorphism from $T_xM \to T_x^*M$ extends to higher tensor spaces such as $\otimes^k T_xM \otimes^l T_x^*M$, consisting of k-contravariant and l-covariant tensors of order $k + l$, and allows us to "raise or lower the indices".

2.2.1 Covariant tensors at a point

The dual T_x^*M is also a linear space over $\mathbf{R}$, of the same dimension as T_xM, called the *cotangent space at* x, and its elements are called *covariant (cotangent) vectors* at x.

Denote further by $\wedge^k T_x^*M$, with $0 \leq k \leq n$, the space of totally antisymmetric covariant tensors of order k at x. Elements of $\wedge^k T_x^*M$ are sometimes called in the Clifford algebra literature *covariant k-vectors*. In particular, covariant 0-vectors are again identified with the base field $\mathbf{R}$ and covariant 1-vectors with elements of the cotangent space, i.e., $\wedge^1 T_x^*M \cong T_x^*M$.

With respect to the natural cobasis $B_x^* \triangleq \{dx^\mu, \forall \mu \in \mathbf{Z}_{[1,n]}\}$ for T_x^*M, naturally induced by the basis B_x for T_xM by defining $\langle dx^\mu, \partial_\nu \rangle_x = \delta^\mu_\nu$, any covariant vector u^* has the representative $u^* = u^*_\mu dx^\mu$ with components $\{u^*_\mu\}$. Any basis B_x^* for T_x^*M induces a basis $\wedge^k B_x^* \triangleq \{dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k}, \forall \mu_1 < \ldots < \mu_k \in \mathbf{Z}_{[1,n]}\}$ for $\wedge^k T_x^*M$, $\forall k \in \mathbf{Z}_{[2,n]}$. Any $t^* \in \wedge^k T_x^*M$ has, with respect to $\wedge^k B_x^*$, the representative $t^* = t_{\mu_1 \ldots \mu_k} (dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k})$ with strict components $\{t_{\mu_1 \ldots \mu_k}, \forall \mu_1 < \ldots < \mu_k \in \mathbf{Z}_{[1,n]}\}$. An equivalent expression for t^* is the expansion

$$t^* = \frac{1}{k!} t_{\mu_1 \cdots \mu_k} (dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k}), \qquad (4)$$

in terms of non-strict indices.

2.2.2 Covariant inner product

The canonical isomorphism from $T_xM \to T_x^*M$, $\forall x \in M$, induced by g, together with the non-degeneracy of g, enables us to define also an inner product on T_x^*M, called *covariant inner product*, by $\cdot : T_x^*M \times T_x^*M \to \mathbf{R}$ such that $(u^*, v^*) \mapsto u^* \cdot v^* = g_x^{-1}(u^*, v^*) \triangleq g_x(u, v)$.

With respect to natural bases, $u^* \cdot v^* = g^{\mu\nu}(x) u^*_\mu v^*_\nu$, with the $n \times n$ matrix $[g^{\mu\nu}(x)] \triangleq [g_{\mu\nu}(x)]^{-1}$. The non-degeneracy condition on g ensures that, $\forall x \in M$,

$$\begin{aligned} \det [g_x] &\triangleq \det [g_{\mu\nu}(x)], \\ &= \delta^{\mu_1 \ldots \mu_n}_{1 \ldots n} g_{\mu_1 1}(x) \ldots g_{\mu_n n}(x) \neq 0, \end{aligned} \qquad (5)$$

with δ the generalized Kronecker tensor, [6, p. 142]. We will use the same product symbol $\cdot$ for the inner product on both the tangent and cotangent spaces, as the distinction will be clear from the context. In particular, $dx^\mu \cdot dx^\nu = g^{\mu\nu}(x)$. Then, the image of a covariant vector, with representative $u^* = u^*_\mu dx^\mu$, under the inverse canonical isomorphism from $T_x^*M \to T_xM$ such that $u^* \mapsto u$, has the representative $u = u^\mu \partial_\mu$, with $u^\mu = g^{\mu\nu}(x) u^*_\nu$.

The commutative inner product of any pair of covariant k-vectors, with respect to a basis for $\wedge^k T_x^*M$, is defined by

$$\begin{aligned} (\alpha, \beta) \mapsto \alpha \cdot \beta = &\frac{1}{k!} \alpha_{\mu_1 \ldots \mu_k} \beta_{\nu_1 \ldots \nu_k} \\ &g^{\mu_1 \nu_1}(x) \ldots g^{\mu_k \nu_k}(x). \end{aligned} \qquad (6)$$

2.2.3 Covariant tensor fields

The manifold M together with the set of linear spaces $\wedge^k T_x^*M$, $\forall x \in M$, can be given the structure of a linear bundle, denoted $\wedge^k T^*M$ and called the *k-th exterior power of the cotangent bundle of* M. Any section

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of the linear bundle $\wedge^k T^*M$ is called a totally antisymmetric covariant tensor field of order (or grade) k on M, or in short a *covariant k-vector field*. In the mathematical literature, a covariant k-vector field is usually called a *k-form*. We will denote the set of k-forms by $\Gamma\left(\wedge^k T^*M\right)$.

In particular, covariant 0-vector fields are by definition also identified with scalar functions from $M \to \mathbf{R}$ and covariant 1-vector fields are identified with sections of the cotangent bundle, i.e., covariant vector fields.

The manifold M together with the set of bases for $\wedge^k T_x^* M$, $\forall x \in M$, can also be given the structure of a linear bundle, called the *frame bundle for $\wedge^k T^*M$*. Any section of this frame bundle is called a covariant frame field of order (or grade) k on M, denoted $\wedge^k B^*$, or in short a *(moving) covariant k-frame*. Any covariant k-vector field has a representative with respect to any covariant k-frame.

