Theory of Multivalent Delta-Fuzzy Measures and its Application

Hsiang-Chuan Liu, Der-Bang Wu, Yu-Du Jheng, and Tian-Wei Sheu

***Abstract*—**The well known fuzzy measures, Lambda-measure and P-measure, have only one formulaic solution, the former is not a closed form, and the latter is not sensitive enough. In this paper, a novel fuzzy measure, called Delta-measure, is proposed. This new measure proves to be a multivalent fuzzy measure which provides infinitely many solutions to closed form, and it can be considered as an extension of the above two measures. In other words, the above two fuzzy measures can be treated as the special cases of Delta-measure. For evaluating the Choquet integral regression models with our proposed fuzzy measure and other different ones, a real data experiment by using a 5-fold cross-validation mean square error (MSE) is conducted. The performances of Choquet integral regression models with fuzzy measure based on Delta-measure, Lambda-measure and P-measure, respectively, a ridge regression model and a multiple linear regression model are compared. Experimental result shows that the Choquet integral regression models with respect to Delta-measure based on Gamma-support outperforms other forecasting models.

***Keywords*—**Lambda-measure, P-measure, Delta-measure, Gamma-support, Choquet integral regression model.

I. Introduction

When there are interactions among independent variables, traditional multiple linear regression models do not perform well enough. The traditional improved methods exploited ridge regression models [1]. In this paper, we suggest using the Choquet integral regression models [5,6,7,8,9,10] based on some single or compounded fuzzy measures [2,3,4, 12,13] to improve this situation. The well-known fuzzy measures, λ-measure [2,3] and P-measure [4], have only one formulaic solution of fuzzy measure, the former is not a closed form, and the latter is not sensitive enough. In this paper, we proposed a new fuzzy measure, δ-measure, which offers infinitely many solutions to a fuzzy measure with closed form and without changing the given singleton measure, and thereby, we can obtain an improved Choquet integral regression model with respect to this new fuzzy measure.

This paper is organized as follows: The multiple linear regression and ridge regression [1] are introduced in section II; two well known fuzzy measure, λ-measure [2] and P-measure [4], are introduced in section III; our new measure, δ-measure, is introduced in section IV; the fuzzy support, γ-support [7] is described in section V; the Choquet integral regression model [6],[7],[8] based on fuzzy measures are described in section VI; experiment and result are described in section VII; and final section is for conclusions and future works.

Manuscript received March 20, 2009: Revised version received May 20, 2009. This work was supported in part by the National Science Council grant of Taiwan Government (NSC 97-2410-H-468-014).

H. C. Liu is with the Department of Bioinformatics, Asia University, Taichung, 41345 Taiwan (e-mail: lhc@asia.edu.tw).

D. B. Wu, is with National Taichung University, Taichung, Taiwan 40306. He is now with the Graduate Institute of Educational Measurement and Department of Mathematics Education, National Taichung University, Taichung, 40306 Taiwan. (phone: 886-93673-1088; fax: 886-42218-3500; e-mail:wudb@hotmail.com).

Y. D. Jheng with National Taichung University, Taichung, Taiwan 40306. He is now with the Graduate Institute of Educational Measurement, National Taichung University, Taichung, 40306 Taiwan. (e-mail: doora0622@yahoo.com.tw).

T. W. Sheu with National Taichung University, Taichung, Taiwan 40306. He is now with the Graduate Institute of Educational Measurement, National Taichung University, Taichung, 40306 Taiwan. (e-mail:sheu@mail.ntcu.edu.tw).

II. The Multiple Linear Regression, Ridge Regression

Let $\underline{Y} = X\underline{\beta} + \underline{\varepsilon}$, $\underline{\varepsilon} \sim \mathrm{N}\left(\underline{0},\sigma^2 I_n\right)$ be a multiple linear model, $\underline{\hat{\beta}} = \left(X'X\right)^{-1} X'Y$ be the estimated regression coefficient vector, and $\underline{\hat{\beta}_k} = \left(X'X + kI_n\right)^{-1} X'Y$ be the estimated ridge regression coefficient vector, Kenard and Baldwin [1] suggested

$$\hat{k} = \frac{n\hat{\sigma}^2}{\underline{\hat{\beta}}'\underline{\hat{\beta}}} \tag{1}$$

III. Fuzzy Measures

The two well known fuzzy measures, the λ-measure proposed by Sugeno in 1974, and P-measure proposed by Zadah in 1978, are concise introduced as follows.

A. Axioms of Fuzzy Measures [2, 3, 4]

A fuzzy measure μ on a finite set X is a set function $\mu : 2^X \to [0,1]$ satisfying the following axioms:

1) $\mu(\phi) = 0$, $\mu(X) = 1$ (boundary conditions) (2)

2) $A \subseteq B \Rightarrow \mu(A) \le \mu(B)$ (monotonicity) (3)

B. *Singleton Measures* [2, 6, 7]

A singleton measure of a fuzzy measure μ on a finite set X is a function $s : X \to [0,1]$ satisfying:

$$s(x) = \mu(\{x\}),\ x \in X \quad (4)$$

$s(x)$ is called the fuzzy density of singleton x.