2.3 Exterior differential forms on M

Let $0 \leq k \leq n$. A *totally antisymmetric covariant tensor field of order k* is called a *k-form of grade k*.

Let $\mathcal{F}_M \triangleq (C^\infty(M, \mathbf{R}), +,)$ denote the unital ring of *smooth real functions* defined on M, $C^\infty(M, \mathbf{R})$, together with function pointwise addition $+$ and function pointwise multiplication (denoted by juxtaposition).

Let $\mathcal{D}'_M \triangleq \left(\mathcal{D}'_+(M), +, \star\right)$ denote the integral domain of *distributions* based on M, with support in a closed forward (or causal) null (or light) cone (assuming $p > 0$ and $q > 0$), together with distributional addition $+$ and distributional convolution $\star$.

Hereafter, the generic ring $\mathcal{R}$ stands for either $\mathcal{F}_M$ or $\mathcal{D}'_M$.

The set of k-forms $\Gamma\left(\wedge^k T^*M\right)$, $\forall k \in \mathbf{Z}_{[0,n]}$, together with $\mathcal{R}$ and a left external operation from $\mathcal{R} \times \Gamma\left(\wedge^k T^*M\right) \to \Gamma\left(\wedge^k T^*M\right)$, is a left module. The elements of this structure are called *left k-forms over $\mathcal{R}$*. We can equally consider the set of k-forms $\Gamma\left(\wedge^k T^*M\right)$, $\forall k \in \mathbf{Z}_{[0,n]}$, together with $\mathcal{R}$ and a right external operation from $\Gamma\left(\wedge^k T^*M\right) \times \mathcal{R} \to \Gamma\left(\wedge^k T^*M\right)$, which is a right module. The elements of this structure are called *right k-forms over $\mathcal{R}$*. From now on, we will use the module of *k-forms over $\mathcal{R}$*, which is both a left and right module over $\mathcal{R}$, and this will be denoted by $\left(\Gamma\left(\wedge^k T^*M\right), \mathcal{R}\right)$.

The module of k-forms over $\mathcal{R}$ is further equipped with the following useful operations. The resulting structure is then called the exterior algebra of differential forms (with inner product) on M.

2.3.1 The exterior product

The *exterior product* is a bilinear map $\wedge : \Gamma\left(\wedge^l T^*M\right) \times \Gamma\left(\wedge^k T^*M\right) \to \Gamma\left(\wedge^{l+K} T^*M\right)$, which is just the antisymmetric tensor product $\alpha \wedge \beta \triangleq \alpha \otimes \beta - \beta \otimes \alpha$. With respect to natural covariant frames, the wedge product of any l-form α and any k-form β is given by

$$\begin{aligned}\alpha \wedge \beta &= \frac{1}{(l+k)!}\frac{1}{l!}\frac{1}{k!}\delta^{\kappa_1\ldots\kappa_l\lambda_1\ldots\lambda_k}_{\mu_1\ldots\mu_{l+k}}\alpha_{\kappa_1\ldots\kappa_l} \\ &\quad \beta_{\lambda_1\ldots\lambda_k}\left(dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_{l+k}}\right), \quad (7)\end{aligned}$$

wherein δ stands for the generalized Kronecker tensor, [6, p. 142]. The exterior product inherits distributivity with respect to addition from the tensor product.

If $l = 0$ (or $k = 0$), the tensor product of a scalar field α with any k-form β (or any l-form α with a scalar field β) is defined to equal the external product of the module of k-forms (or l-forms). Hence, the exterior product, being the antisymmetrization, is zero (notice that this requires that k-forms form both a left module and a right module over $\mathcal{R}$). If $l + k > n$, the wedge product is defined to be zero, since there are no totally antisymmetric tensor fields of order greater than the dimension of the manifold.

The exterior product is associative, but generally not commutative since

$$\alpha \wedge \beta = (-1)^{kl}\, \beta \wedge \alpha. \quad (8)$$

2.3.2 Hodge's left covariant star operator

This is a very practical grade mapping operator. *Hodge's left covariant star operator* is a linear map $* : \Gamma\left(\wedge^k T^*M\right) \to \Gamma\left(\wedge^{n-k} T^*M\right)$ such that $\beta \mapsto *\beta$ and is defined by

$$\begin{aligned}*1 &= \omega, & (9)\\ \alpha \wedge (*\beta) &= (*\alpha) \wedge \beta = (\alpha \cdot \beta)\, \omega, & (10)\end{aligned}$$

$\forall \alpha, \beta \in \Gamma\left(\wedge^k T^*M\right)$ with $k > 0$. The inner product $\cdot$ in (10) is given by (6) and ω is the oriented volume n-(pseudo)form on M, [6, p. 294],

$$\omega \triangleq \sqrt{|\det[g]|}\left(dx^1 \wedge \ldots \wedge dx^n\right), \quad (11)$$

and equals the Levi-Civita pseudotensor ϵ. With respect to natural covariant frames,

$$\begin{aligned}*\beta &= \frac{1}{(n-k)!}\frac{1}{k!}\epsilon_{\mu_1\ldots\mu_k\mu_{k+1}\ldots\mu_n} \\ &\quad g^{\mu_1\nu_1}\ldots g^{\mu_k\nu_k} \\ &\quad \beta_{\nu_1\ldots\nu_k}\left(dx^{\mu_{k+1}}\wedge\ldots\wedge dx^{\mu_n}\right), \quad (12)\end{aligned}$$

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The inverse star operator, acting on any k-form, is given by

$$*^{-1} = (-1)^{k(n-k)+q} *, \tag{13}$$

with q the number of space dimensions (see (3), the signature (p, q) of M).

The presence of the Levi-Civita pseudotensor ϵ in (12) makes that $*\beta$ has the opposite parity of β. For instance, if β is an ordinary (even) form, then $*\beta$ is an odd form (also called a twisted form). Contrary to an even form, an odd form, transforming under a change of basis with Jacobian determinant J, picks up an extra factor $\operatorname{sgn}(J)$, see [6, p. 294].

To define Hodge's star operator requires that an inner product structure is given on M. We call the map $*$ local since it depends on ω, which in turns depends on $\sqrt{|\det[g_x]|}$ at $x \in M$. This is the place where gravity enters our mathematical formulation, since the inner product structure g plays the role of gravitational potential tensor in the framework of Einstein's General Theory of Relativity. On a flat manifold M, gravity is absent, but we can still have a non-trivial inner product structure g, defined by the local coordinate system used on M. Hence, formulating electromagnetism in terms of Hodge's star operator ensures that the resulting model will be valid in any gravitational environment and for any system of local manifold coordinates.

2.3.3 The interior product

We can use Hodge's star operator to define an interior product, based on the exterior product $\wedge$.

The *interior product* is a bilinear map $\cdot$: $\Gamma\left(\wedge^l T^*M\right) \times \Gamma\left(\wedge^k T^*M\right) \to \Gamma\left(\wedge^{|k-l|} T^*M\right)$ such that $(\alpha, \beta) \mapsto \alpha \cdot \beta$ with, for $0 \le l \le k$,

$$\begin{aligned}\alpha \cdot \beta &\triangleq (-1)^{l(k-l)} *^{-1} (\alpha \wedge (*\beta)), \\ \beta \cdot \alpha &\triangleq (-1)^{l(k+1)} \alpha \cdot \beta.\end{aligned} \tag{14}$$

With respect to natural covariant frames, (14) becomes

$$\begin{aligned}&\alpha \cdot \beta \\ =\ &\frac{1}{(k-l)!}\frac{1}{l!} g^{\kappa_1\tau_1} \ldots g^{\kappa_l\tau_l} \alpha_{\tau_1\ldots\tau_l} \beta_{\kappa_1\ldots\kappa_l\nu_{l+1}\ldots\nu_k} \\ &\left(dx^{\nu_{l+1}} \wedge \ldots \wedge dx^{\nu_k}\right),\end{aligned} \tag{15}$$

so the interior product is just the tensor contraction product.

If $l = 0$, we get $\alpha \cdot \beta = *^{-1}(\alpha \wedge (*\beta)) = 0$, since the exterior product of a scalar function with a k-form is zero. For $l = k$, (15) coincides with the inner product (6). For $l < k$, $\alpha \cdot \beta$ is a form of the same parity of β. The interior product is not associative and generally not commutative.

The interior product, as we have defined it here, is not part of the classical algebra of exterior differential forms. The latter is usually supplemented with a left interior product between a contravariant vector field and any k-form as a bilinear map i_a : $\Gamma(TM) \times \Gamma\left(\wedge^k T^*M\right) \to \Gamma\left(\wedge^{k-1} T^*M\right)$, see e.g., [6]. The operation i_a can be defined in general, i.e., even if M is not equipped with an inner product structure. However in this work, we have assumed the existence of an inner product structure g on M. This structure allows us to naturally define i_a as an inner product between the 1-form α, obtained from a under the canonical isomorphism from $TM \to T^*M$, and any k-form β. With the product (14), we just generalize the classical operation i_a on a pseudo-Riemannian manifold to the full tensor contraction product.

2.3.4 The exterior derivative

The *left exterior derivative* operator is a linear map $d : \Gamma\left(\wedge^k T^*M\right) \to \Gamma\left(\wedge^{k+1} T^*M\right)$ such that $\alpha \mapsto d\alpha$ with its action on any k-form given, with respect to natural covariant frames, by

$$d\alpha = \left(\partial_{\mu_1}\alpha\right) dx^{\mu_1}, k = 0, \tag{16}$$

$$\begin{aligned}d\alpha &= \frac{1}{(k+1)!}\frac{1}{k!}\delta^{\nu\nu_1\ldots\nu_k}_{\mu_1\mu_2\ldots\mu_{k+1}} \left(\partial_\nu \alpha_{\nu_1\ldots\nu_k}\right) \\ &\left(dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{k+1}}\right), k > 0.\end{aligned} \tag{17}$$

When acting on any 0-form f, df is defined by (16) to coincide with the ordinary differential of the scalar function f. Due to antisymmetry is $d \circ d = 0$, $\forall k \in \mathbf{Z}_{[0,n]}$. The exterior derivative d does not depend of the coordinate system. Further, $\forall \alpha \in \Gamma\left(\wedge^l T^*M\right)$ and $\forall \beta \in \Gamma\left(\wedge^k T^*M\right)$ holds that

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^l \alpha \wedge (d\beta), \tag{18}$$

so d is an antiderivation with respect to $\wedge$. The kernel of d consists of the closed forms (i.e., forms α for which $d\alpha = 0$), and its image are the exact k-forms, $1 \le k \le n$, (i.e., having the form $d\alpha$).

Any pseudo-Riemannian manifold (M, g) has a unique torsion free, metric connection, called the Riemannian connection, [6, p. 308]. In terms of this connection, the directional covariant derivative ∇_u, in the direction of the contravariant vector field u, is determined, with respect to natural frames, by the Christoffel symbols. We will rewrite (17) in terms of the covariant derivatives along frame fields, $\nabla_\mu \triangleq \nabla_{\partial_\mu}$. Due to the antisymmetry of $d\alpha$ (for $k > 0$) and since

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the Riemannian connection is torsion free, the connection terms (involving the Christoffel symbols) in (20) cancel out, so we get

$$d\alpha = \left(\nabla_{\mu_1}\alpha\right) dx^{\mu_1}, k = 0, \tag{19}$$

$$d\alpha = \frac{1}{(k+1)!}\frac{1}{k!}\delta^{\nu\nu_1\ldots\nu_k}_{\mu_1\mu_2\ldots\mu_{k+1}}\left(\nabla_\nu\alpha_{\nu_1\ldots\nu_k}\right)\left(dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_{k+1}}\right), k > 0. \tag{20}$$

Although $d\alpha$ does not depend on the inner product structure g and despite the fact that (17) is simpler than (20), we will find it convenient to add the extra connection terms in order to cast our results in Clifford algebra form later.