For given singleton measures s, a λ-measure, g_λ, is a fuzzy measure on a finite set X, satisfying:

$$\begin{aligned} & A, B \in 2^X,\ A \cap B = \phi,\ A \cup B \neq X \\ & \Rightarrow g_\lambda(A \cup B) \\ & \quad = g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A) g_\lambda(B) \end{aligned} \quad (5)$$

$$\prod_{i=1}^{n}\left[1 + \lambda s(x_i)\right] = \lambda + 1 > 0,\ s(x_i) = g_\lambda(\{x_i\}) \quad (6)$$

Note that once the singleton measure is known, we can obtain the values of λ uniquely by using the previous polynomial equation. In other words, λ-measure has a unique solution without closed form. Moreover, for given singleton measures s, *If* $\sum_{x \in X} s(x) = 1$ *then* $g_\lambda(A) = \sum_{x \in A} s(x)$, in other word, if $\sum_{x \in X} s(x) = 1$ then λ-measure is just the additive measure

C. *P-measure [4]*

For given singleton measures s, a P-measure, g_P, is a fuzzy measure on a finite set X, satisfying:

$$\begin{aligned} & {}^\forall A \in 2^X \\ & \Rightarrow g_P(A) = \max_{x \in A}\{s(x)\} = \max_{x \in A}\{g_P(\{x\})\} \end{aligned} \quad (7)$$

Note that for any subset of X, A, P-measure considers only the maximum value and will lead to insensitivity.

IV. A NEW METHOD - DELTA-MEASURES

A. *Definition of δ-measure*

For given singleton measure s, a δ-measure, g_δ, is a fuzzy measure on a finite set X, $|X| = n$, satisfying:

1) $\delta \in [-1,1],\ \sum_{x \in X} s(x) = 1$

2) $g_\delta(\phi) = 0,\ g_\delta(X) = 1$

3) $\forall A \subset X, A \neq X \Rightarrow$

$$g_\delta(A) = \left[1 + \delta \max_{x \in A} s(x)\right] \frac{(1+\delta)\sum_{x \in A} s(x)}{1 + \delta \sum_{x \in A} s(x)} - \delta \max_{x \in A} s(x) \quad (8)$$

B. *Important Properties of δ-measure*

To prove that δ-measure is a fuzzy measure, we need to prove the following theorem 1 firstly.

Theorem 1

For given singleton measure s,

If $A \subseteq B \subseteq X$ then

$$\sum_{x \in B} s(x) - \sum_{x \in A} s(x) \geq \max_{x \in B}\{s(x)\} - \max_{x \in A}\{s(x)\} \geq 0 \quad (9)$$

[Proof]

Let $B = A \cup C = A \cup \{x_1, x_2, ..., x_n\},\ C = \{x_1, x_2, ..., x_n\}$

If $\max_{x \in B}\{s(x)\} = \max_{x \in A}\{s(x)\}$, then

$\sum_{x \in B} s(x) - \sum_{x \in A} s(x) \geq 0 = \max_{x \in B}\{s(x)\} - \max_{x \in A}\{s(x)\}$,

its true,

now suppose that $\max_{x \in B}\{s(x)\} > \max_{x \in A}\{s(x)\}$

(I) If n=1, let $B = A \cup C = A \cup \{x_1\}$, then

(i)

$$\begin{aligned} & \sum_{x \in B} s(x) = \sum_{x \in A} s(x) + \sum_{x \in C} s(x) = \sum_{x \in A} s(x) + s(x_1) \\ & \Rightarrow s(x_1) = \sum_{x \in B} s(x) - \sum_{x \in A} s(x) \geq 0 \end{aligned}$$

(ii) Since

$$\max_{x \in B}\{s(x)\} = \max\left\{\max_{x \in A}\{s(x)\}, s(x_1)\right\} > \max_{x \in A}\{s(x)\}$$

$$\Rightarrow s(x_1) = \max_{x \in B}\{s(x)\} \geq \max_{x \in B}\{s(x)\} - \max_{x \in A}\{s(x)\}$$

(iii) from (i) and (ii), we can obtain

$$\sum_{x \in B} s(x) - \sum_{x \in A} s(x) \geq \max_{x \in B}\{s(x)\} - \max_{x \in A}\{s(x)\}$$

(II) If n=k , let $B = A \bigcup C = A \bigcup \{x_1, x_2, ..., x_k\}$ satisfying

$$\sum_{x\in B} s(x) - \sum_{x\in A} s(x) \ge \max_{x\in B}\{s(x)\} - \max_{x\in A}\{s(x)\} \quad (10)$$

To prove that if n=k+1,

$B' = A \bigcup C = A \bigcup \{x_1, x_2, ..., x_k, x_{k+1}\} = B \bigcup \{x_{k+1}\}$

Satisfying

$$\sum_{x\in B'} s(x) - \sum_{x\in A} s(x) \ge \max_{x\in B'}\{s(x)\} - \max_{x\in A}\{s(x)\} \quad (11)$$

Since $B' = B \bigcup \{x_{k+1}\}$, and $s(x_{k+1}) \le \max_{x\in B'}\{s(x)\}$

(i) if $\max_{x\in B}\{s(x)\} = \max_{x\in B'}\{s(x)\}$, then

$$\begin{aligned}
&\sum_{x\in B'} s(x) - \sum_{x\in A} s(x) = s(x_{k+1}) + \sum_{x\in B} s(x) - \sum_{x\in A} s(x) \\
&\ge s(x_{k+1}) + \max_{x\in B}\{s(x)\} - \max_{x\in A}\{s(x)\} \\
&= s(x_{k+1}) + \max_{x\in B'}\{s(x)\} - \max_{x\in A}\{s(x)\} \\
&\ge \max_{x\in B'}\{s(x)\} - \max_{x\in A}\{s(x)\} \\
&\Rightarrow \sum_{x\in B'} s(x) - \sum_{x\in A} s(x) \\
&\ge \max_{x\in B'}\{s(x)\} - \max_{x\in A}\{s(x)\}
\end{aligned}$$