Define a 1-form ∂ by

$$\partial \triangleq dx^\mu \nabla_\mu. \tag{21}$$

The operator ∂, defined in (21), generalizes the ∂ operator encountered in Clifford analysis over Euclidean spaces (and which is there called Dirac operator, [5]) to the setting of contravariant and covariant k-vector fields on pseudo-Riemannian manifolds. When ∂ is applied to differential k-forms over $\mathcal{D}'_M$, the generalized partial derivative D_μ replaces the ordinary partial derivative ∂_μ in ∇_μ.

We can now write (20), for $1 \leq k \leq n$, in terms of the exterior product (7) as

$$d\alpha = \partial \wedge \alpha, \tag{22}$$

Thus, the exterior derivative operator d is at the same time an analytical directional covariant derivative on the components of α and an algebraic wedge operator on the natural covariant frame of α.

When acting on any n-form ω, $d\omega = \partial \wedge \omega = 0$, because $d\omega$ has grade $n+1$.

The operation $d = \partial\wedge$ appropriately defines and generalizes the curl operation (for $1 \leq k \leq n$), defined in classical vector calculus, to totally antisymmetric covariant tensor fields on pseudo-Riemannian manifolds.

2.3.5 The interior derivative

We can use Hodge's left star operator and its inverse to define the *left interior derivative* operator (also called codifferential) $\delta : \Gamma\left(\wedge^k T^*M\right) \to \Gamma\left(\wedge^{k-1}T^*M\right)$, from the left exterior differential d, such that

$$\alpha \mapsto \delta\alpha = -(-1)^k *^{-1} d * \alpha. \tag{23}$$

Our definition (23) differs by an extra minus sign from the standard definition in the theory of exterior differential forms, in order to let our results agree with standard conventions in Clifford analysis.

When acting on any 0-form f, $\delta f = -(-1)^k *^{-1} d(*f) = 0$, because $d(*f)$ has grade $n+1$. Due to antisymmetry, $\delta \circ \delta = 0$, $\forall k \in \mathbf{Z}_{[0,n]}$. Since Hodge's left star operator is applied twice in (23), δ is independent of any chosen orientation on M.

With respect to natural covariant frames, we get

$$\alpha \mapsto \delta\alpha = \frac{1}{(k-1)!}\left(g^{\tau\sigma}\nabla_\tau\alpha_{\sigma\nu_2\ldots\nu_k}\right)\left(dx^{\nu_2}\wedge\ldots\wedge dx^{\nu_k}\right). \tag{24}$$

Similarly as for the exterior derivative, we can write the action of δ on any k-form α in terms of the operator defined in (21) and the interior product (15), for $1 \leq k \leq n$,

$$\delta\alpha = \partial \cdot \alpha, \tag{25}$$

The interior derivative operator δ is at the same time an analytical directional covariant derivative on the components of α and an algebraic contraction operator on the natural covariant frame of α.

When acting on any n-form $\varpi = f\left(dx^1\wedge\ldots\wedge dx^n\right)$,

$$\begin{aligned}\delta\varpi &= \partial\cdot\varpi,\\ &= \left(dx^\mu\nabla_\mu\right)\cdot\left(f\left(dx^1\wedge\ldots\wedge dx^n\right)\right),\\ &= \left(\nabla_\mu f\right)\left(dx^\mu\cdot\left(dx^1\wedge\ldots\wedge dx^n\right)\right),\\ &= \left(\nabla_\mu f\right)g^{\mu\nu}\left(dx^1\wedge\ldots\widehat{dx^\nu}\ldots\wedge dx^n\right),\\ &= g^{\mu\nu}\left(\partial_\mu f\right)\left(dx^1\wedge\ldots\widehat{dx^\nu}\ldots\wedge dx^n\right),\end{aligned} \tag{26}$$

wherein $\widehat{dx^\nu}$ denotes the absence of the frame factor dx^ν in the exterior product in (26).

The *contravariant* vector field $\operatorname{grad} f \triangleq g^{\mu\nu}\left(\partial_\mu f\right)\partial_\nu$ is called the gradient of the scalar function f and is readily seen to be the image of the differential df under the inverse canonical isomorphism from $T^*M \to TM$. Eq. (26) shows that its contravariant components also arise as the components of the $(n-1)$-form $\delta\varpi$. The gradient of a function is a less general concept than the differential of a function, since it is only defined for functions on a manifold with an inner product structure (e.g., on a pseudo-Riemannian manifold), while the latter exists for functions on any differential manifold.

Further, we will need $\delta df = \left(dx^\mu\nabla_\mu\right)\cdot\left(dx^\nu\left(\partial_\nu f\right)\right) = g^{\mu\nu}\nabla_\mu\left(\partial_\nu f\right)$, explicitly given by, see e.g., [6, p. 319],

$$\delta df = \frac{1}{\sqrt{|\det[g]|}}\partial_\mu\left(\sqrt{|\det[g]|}g^{\mu\nu}\partial_\nu f_x\right). \tag{27}$$

The operation $\delta = \partial\cdot$ generalizes the divergence operation (for $1 \leq k \leq n$), defined in classical vector calculus, to totally antisymmetric covariant tensor fields on pseudo-Riemannian manifolds.

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2.4 Differential multiforms on M

We now consider objects that are collections of k-forms.