(ii) Now suppose that $\max_{x\in B}\{s(x)\} < \max_{x\in B'}\{s(x)\}$,

then $s(x_{k+1}) = \max_{x\in B'}\{s(x)\}$, and

$$\begin{aligned}
&\sum_{x\in B'} s(x) - \sum_{x\in A} s(x) \\
&= s(x_{k+1}) + \sum_{x\in B} s(x) - \sum_{x\in A} s(x) \\
&\ge s(x_{k+1}) = \max_{x\in B'}\{s(x)\} \\
&\ge \max_{x\in B'}\{s(x)\} - \max_{x\in A}\{s(x)\} \\
&\Rightarrow \sum_{x\in B'} s(x) - \sum_{x\in A} s(x) \\
&\ge \max_{x\in B'}\{s(x)\} - \max_{x\in A}\{s(x)\}
\end{aligned}$$

(III) By mathematical induction, from (I) and (II), the proof is completed.

Theorem 2

For given singleton measure s, $\forall\delta \in [-1,1]$, δ-measure is a fuzzy measure.

[Proof];

(I) To prove the boundary conditions; $0 \le g_\delta(A) \le 1$

(i) if $\delta = -1$ it is trivial

(ii) if $\delta > -1$

$$0 \le \sum_{x\in A} s(x) \le 1 \Rightarrow \quad 1 + \delta \sum_{x\in A} s(x) > 0$$

Since

$$\begin{aligned}
&g_\delta(A)\left[1 + \delta \sum_{x\in A} s(x)\right] = \\
&\left[1 + \delta \max_{x\in A}\{s(x)\}\right]\left[(1+\delta)\sum_{x\in A} s(x)\right] \\
&-\left[\delta \max_{x\in A}\{s(x)\}\right]\left[1 + \delta \sum_{x\in A} s(x)\right]
\end{aligned}$$

and

$$\begin{aligned}
&\delta > -1, \\
&\left[1 + \delta \sum_{x\in A} s(x)\right] \ge \left[1 + \delta \max_{x\in A}\{s(x)\}\right] > 0, \\
&0 < \max_{x\in A}\{s(x)\} \le \sum_{x\in A} s(x) \le 1
\end{aligned}$$

Hence

$$\begin{aligned}
&g_\delta(A)\left[1 + \delta \sum_{x\in A} s(x)\right] \\
&= (1+\delta)\sum_{x\in A} s(x) + \delta \max_{x\in A}\{s(x)\}\sum_{x\in A} s(x) \\
&\quad - \delta \max_{x\in A}\{s(x)\}
\end{aligned}$$

$if\ -1 \le \delta \le 0,\ then$

$$\begin{aligned}
&g_\delta(A)\left[1 + \delta \sum_{x\in A} s(x)\right] = (1+\delta)\sum_{x\in A} s(x) \quad (12) \\
&+(-\delta)\max_{x\in A}\{s(x)\}\left[1 - \sum_{x\in A} s(x)\right] \ge 0
\end{aligned}$$

$if\ \delta > 0,\ then$

$$\begin{aligned}
&g_\delta(A)\left[1 + \delta \sum_{x\in A} s(x)\right] \\
&= \left(1 + \delta \max_{x\in A}\{s(x)\}\right)\sum_{x\in A} s(x) + \\
&\delta\left[\sum_{x\in A} s(x) - \max_{x\in A}\{s(x)\}\right] \ge 0
\end{aligned}$$

Therefore

$$g_\delta(A)\left[1 + \delta \sum_{x\in A} s(x)\right] \ge 0 \text{ , and } \quad g_\delta(A) \ge 0 \quad (13)$$

(iii)

$$\left[1+\delta\max_{x\in A}\{s(x)\}\right](1+\delta)\sum_{x\in A}s(x)-$$

$$\delta\max_{x\in A}\{s(x)\}\left[1+\delta\sum_{x\in A}s(x)\right]-\left[1+\delta\sum_{x\in A}s(x)\right]$$

$$=\left[1+\delta\max_{x\in A}\{s(x)\}\right]\left[\sum_{x\in A}s(x)-1\right]\le 0$$

$$\Rightarrow\left[1+\delta\max_{x\in A}\{s(x)\}\right](1+\delta)\sum_{x\in A}s(x)-$$

$$\delta\max_{x\in A}\{s(x)\}\left[1+\delta\sum_{x\in A}s(x)\right]$$

$$\le\left[1+\delta\sum_{x\in A}s(x)\right]$$

$$\Rightarrow\left[1+\delta\max_{x\in A}\{s(x)\}\right]\frac{(1+\delta)\sum_{x\in A}s(x)}{\left[1+\sum_{x\in A}s(x)\right]}-\delta\max_{x\in A}\{s(x)\}\le 1$$

Therefore $0\le g_\delta(A)\le 1,\ \forall A\subset X$

(II) To prove the monotonicity;

$$A\subset B\subset X\Rightarrow g_\delta(A)\le g_\delta(B)$$

(i) Let $\mathrm{g_P}(A)=\max_{x\in A}\{s(x)\},\mathrm{g}_\sigma(A)=\sum_{x\in A}s(x)$