The direct (or geometric) sum of the sets of k-forms over all grades k, $\Sigma(T^*M) \triangleq \oplus_{k=0}^{n}\Gamma\left(\wedge^k T^*M\right)$, will be called the set of *multiforms*.

Since k-forms are covariant k-vector fields, multiforms are just covariant multivector fields. The concept of a multiform is the covariant analogue of the more common concept of a contravariant multivector field in Clifford analysis over a manifold, and there called a (contravariant) Clifford-valued function on M. We can thus similarly call a multiform a covariant Clifford-valued function on M.

The modules of k-forms over $\mathcal{R}$ naturally combine into the module of multiforms over $\mathcal{R}$, denoted $(\Sigma(T^*M), \mathcal{R})$.

The interior and exterior products and derivatives, defined for k-forms, naturally extend by linearity to multiforms. We will call the elements of this final structure *differential multiforms over R*.

Contravariant multivector fields over $\mathcal{R}$ on M can also be defined, but will not be needed here.

2.5 Clifford product of a 1-form and a multiform

The *Clifford product* (or geometrical product) combines the interior and exterior products into a single product.

We define a left Clifford product (denoted by juxtaposition) of any 1-form over $\mathcal{R}$ with any k-form over $\mathcal{R}$ as a bilinear map from $\Gamma(T^*M) \times \Gamma\left(\wedge^k T^*M\right) \to \Gamma\left(\wedge^{k-1}T^*M\right) \oplus \Gamma\left(\wedge^{k+1}T^*M\right)$ such that $(\alpha, \beta) \mapsto \alpha\beta$ with

$$\alpha\beta \triangleq \alpha \cdot \beta + \alpha \wedge \beta. \tag{28}$$

Herein is $\alpha \cdot \beta$ the in (15) defined left inner product defined between any 1-form and any k-form and $\wedge$ the exterior product defined by (7). The product between the components of α and β in $\alpha \cdot \beta$ and $\alpha \wedge \beta$ is the multiplication product of the ring $\mathcal{R}$.

The Clifford product, defined in (28), for a given 1-form α, is readily extended to any multiform β by linearity.

The Clifford product is associative, but generally not commutative.

We will denote the resulting (here restricted) Clifford algebra by $Cl(T^*M, g)$. The Clifford product can be defined more generally between arbitrary multiforms, but we do not need this generalization here. It would then generate the covariant Clifford bundle on M.

3 Electromagnetism in vacuum

3.1 Formulation

The formulation of electromagnetism in terms of exterior differential forms has been considered by several authors, e.g., [20], [8], [30], [9]. This has not however resulted in much progress in solving real world electromagnetic problems with this algebra.

The algebra of exterior differential forms goes back, in its most general form, to E. J. Cartan, who developed it in the period 1894–1904 based on the work of Grassmann. It is often overlooked that this algebra, in the form used by Cartan, is too general to even start formulating electromagnetism in this language. Being general means that it can be applied to more general differential manifolds (which have less structure) than inner product differential manifolds such as pseudo-Riemannian manifolds. Cartan's version lacks a specific structure which is an essential part of electromagnetism, namely an inner product g^{-1}. Once an inner product is added to Cartan's algebra of exterior differential forms, Hodge's star operator can be defined. Now, both an interior product and an interior derivative can be defined by combining Hodge's star operator with the exterior product and exterior derivative, respectively. At this point, we have the necessary ingredients to formulate electromagnetism. But this is not yet sufficient to solve the resulting equations (29)–(30), below (we want to avoid potentials). We have to extend our mathematical language further by combining the interior and exterior product into a geometrical product (the Clifford product) and we are also forced to introduce multiforms. The resulting Clifford algebra of multiforms, $Cl(T^*M, g)$, is finally powerful enough to elegantly solve eq. (31) below, as we will see in the next section. In fact, for the problem considered in this paper, a restricted Clifford algebra of 1-forms and multiforms is sufficient. We now proceed along these lines.

Electromagnetism in time-space can be correctly described, i.e., with respect for its geometrical content and physical invariances, in terms of an inner product specialization of the algebra of exterior differential forms. We get the following two equations,

$$dF = -K, \tag{29}$$

$$\delta F = -J. \tag{30}$$

Herein stands $J \in (\Gamma(T^*M), \mathcal{F}_M)$ for the electric monopole charge-current density source field, $K \in \left(\Gamma\left(\wedge^3 T^*M\right), \mathcal{F}_M\right)$ for the magnetic monopole charge-current density source field and $F \in \left(\Gamma\left(\wedge^2 T^*M\right), \mathcal{F}_M\right)$ for the resulting electromagnetic field. We will make the additional reasonable phys-

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ical assumption that (the components of) both J and K are of compact support in M.

In expectation that any magnetic monopoles are discovered in our universe, we can always put $K = 0$. However, for the mathematical structure that we wish to expose here, it is instructive to keep K in our model.

Eqs. (29)–(30) hold in the presence of any gravitational field, which is represented by the inner product structure g on M. The first equation (29) is independent of g, but the second equation (30) depends on g since the interior derivative δ depends on it, through Hodge's star operator.

Being both tensor equations, eqs. (29)–(30) are form invariant under any change of bases, so they are in particular invariant under any change of coordinates. Hence, (29)–(30) hold for any coordinate system.

Eqs. (25), (22) and (21) allow us to consider the direct sum of eqs. (29)–(30) and combine them into the single equation,

$$\partial F = -(J + K). \tag{31}$$

In the process of adding we have extended our set of mathematical quantities, k-forms, to the set of multiforms. For instance, $J + K$ is a multiform consisting of the 1-form J and the 3-form K. Clearly, the left-hand side of (31) also contains a multiform consisting of the 1-form $\partial \cdot F$ and the 3-form $\partial \wedge F$.

Eq. (31) is a very compact formulation for electromagnetism on a pseudo-Riemannian (vacuum) manifold. In addition to being compact, eq. (31) is also a fertile starting point to derive an analytical expression for the solution of electromagnetic radiation problems in vacuum, in the presence of any gravity field, in terms of any coordinates, and for any smooth compact sources J and K, as will be explained in the next section.

Additional information about other uses of Clifford algebra in electromagnetism can be found in, e.g., [2], [18], [21].

3.2 Radiation problem

3.2.1 Method

We will base our solution method on a local reciprocity relation.

By definition of the directional covariant derivative ∇_u along a contravariant vector field u, [6, p. 303], ∇_u commutes with contracted tensor multiplication (in particular, with the interior product (15)). Further, ∇_u is a derivation with respect to the tensor product $\otimes$ and by linearity also with respect to the wedge product $\wedge$. Combining both properties, shows that ∇_u is a derivation with respect to the Clifford product (28) for multiforms. Therefore, for any 1-form α over $\mathcal{D}'_M$ and any multiform β over $\mathcal{F}_M$, Leibniz' rule holds,

$$\nabla_u (\alpha\beta) = (\nabla_u \alpha)\beta + \alpha(\nabla_u \beta). \tag{32}$$

The product between the components of α and β in the Clifford products $\alpha\beta$, $(\nabla_u \alpha)\beta$ and $\alpha(\nabla_u \beta)$ in (32) is the multiplication product between distributions and smooth functions and is always defined (see (38)).

Let C_{x_0} denote a still to be determined 1-form over $\mathcal{D}'_M$. Substituting $u = \partial_\mu$, $\alpha = C_{x_0}$ and $\beta = dx^\mu F$ in (32) and contracting over μ, we get

$$\begin{aligned} & \nabla_\mu (C_{x_0} dx^\mu F) \\ = \; & ((\nabla_\mu C_{x_0}) dx^\mu) F + C_{x_0} ((\nabla_\mu dx^\mu) F), \end{aligned}$$

or

$$\nabla_\mu (C_{x_0} dx^\mu F) = \left(C_{x_0} \underset{\leftarrow}{\partial} \right) F + C_{x_0} \left(\underset{\rightarrow}{\partial} F \right). \tag{33}$$

In (33) the under arrows indicate the direction of operation of ∇_ν in the 1-form operator ∂. We need this notation due to the non-commutativity of the Clifford product between C_{x_0} and ∂.

Substituting the equation for electromagnetism, (31), in (33) gives

$$\left(C_{x_0} \underset{\leftarrow}{\partial} \right) F = C_{x_0} (J + K) + \nabla_\mu (C_{x_0} dx^\mu F). \tag{34}$$

This is the sought local reciprocity relation between the electromagnetic field 2-form F over $\mathcal{F}_M$ and the 1-form C_{x_0} over $\mathcal{D}'_M$.

We now choose C_{x_0} such that

$$C_{x_0} \underset{\leftarrow}{\partial} = \underset{\rightarrow}{\partial} C_{x_0} = \delta_{x_0}, \tag{35}$$

with δ_{x_0} the delta distribution concentrated at a parameter point $x_0 \in M$. Eq. (35) only determines C_{x_0} up to a closed 1-form α over $\mathcal{D}'_M$. For our purpose however, any fundamental solution C_{x_0} of (35) will do and any such C_{x_0} is a realization of the inverse operator ∂^{-1}.

Consider an open bounded region $\Omega \subset M$, with boundary $\partial\Omega$ and closure $\overline{\Omega} = \Omega \cup \partial\Omega$, such that $\Omega \supset \operatorname{supp}(J + K)$ and $x_0 \in \Omega$ for δ_{x_0} in (35). Let $C_c^\infty(M, \mathbf{R})$ denote the set of real smooth function of compact support defined on M and $\varphi \in C_c^\infty(M, \mathbf{R})$ a test function equaling 1 over $\overline{\Omega}$. Let further $\langle , \rangle : \mathcal{D}'_+(M) \times C_c^\infty(M, \mathbf{R}) \to \mathbf{R}$ be the *scalar product over M* between our set of distributions $\mathcal{D}'_+(M)$ and the set of test functions $C_c^\infty(M,\mathbf{R})$. This scalar product extends to k-forms over $\mathcal{D}'_M$ as

$$\left\langle \frac{1}{k!}\alpha_{\mu_1\mu_2\ldots\mu_k}\left(dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_k}\right),\varphi\right\rangle \triangleq \frac{1}{k!}\left\langle \alpha_{\mu_1\mu_2\ldots\mu_k},\varphi\right\rangle\left(dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_k}\right) \quad (36)$$

and to multiforms by linearity.

Substituting (35) in (34) and calculating the scalar product of eq. (34) with φ gives

$$\langle\delta_{x_0}F,\varphi\rangle = \langle C_{x_0}(J+K),\varphi\rangle + \langle\nabla_\mu\left(C_{x_0}dx^\mu F\right),\varphi\rangle. \quad (37)$$