Let

$$A\subset B\subset X$$

$$\Rightarrow\ \mathrm{g_P}(A)\le\mathrm{g_P}(B),\ \mathrm{g}_\sigma(A)\le\mathrm{g}_\sigma(B)$$

$$\mathrm{g_P}(B)=\mathrm{g_P}(A)+c\le\mathrm{g}_\sigma(B)=\mathrm{g}_\sigma(A)+d$$

From theorem 1 we know that $0\le c\le d\le 1$, then

$$-\left[\left[1+\delta g_P(A)\right]\frac{(1+\delta)g_\sigma(A)}{\left[1+\delta g_\sigma(A)\right]}-\delta g_P(A)\right]$$

$$=\left[1+\delta\mathrm{g}_P(A)\right]\frac{(1+\delta)g_\sigma(B)}{1+\delta g_\sigma(B)}$$

$$+\delta c\frac{(1+\delta)g_\sigma(B)}{\left[1+\delta g_\sigma(B)\right]}-\delta c$$

$$-\left[\left(1+\delta\mathrm{g}_P(A)\right)\frac{(1+\delta)\mathrm{g}_\sigma(A)}{1+\delta\mathrm{g}_\sigma(A)}\right]\tag{14}$$

$$\Rightarrow D'=\left[1+\delta g_\sigma(B)\right]D$$

$$=\left[1+\delta\mathrm{g}_P(A)\right](1+\delta)g_\sigma(B)+\delta c(1+\delta)g_\sigma(B)$$

$$-\delta c\left[1+\delta g_\sigma(B)\right]$$

$$-\left[1+\delta g_\sigma(B)\right]\left[\left(1+\delta\mathrm{g}_P(A)\right)\frac{(1+\delta)\mathrm{g}_\sigma(A)}{1+\delta\mathrm{g}_\sigma(A)}\right]$$

$$=\left[1+\delta\left[\mathrm{g}_P(A)\right]\right](1+\delta)[d]$$

$$+\left[1+\delta\left[\mathrm{g}_\sigma(A)\right]\right]\left[\delta c\left[\mathrm{g}_\sigma(B)-1\right]\right]\tag{15}$$

$if\ \delta\in[-1,0]\ then\ \left[1+\delta\left[\mathrm{g}_P(A)\right]\right]\ge 0,$

$(1+\delta)\ge 0,$

$\left[1+\delta\left[\mathrm{g}_\sigma(A)\right]\right]\ge 0,$

$\left[\delta c\left[\mathrm{g}_\sigma(B)-1\right]\right]\ge 0,$

$where\left[\mathrm{g}_\sigma(B)\right]\in[0,1],$

$d>0\quad then\ D'\ge 0,$

$and\ g_\delta(A)\le g_\delta(B)$

$if\ 0\le\delta\le 1, since\ d\ge c\ge 0$

$$\Rightarrow\left[1+\delta\left[\mathrm{g}_P(A)\right]\right](1+\delta)[d]$$

$$+\left[1+\delta\left[\mathrm{g}_\sigma(A)\right]\right]\left[\delta c\left[\mathrm{g}_\sigma(B)-1\right]\right]$$

$$\ge\left[1+\delta\left[\mathrm{g}_P(A)\right]\right](1+\delta)[c]$$

$$+\left[1+\delta\left[\mathrm{g}_\sigma(A)\right]\right]\left[\delta c\left[\mathrm{g}_\sigma(B)-1\right]\right]\ge 0$$

where $(1+\delta)\ge\left[1+\delta\left[\mathrm{g}_\sigma(A)\right]\right]\ge 0, and$

$$\left[1+\delta\left[\mathrm{g}_P(A)\right]\right]\ge\left[\delta\left[1\text{-}\mathrm{g}_\sigma(B)\right]\right]\ge 0$$

$then\ A\subset B\Rightarrow\mathrm{g}_\delta(A)\le\mathrm{g}_\delta(B)$

Theorem 3

(i) δ-measure is increasing function on δ

(ii) if $\delta=-1$ then δ-measure is just the P-measure

(iii) if $\delta=0$ then δ-measure is just the additive measure

(iv) if $-1<\delta<0$ then δ-measure is a sub-additive measure

(v) if $0<\delta<1$ then δ-measure is a supper-additive measure

[Proof];

(i) δ-measure is increasing function onδ

Let $-1\le\delta_1<\delta_2\le 1$ to prove that for each

$$A\subset X\Rightarrow g_{\delta_1}(A)\le g_{\delta_2}(A)$$

Let $f(\delta)=g_\delta(A)$

$$=\left[1+\delta g_P(A)\right]\frac{(1+\delta)g_\sigma(A)}{1+\delta g_\sigma(A)}-\delta g_P(A)\tag{16}$$

Since $1-g_\sigma(A)\ge 0,\ g_\sigma(A)\ge g_P(A)$

Then

$$= \ f'(\delta)=\frac{[1-g_\sigma(A)][g_\sigma(A)-g_P(A)]}{[1+\delta g_\sigma(A)]^2}\ge 0 \quad (17)$$

$$f''(\delta)=\frac{-2g_\sigma(A)[1-g_\sigma(A)][g_\sigma(A)-g_P(A)]}{[1+\delta g_\sigma(A)]^3}<0 \quad (18)$$

Therefore δ-measure is a concaved downward and increasing function on δ.

(ii) and (iii) are trivial

(iv) If $-1<\delta<0$, since δ-measure is increasing function on δ

then $\forall A\subset X \Rightarrow g_\delta(A)\le g_0(A)=\sum_{x\in A}s(x)$,in other word , δ-measure is sub-additive

(v) If $0<\delta<1$, since δ-measure is increasing function on δ

then $\forall A\subset X \Rightarrow g_\delta(A)\ge g_0(A)=\sum_{x\in A}s(x)$,in other word , δ-measure is supper-additive.

Theorem 4

If $\sum_{x\in X}s(x)=1$ and $\delta=0$ then δ-measure is just the λ-measure

Theorem 5

P -measure, additive measure and λ-measure are the special cases of δ-measure

V. Γ - SUPPORT [7]

For given singleton measure s of a fuzzy measure μ on a finite set X, if $\sum_{x\in X}s(x)=1$, then s is called a fuzzy support measure of μ, or a fuzzy support of μ, or a support of μ. Two kinds of fuzzy supports are introduced as below.

Let μ be a fuzzy measure on a finite set $X=\{x_1,x_2,\dots,x_n\}$, y_i be global response of subject i and $f_i(x_j)$ be the evaluation of subject i for singleton x_j, satisfying:

$$0<f_i(x_j)<1,\ i=1,2,\dots,N,\ j=1,2,\dots,n$$

$$\gamma(x_j)=\frac{1+r(f(x_j))}{\sum_{k=1}^{n}[1+r(f(x_k))]},\ j=1,2,\dots,n \quad (19)$$

where $r(f(x_j))=\frac{S_{y,x_j}}{S_yS_{x_j}}$ (20)

$$S_y^2=\frac{1}{N}\sum_{i=1}^{n}\left(y_i-\frac{1}{N}\sum_{i=1}^{N}y_i\right)^2 \quad (21)$$

$$S_{x_j}^2=\frac{1}{N}\sum_{i=1}^{n}\left[f_i(x_j)-\frac{1}{N}\sum_{i=1}^{N}f_i(x_j)\right]^2 \quad (22)$$

$$S_{y,x_j}=\frac{1}{N}\sum_{i=1}^{n}\left(y_i-\frac{1}{N}\sum_{i=1}^{N}y_i\right)\left[f_i(x_j)-\frac{1}{N}\sum_{i=1}^{N}f_i(x_j)\right] \quad (23)$$

satisfying $0\le\gamma(x_j)\le 1$ and $\sum_{j=1}^{n}\gamma(x_j)=1$ (24)

then the function $\gamma: X\to[0,1]$ satisfying $\mu(\{x\})=\gamma(x)$, $\forall x\in X$ is a fuzzy support of μ, called γ-support of μ.

VI. CHOQUET INTEGRAL REGRESSION MODELS

A. *Choquet Integral [3, 5, 9, 10]*

Let μ be a fuzzy measure on a finite set X. The Choquet integral of $f_i: X\to R_+$ with respect to μ for individual i is denoted by

$$\int_C f_i d\mu=\sum_{j=1}^{n}\left[f_i(x_{(j)})-f_i(x_{(j-1)})\right]\mu(A_{(j)}^i)\ ,i=1,2,\dots,N \quad (25)$$

where $f_i(x_{(0)})=0$, $f_i(x_{(j)})$ indicates that the indices have been permuted so that

$$0\le f_i(x_{(1)})\le f_i(x_{(2)})\le\dots\le f_i(x_{(n)}) \quad (26)$$

$$A_{(j)}=\{x_{(j)},x_{(j+1)},\dots,x_{(n)}\} \quad (27)$$

B. *Choquet Integral Regression Models [6 - 12]*

Let $y_1,y_2,\dots,y_N$ be global evaluations of N objects and $f_1(x_j),f_2(x_j),\dots,f_N(x_j),\ j=1,2,\dots,n$, be their evaluations of x_j, where $f_i: X\to R_+$, $i=1,2,\dots,N$.

Let μ be a fuzzy measure, $\alpha,\beta\in R$,

$$y_i=\alpha+\beta\int_C f_i dg_\mu+e_i\ ,\ e_i\sim N(0,\sigma^2)\ ,\ i=1,2,\dots,N \quad (28)$$

$$\left(\hat{\alpha},\hat{\beta}\right)=\arg\min_{\alpha,\beta}\left[\sum_{i=1}^{N}\left(y_i-\alpha-\beta\int_C f_i dg_\mu\right)^2\right] \quad (29)$$

then $\hat{y}_i=\hat{\alpha}+\hat{\beta}\int f_i dg_\mu,\ i=1,2,...,N$ is called the Choquet integral regression equation of μ, where

$$\hat{\beta}=S_{yf}/S_{ff} \quad (30)$$

$$\hat{\alpha}=\frac{1}{N}\sum_{i=1}^{N}y_i-\hat{\beta}\frac{1}{N}\sum_{i=1}^{N}\int f_i dg_\mu \quad (31)$$

$$S_{yf}=\frac{\sum_{i=1}^{N}\left[y_i-\frac{1}{N}\sum_{i=1}^{N}y_i\right]\left[\int f_i dg_{\mu^*}-\frac{1}{N}\sum_{k=1}^{N}\int f_k dg_{\mu^*}\right]}{N-1} \quad (32)$$

$$S_{ff}=\frac{\sum_{i=1}^{N}\left[\int f_i dg_{\mu^*}-\frac{1}{N}\sum_{k=1}^{N}\int f_k dg_{\mu^*}\right]^2}{N-1} \quad (33)$$

VII. Experiment and Result

A. Education Data

The total scores of 60 students from a junior high school in Taiwan are used for this research. The examinations of four courses, physics and chemistry, biology, geoscience and mathematics, are used as independent variables, the score of the Basic Competence Test of junior high school is used as a dependent variable.