Recall from the theory of distributions the definition for the product of a distribution $f\in\mathcal{D}'_+(M)$ with a smooth function $h\in C^\infty(M,\mathbf{R})$,

$$\langle hf,\varphi\rangle \triangleq \langle f,h\varphi\rangle, \quad (38)$$

a definition which is legitimate since $h\varphi \in C_c^\infty(M,\mathbf{R})$. Using (36) and (38), (37) can now be written out explicitly as

$$\begin{aligned}&\frac{1}{2!}\langle\delta_{x_0},F_{\nu_1\nu_2}\varphi\rangle\left(dx^{\nu_1}\wedge dx^{\nu_2}\right)\\ =&\left(\begin{array}{c}\left\langle\left(C_{x_0}\right)_{\mu_1},J_{\nu_1}\varphi\right\rangle\left(dx^{\mu_1}dx^{\nu_1}\right)\\ +\frac{1}{3!}\left\langle\left(C_{x_0}\right)_{\mu_1},K_{\nu_1\nu_2\nu_3}\varphi\right\rangle\\ \left(dx^{\mu_1}\left(dx^{\nu_1}\wedge dx^{\nu_2}\wedge dx^{\nu_3}\right)\right)\end{array}\right)\\ &+\frac{1}{2!}\left\langle\nabla_\mu\left(\left(C_{x_0}\right)_{\mu_1}F_{\nu_1\nu_2}\right),\varphi\right\rangle\\ &\left(dx^{\mu_1}dx^\mu\left(dx^{\nu_1}\wedge dx^{\nu_2}\right)\right). \end{aligned} \quad (39)$$

In (39), the product between the frame fields is the Clifford product between a 1-form and a k-form (with $k=1$ and $k=3$).

By definition of the delta distribution and the choice of support for the test function φ, (39) reduces to

$$\begin{aligned}&F(x_0)\\ =&\left(\begin{array}{c}\left\langle\left(C_{x_0}\right)_{\mu_1},J_{\nu_1}\right\rangle\left(dx^{\mu_1}dx^{\nu_1}\right)\\ +\frac{1}{3!}\left\langle\left(C_{x_0}\right)_{\mu_1},K_{\nu_1\nu_2\nu_3}\right\rangle\\ \left(dx^{\mu_1}\left(dx^{\nu_1}\wedge dx^{\nu_2}\wedge dx^{\nu_3}\right)\right)\end{array}\right)\\ &+\frac{1}{2!}\left\langle\nabla_\mu\left(\left(C_{x_0}\right)_{\mu_1}F_{\nu_1\nu_2}\right),\varphi\right\rangle\\ &\left(dx^{\mu_1}dx^\mu\left(dx^{\nu_1}\wedge dx^{\nu_2}\right)\right). \end{aligned} \quad (40)$$

The second term in the right-hand side of (40) containing ∇_μ can be converted by Stokes' theorem to a scalar product over $\partial\Omega$ of a multiform concentrated on $\partial\Omega$ with the 1 function on $\partial\Omega$. Then, eq. (40) gives the general solution of the boundary value problem, consisting of eq. (31) together with prescribed boundary values for the electromagnetic field F on $\partial\Omega$. The outward radiation condition corresponds to putting this converted second term equal to zero. All this can be made more explicit, but this development requires a somewhat more advanced derivation, based on a generalization of Stokes' theorem, and will be presented elsewhere. Hence, the particular solution caused by the sources J and K, denoted by F^{src}, is thus given by

$$\begin{aligned}&F^{src}(x_0)\\ =&\left\langle\left(C_{x_0}\right)_{\mu_1},J_{\nu_1}\right\rangle\left(dx^{\mu_1}dx^{\nu_1}\right)\\ &+\frac{1}{3!}\left\langle\left(C_{x_0}\right)_{\mu_1},K_{\nu_1\nu_2\nu_3}\right\rangle\\ &\left(dx^{\mu_1}\left(dx^{\nu_1}\wedge dx^{\nu_2}\wedge dx^{\nu_3}\right)\right). \end{aligned} \quad (41)$$

3.2.2 Construction of C_{x_0}

The construction of C_{x_0} is simplified by noting that $dC_{x_0}=\partial\wedge C_{x_0}=0$ (i.e., C_{x_0} is closed) implies, by Poincaré's lemma, that (locally)

$$C_{x_0}=df_{x_0}=\partial\wedge f_{x_0}, \quad (42)$$

(i.e., C_{x_0} is exact) for some 0-form f_{x_0} over $\mathcal{D}'_M$. Since $\delta f_{x_0}=\partial\cdot f_{x_0}=0$, we can write (42) also in Clifford form as

$$C_{x_0}=\partial f_{x_0}. \quad (43)$$

Substituting this representation for C_{x_0} in its defining equation (35) gives the equation to be satisfied by f_{x_0},

$$\partial\left(\partial f_{x_0}\right)=\delta_{x_0}. \quad (44)$$

The grade preserving operator $\partial\circ\partial=(d+\delta)\circ(d+\delta)=d\circ\delta+\delta\circ d$ is just the Laplace-de Rham operator. Since $\delta f_{x_0}=0$ and by using expression (27) for δdf_{x_0}, we get for (44), with respect to natural frames,

$$\frac{1}{\sqrt{|\det[g]|}}D_\mu\left(\sqrt{|\det[g]|}g^{\mu\nu}D_\nu f_{x_0}\right)=\delta_{x_0}, \quad (45)$$

which is the generalized (i.e., the distributional) scalar wave equation in curved time-space.

Collecting results, we see that any fundamental solution f_{x_0} of the generalized scalar wave equation in curved time-space generates a 1-form C_{x_0} over $\mathcal{D}'_M$, which realizes the inverse operator ∂^{-1} as

$$\partial^{-1}=-\langle C_{x_0},\_\rangle \quad (46)$$

and so in turn generates the general solution by (40).

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It can be shown that the general solution F given by (40) is independent of the particular choice of fundamental solution f_{x_0} of (44).

Our main result (40) would be most useful in gravity fields for which f_{x_0} could be obtained in analytical form. It is not known how to analytically solve eq. (45) for f_{x_0} in a general gravity field g. The construction for general g, given by Hadamard in [13], proofs the existence and uniqueness of the solution of the Cauchy problem for (45), but is of limited value to calculate f_{x_0} in general. For some specific backgrounds however, such as the de Sitter metric, [11], some Bianchi-type I universes, [26], and a class of Robertson-Walker metrics [22], an analytical expression for f_{x_0} can be obtained exactly.