The data of all variables listed in Table III is applied to evaluate the performances of four Choquet integral regression models with P-measure, λ-measure and δ-measure based on γ-support respectively, a ridge regression model, and a multiple linear regression model by using 5-fold cross validation method to compute the mean square error (MSE) of the dependent variable. The formula of MSE is

$$MSE=\frac{1}{N}\sum_{i=1}^{N}(y_i-\hat{y}_i)^2 \quad (34)$$

The singleton measures, γ-support of the P-measure, λ-measure and δ-measure are listed as follows which can be obtained by using the formula (19).

{0.2488, 0.2525, 0.2439, 0.2547}

For any fuzzy measure, μ-measures, once the fuzzy support of the μ-measure is given, all event measures of μ can be found, and then, the Choquet integral based on μ and the Choquet integral regression equation based on μ can also be found by using above corresponding formulae.

The experimental results of five forecasting models are listed in Table I. We find that the Choquet integral regression model with δ-measure based on γ-support outperforms other forecasting regression models.

TABLE I MSE OF REGRESSION MODELS

Regression model		5-fold CV MSE
Choquet Integral Regression model	measure	
	δ	48.7672
	λ	49.1832
	p	53.9582
Ridge regression		59.1329
Multiple linear regression		65.0664

B. *Fat Data* [3, 5, 9, 10, 11]

In this study, anthropometric dimensions were measured following a standard protocol [11]. High was measured to the nearest 0.1 cm using anthropometers. Body weight was measured to the nearest 0.1 kg at the same time the bioelectric impedance was measured using a body fat analyzer (TBF310; Tanita, Tokyo, Japan) to estimate the percentage of body fat (%fat). Skinfold thicknesses at biceps, triceps, subscapular, and suprailiac of the right side of body were measured with GMP skinfold calipers (Siber Hegener and Co. Ltd, Switzerland). The measurements were performed by one experienced operator that took two repeated measurements at the test site of the same subject. The mean of the two readings from each site was used to calculate body composition.

A real data set with 128 samples from a elementary school in Taiwan including the independent variables, 4 Skinfold determination values, and the dependent variable, the measurements of the BIA of each student listed in Table IV is applied to evaluate the performances of three Choquet integral regression models with P-measure, λ-measure and L-measure based on γ-support respectively, a ridge regression model, and a multiple linear regression model by using 5-fold cross validation method to compute the mean square error (MSE) of the dependent variable.

The singleton measures, γ-support of the P-measure, λ-measure and δ-measure are listed as follows which can be obtained by using the formula (19).

{0.2396, 0.2466, 0.254, 0.2596}

The formulas of MSE is by using 5-fold cross validation method to compute the mean square error (MSE) of the depen dent variable.

For any fuzzy measure, μ-measures, once the fuzzy support of the μ-measure is given, all event measures of μ can be found, and then, the Choquet integral based on μ and the Choquet integral regression equation based on μ can also be found.

The singleton measures, γ-support of the P-measure, λ-measure and L-measure can be obtained by using the formulas (6).

The experimental results of five forecasting models are listed in Table II. We find that the Choquet integral regression model with δ-measure based on γ-support outperforms other forecasting regression models.

TABLE II MSE OF REGRESSION MODELS

Regression model		5-fold CV MSE
Choquet Integral Regression model	measure	
	δ	14.4228
	λ	14.9218
	p	18.3846
Ridge regression		15.7434
Multiple linear regression		16.1122

VIII. CONCLUSION S

In this paper, multivalent fuzzy measure, δ-measure, is proposed. This new measure is proved that it is of closed form with infinitely many solutions, and it can be considered as an extension of the two well known fuzzy measures, λ-measure and P-measure. By using 5-fold cross-validation RMSE, an experiment is conducted for comparing the performances of a multiple linear regression model, a ridge regression model, and the Choquet integral regression model with respect to P-measure, λ-measure, and our proposed δ-measure based on γ-support respectively. The result shows that the Choquet integral regression models with respect to the proposed δ-measure based on γ-support outperforms other forecasting models.

In the future, we will apply the proposed Choquet integral regression model with fuzzy measure based on γ-support to develop multiple classifier system.

REFERENCES

[1] A. E. Hoerl, R. W. Kenard, and K. F. Baldwin, Ridge Regression: Some Simulation, *Communications in Statistics*, vol. 4, No. 2, pp. 105-123, 1975.

[2] Z. Wang, and G. J. Klir, *Fuzzy Measure Theory*, Plenum Press, New York, 1992.

[3] M. Sugeno, *Theory of Fuzzy Integrals and its Applications*, unpublished doctoral dissertation, Tokyo Institute of Technology, Tokyo, Japan, 1974.

[4] L. A. Zadeh, *Fuzzy Sets and Systems*, vol. 1, pp. 3, 1978.

[5] G. Choquet, Theory of Capacities, *Annales de l'Institut Fourier*, vol. 5, pp. 131-295, 1953.

[6] H.-C. Liu, C.-C. Chen, D.-B. Wu, and Y.-D. Jheng, A New Weighting Method for Detecting Outliers in IPA Based on Choquet Integral, *IEEE International conference on Industrial Engineering and Engineering Management* 2007, Singapore, December 2007.

[7] Liu, H.-C. Tu, Y.-C. Chen, C.-C. and Weng, W.-S. The Choquet Integral with Respect to λ-Measure Based on γ-support, *2008 International Conferences on Machine Learning and Cybernetics*, Kunming, China, July 2008.

[8] J.-I Shieh, H.-H. Wu, H.-C. Liu, Applying Complexity-based Choquet Integral to Evaluate Students' Performance, *Expert Systems with Applications*, 36, pp. 5100-5106, 2009.

[9] S. Srivastava, A. Bansal, D. Chopra, G. Goe, Modeling and Control of a Choquet Fuzzy Integral Based Controller on a Real Time System. *WSEAS Transactions on Systems*, vol. 7, no. 5, pp. 1571-1578, July 2006.

[10] H.-C. Liu, Chen, C.-C., Chen, G.-S., & Jheng, Y.-D. (2007). Power-transformed-measure and its Choquet Integral Regression Model, *Proc. of 7th WSEAS International Conference on Applied Computer Science* (ACS'07), pp.101-104, 2007.

[11] J. S. Weiner, and J. A. Lourie, *Practical Human Biology*, Academic Press, London, 1981.

[12] H.-C. Liu, C. C. Chen, D. B. Wu, & Y. D. Jheng, A New Weighting Method for Detecting Outliers in IPA Based on Choquet Integral. *Proceedings of International Conference on Industrial Engineering & Engineering Management* (IEEM2007), pp. 2015-2019, Dec., 2007.

[13] G.S. Chen, Y.-D. Jheng, H.-C. Yao, and H.-C. Liu, Stroke Order Computer-based Assessment with Fuzzy Measure Scoring. *WSEAS Transactions Information Science & Applications*, vol. 2, no. 5, 62-68, 2008.

TABLE III THE DATA SET WITH FOUR COURSES AND SCIENCE SCORES OF THE BCT

No.	C1	C2	C3	C4	BCT	No.	C1	C2	C3	C4	BCT
1	72	66	78	72	19	31	66	68	75	74	25
2	86	80	82	81	35	32	68	70	74	76	40
3	56	63	69	75	21	33	57	65	75	70	24
4	78	86	86	86	33	34	74	70	80	75	35
5	66	72	80	76	23	35	49	60	69	64	13
6	68	74	77	80	28	36	51	60	63	64	18
7	74	86	87	88	44	37	58	64	68	66	32
8	54	56	62	68	7	38	73	78	84	81	39
9	71	74	80	77	26	39	56	56	65	61	6
10	68	70	80	75	33	40	61	62	70	70	25
11	53	56	70	63	22	41	57	60	68	64	23
12	67	70	80	75	35	42	57	64	67	70	26
13	70	66	70	74	13	43	50	52	68	60	7
14	60	65	75	70	23	44	84	80	76	72	49
15	68	68	78	76	35	45	62	66	76	71	22
16	58	66	76	71	37	46	70	74	78	82	32
17	61	66	72	78	33	47	69	70	80	75	26
18	68	68	80	74	26	48	63	74	74	74	42
19	56	66	76	71	21	49	66	78	80	82	39
20	59	62	70	78	29	50	67	70	80	75	31
21	62	64	76	70	36	51	56	65	75	70	23
22	71	72	78	75	26	52	50	54	66	60	18
23	74	63	69	75	12	53	71	75	85	80	41
24	59	70	80	76	37	54	74	77	80	85	26
25	75	75	85	80	39	55	71	72	76	80	31
26	73	78	84	81	24	56	60	65	75	70	21
27	62	68	72	74	29	57	59	57	70	68	17
28	77	74	80	76	42	58	50	56	65	68	13
29	63	60	68	69	17	59	72	76	80	78	38
30	56	61	75	68	22	60	81	76	78	80	33

C1 : physics and chemistry
C2 : biology
C3 : geoscience
C4 : mathematics
BCT : Basic Competence Test of nature science

Table IV Measurements of BIA and four skinfold determinations of percent body fat