3.2.3 Integrability conditions

It is remarkable that the generally accepted mathematical model for electromagnetism has in general no particular solution (this also holds for Heaviside's model). Indeed, operating on the left with ∂ shows that any solution of $\partial F = -(J+K)$ is necessarily also a solution of

$$\partial^2 F = -\partial (J+K). \tag{47}$$

We used in (47) the associativity of the Clifford product and of ∇_μ when we equaled $\partial(\partial F)$ to $\partial^2 F$. The operator ∂^2 is grade preserving, hence the grade of the left-hand side of eq. (47) equals the grade of F, which is 2. The right-hand side of eq. (47) has a grade 0 part, $-(\partial \cdot J)$, a grade 2 part, $-(\partial \wedge J + \partial \cdot K)$, and a grade 4 part, $-(\partial \wedge K)$. For eq. (47) to have a solution it is thus necessary that both the grade 0 part and the grade 4 part vanishes. This requires that J and K must satisfy $\delta J = \partial \cdot J = 0$ and $dK = \partial \wedge K = 0$, or with respect to natural frames, that

$$\frac{1}{\sqrt{|\det[g]|}}\frac{\partial}{\partial x^\tau}\left(\sqrt{|\det[g]|}g^{\tau\sigma}J_\sigma\right) = 0, \tag{48}$$

$$\frac{1}{3!}\epsilon^{\tau\nu_1\nu_2\nu_3}\frac{\partial K_{\nu_1\nu_2\nu_3}}{\partial x^\tau} = 0. \tag{49}$$

Eqs. (48)–(49) are the necessary integrability conditions of our model for electromagnetism on a pseudo-Riemannian (vacuum) manifold in the presence of gravity. Eqs. (48)–(49) amount physically to the local conservation of electric monopole charge and of magnetic monopole charge, respectively. Although it is well-known that from the equation(s) for electromagnetism conservation of charge can be derived, the mathematical implication of this fact, namely that local conservation of charge is a necessary integrability condition for the equation(s) for electromagnetism, is rarely mentioned in the electromagnetics literature.

It can be shown that conditions (48)–(49) are also sufficient for the existence of a particular solution of our model for electromagnetism.

3.2.4 Solution

General form Evaluating the Clifford products in (41) gives

$$\begin{aligned} & F^{src}(x_0) \\ = & \left\langle (C_{x_0})_{\mu_1}, J_{\nu_1}\right\rangle g^{\mu_1\nu_1} \\ & + \begin{pmatrix} \frac{1}{2!}\delta^{\kappa_1\lambda_1}_{\mu_1\nu_1}\left\langle (C_{x_0})_{\kappa_1}, J_{\lambda_1}\right\rangle (dx^{\mu_1}\wedge dx^{\nu_1}) \\ +\frac{1}{3!}\left\langle (C_{x_0})_{\mu_1}, K_{\nu_1\nu_2\nu_3}\right\rangle \\ (dx^{\mu_1}\cdot(dx^{\nu_1}\wedge dx^{\nu_2}\wedge dx^{\nu_3})) \end{pmatrix} \\ & +\frac{1}{3!}\delta^{\mu_1\nu_1\nu_2\nu_3}_{1234}\left\langle (C_{x_0})_{\mu_1}, K_{\nu_1\nu_2\nu_3}\right\rangle \\ & \left(dx^1\wedge dx^2\wedge dx^3\wedge dx^4\right). \end{aligned} \tag{50}$$

It can be shown that conditions (48)–(49) also guarantee that in (40) only the grade 2 part remains. Then, (50) reduces to

$$\begin{aligned} & F^{src}(x_0) \\ = & \frac{1}{2!}\delta^{\kappa_1\lambda_1}_{\mu_1\nu_1}\left\langle (C_{x_0})_{\kappa_1}, J_{\lambda_1}\right\rangle (dx^{\mu_1}\wedge dx^{\nu_1}) \\ & +\frac{1}{3!}\left\langle (C_{x_0})_{\mu_1}, K_{\nu_1\nu_2\nu_3}\right\rangle \\ & (dx^{\mu_1}\cdot(dx^{\nu_1}\wedge dx^{\nu_2}\wedge dx^{\nu_3})). \end{aligned} \tag{51}$$

In the absence of a magnetic monopole charge-current density source field K, (51) further reduces to

$$F^{src}(x_0) = \frac{1}{2!}\delta^{\kappa_1\kappa_2}_{\mu_1\mu_2}\left\langle (C_{x_0})_{\kappa_1}, J_{\kappa_2}\right\rangle (dx^{\mu_1}\wedge dx^{\mu_2}). \tag{52}$$

By substituting expression (43) for C_{x_0}, (52) is equivalent to

$$F^{src}(x_0) = \frac{1}{2!}\delta^{\kappa_1\kappa_2}_{\mu_1\mu_2}\left\langle D_{\kappa_1}f_{x_0}, J_{\kappa_2}\right\rangle (dx^{\mu_1}\wedge dx^{\mu_2}). \tag{53}$$

We can convert (53) to a simpler form by using the definition of the generalized derivative D_{κ_1}. Since we assumed that J is smooth, we get

$$F^{src}(x_0) = -\frac{1}{2!}\delta^{\kappa_1\kappa_2}_{\mu_1\mu_2}\left\langle f_{x_0}, d_{\kappa_1}J_{\kappa_2}\right\rangle (dx^{\mu_1}\wedge dx^{\mu_2}), \tag{54}$$

with d_{κ_1} the ordinary partial derivative with respect to the coordinate x^{κ_1}. In more compact notation, (54) reads

$$F^{src}(x_0) = -\left\langle f_{x_0}, dJ\right\rangle, \tag{55}$$