No	BIA	biceps	triceps	Sub-scapular	Sup-railiac	No	BIA	biceps	triceps	Sub-scapular	Sup-railiac
1	11.8	15.8	16.2	27.6	16.3	65	22.8	27.4	32.2	33.6	24.5
2	17.8	20.4	12.8	35.6	23.0	66	5.4	14.4	8.0	17.2	17.5
3	6.2	10.4	9.4	12.6	8.2	67	8.6	9.6	11.4	14.4	14.8
4	7.0	10.2	5.6	10.6	13.6	68	7.4	9.6	8.0	10.2	13.4
5	5.2	10.8	12.0	13.8	11.8	69	20.2	29.2	28.2	40.0	23.9
6	20.2	24.6	39.8	40.0	34.7	70	28.4	30.6	38.6	39.8	39.5
7	14.8	18.8	22.4	24.8	18.6	71	6.8	10.4	8.8	14.6	15.2
8	12.4	18.2	20.2	27.6	18.4	72	11.0	12.4	10.4	14.4	11.7
9	17.0	20.6	21.2	27.8	21.6	73	20.8	30.8	38.8	40.0	37.7
10	16.6	22.6	18.8	32.6	16.6	74	25.0	27.6	40.0	40.0	29.6
11	10.2	13.4	8.4	12.8	15.1	75	4.8	6.2	5.6	7.0	9.7
12	21.0	26.7	36.2	40.0	23.9	76	8.4	7.0	8.0	11.8	15.6
13	10.0	13.4	17.4	17.8	18.1	77	11.0	14.0	10.0	12.4	17.7
14	9.8	13.8	11.6	19.2	15.5	78	10.6	11.6	7.0	14.4	12.9
15	8.6	10.8	11.4	14.4	8.5	79	12.4	16.8	17.0	25.8	14.9
16	21.8	25.6	36.6	36.8	31.7	80	11.4	14.0	20.8	26.4	25.1
17	25.2	29.4	30.2	30.6	31.2	81	12.8	15.8	20.2	19.8	16.4
18	10.2	11.0	10.2	14.8	12.2	82	12.0	15.6	16.0	22.8	17.3
19	10.4	15.8	13.0	25.6	16.6	83	13.0	20.4	24.0	27.0	26.2
20	17.6	22.6	23.2	34.6	23.2	84	14.2	15.4	22.4	22.8	18.9
21	12.6	12.4	14.2	16.0	14.2	85	11.0	16.4	14.2	15.8	17.9
22	9.0	11.2	9.4	11.8	9.3	86	22.4	29.8	35.0	36.2	28.5
23	12.2	19.2	17.4	27.8	19.7	87	6.4	7.6	8.6	11.4	12.1
24	4.6	7.0	8.8	11.2	7.2	88	6.8	10.6	9.6	14.6	12.2
25	6.4	8.8	11.0	12.6	10.8	89	16.2	18.4	27.2	27.4	24.9
26	23.8	29.0	37.0	35.0	30.7	90	22.4	26.8	25.4	33.4	30.4
27	8.4	15.8	17.8	23.0	21.6	91	9.6	11.2	10.4	18.0	11.7
28	12.2	16.6	16.4	20.6	18.7	92	10.8	17.2	24.0	24.8	21.9
29	7.2	12.8	8.6	18.6	15.2	93	13.0	16.2	12.4	18.4	14.2
30	21.4	31.2	31.4	39.4	28.6	94	5.6	12.4	11.4	15.6	14.5
31	18.2	23.0	40.0	40.0	28.2	95	19.4	25.0	36.2	39.0	29.9
32	9.2	12.6	40.0	17.8	16.0	96	14.4	22.4	29.8	35.0	24.8
33	10.2	18.8	17.8	20.8	18.4	97	25.4	29.4	37.0	40.0	24.6
34	19.2	24.4	35.2	35.0	34.1	98	9.4	11.2	11.4	12.4	8.9
35	6.8	12.0	8.0	14.4	16.1	99	17.4	22.6	19.4	31.6	22.7
36	16.8	20.8	25.6	27.8	20.7	100	24.0	30.8	40.0	40.0	29.4
37	35.8	38.6	40.0	40.0	30.1	101	3.8	6.0	6.4	6.8	10.8
38	10.0	11.6	10.4	18.6	8.3	102	11.0	19.4	11.6	18.4	13.7
39	5.4	12.2	12.4	21.4	19.2	103	22.6	24.4	40.0	40.0	33.3
40	11.2	18.0	23.6	30.8	22.1	104	9.2	10.0	11.0	19.2	13.4
41	5.4	11.2	6.8	11.6	11.9	105	18.2	19.0	31.0	29.4	24.5
42	7.6	8.4	9.4	13.6	12.8	106	6.8	12.4	14.0	17.8	14.1
43	6.6	9.8	9.6	12.0	9.3	107	7.4	11.6	10.0	16.0	11.0
44	32.4	37.2	40.0	40.0	18.2	108	9.2	10.6	12.4	14.4	12.7
45	7.8	14.0	11.0	17.8	31.3	109	29.4	23.6	39.8	40.0	37.4
46	17.8	26.6	34.2	40.0	22.8	110	6.8	7.8	9.8	12.8	10.9
47	22.0	27.8	38.2	39.4	23.6	111	12.4	14.6	15.8	19.8	16.9
48	14.4	15.8	18.8	23.8	14.9	112	8.2	9.8	9.2	16.0	14.5
49	15.8	18.4	21.4	24.0	24.9	113	16.4	20.8	25.2	30.4	24.9
50	7.4	12.8	10.2	17.0	14.3	114	9.4	11.4	12.0	21.8	14.3
51	16.2	29.0	21.6	29.8	24.1	115	16.4	22.4	33.2	36.8	25.1
52	6.0	7.4	7.6	9.8	8.6	116	7.0	11.4	13.8	17.4	11.2

53	12.2	15.4	16.2	18.8	17.8	117	10.4	12.6	14.8	23.8	18.0
54	11.6	12.0	9.8	13.0	8.9	118	5.6	8.2	10.2	8.6	7.7
55	17.8	22.6	38.0	31.0	24.8	119	10.8	11.8	17.8	21.2	19.9
56	13.2	16.8	18.6	23.4	20.7	120	9.6	15.8	14.4	19.4	18.6
57	4.4	7.2	8.2	9.8	14.3	121	5.0	6.8	7.4	9.4	6.0
58	16.2	21.8	28.2	32.6	27.2	122	9.8	12.2	12.4	15.4	13.5
59	11.4	19.4	28.8	32.8	22.3	123	13.8	18.0	16.4	21.0	19.3
60	11.2	13.0	18.8	22.6	21.9	124	8.8	12.8	9.8	11.8	13.3
61	8.6	11.4	7.2	10.2	7.5	125	15.8	21.0	35.4	39.8	27.3
61	20.4	26.2	31.0	32.8	25.8	126	10.8	16.6	15.6	23.2	16.5
63	7.0	8.8	11.6	9.4	12.0	127	9.0	10.6	10.0	16.8	11.9
64	14.6	17.4	12.8	16.8	14.7	128	8.8	12.4	10.0	10.8	11.3