Application Of Alternating Group Explicit Method For Parabolic Equations

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Abstract: - Based on the concept of decomposition, two alternating group explicit methods are constructed for 1D convection-diffusion equation with variable coefficient and 2D diffusion equations respectively. Both the two methods have the property of unconditional stablility and intrinsic parallelism . Numerical results show the two methods are of high accuracy.

Key-Words: - alternating group method; parallel computing; explicit scheme; parabolic equation; finite difference

1 Introduction

Parabolic equations are widely used in describing many physical phenomena such as fluid flowing, river and atmosphere pollution and so on. Researches on finite difference methods for them are getting more and more popular. Many finite difference methods have been presented so far [1-4], which are sorted by explicit and implicit methods in general. As we all know, explicit methods are easy for computing, but are commonly short in stability and accuracy. Most of implicit methods are of good stability, while are not suitable for parallel computing. Thus the task of presenting finite difference methods with good stability and property of parallelism is of important theoretic and practical meaning. D. J. Evans presented an AGE method in [5] originally. The AGE method is used in computing by applying the special combination of several asymmetry schemes to a group of grid points. Then the numerical solutions at the group of points can be obtained independently, and the computation in the whole domain can be divided into many sub-domains. Furthermore, by alternating use of asymmetry schemes at adherent grid points and different time levels, the AGE method can lead to the property of unconditional stability. The AGE method is soon applied to convection-diffusion equations in [6]. The AGE method is widely cared for it is simple for computing, unconditionally stable, and suitable for parallelism. Under the enlightenment of the AGE method, Baolin Zhang and S. Zhu gave alternating block explicit-implicit methods in [7-9] , while Rohallah Tavakoli derived a class of domain-split method for diffusion equations in [10, 11]. Several AGE methods are given for two-point linear and non-linear boundary value problems in [12-13]. We notice most of the AGE methods are aimed at constant coefficient equation and 1D problems. Researches on variable coefficient equations and 2D problems have been scarcely presented.

Results about the existence and uniqueness of theoretic solution for parabolic equations can be found in [14-17]

We organize the paper as follows: First we present a class of alternating group explicit method for 1D convection-diffusion equations with variant coefficient in section 2, and give stability analysis for it in section 3. Then we apply the construction of the method to 2D diffusion equations in section 4, also the stability analysis is finished. In order to verify the effectiveness of the two methods, we present numerical results comparing with other known AGE method [9] and Crank-Nicolson scheme in section 5. Some conclusions are given at the end of the paper.

2 The Parallel Alternating Group Explicit (AGE) Method

In this section, we consider the following convection -diffusion equation

$$\frac{\partial u}{\partial t}+a(x)\frac{\partial u}{\partial x}=b(x)\frac{\partial^2 u}{\partial x^2},0\le x\le 1,0\le t\le T,a(x)\ge a_0>0,b(x)>0 \tag{1}$$

with initial and boundary value

$$\begin{cases} u(0,t)=f_1(x),u(1,t)=f_2(x), \\ u(x,0)=g(t). \end{cases} \tag{2}$$

Let h and τ be the grid step size in the x and t directions respectively. $h=1/m$, $x_i=ih(i=0,1......m)$, $t_n=n\tau(n=0,1......T/\tau)$. The grid point (x_i,t_n) is denoted by (i,n), and the numerical solution is u_i^n, while the exact solution is $u(x_i,t_n)$. Let $r=\tau/h^2$.

In order to get the solution of (n+1)-th time level while the solution of n-th time level known, we present eight basic schemes using the second kind of saul'yev asymmetry schemes, which will be used in the construction of the alternating group method. In simple, let $\frac{a_i rh}{4}=p_i$, $\frac{b_i r}{4}=q_i$.

$$(2p_i-2q_i)u_{i+1}^{n+1}+(4+2q_i)u_i^{n+1} \\ =2q_iu_{i+1}^n+(4-6q_i)u_i^n+(2p_i+4q_i)u_{i-1}^n \tag{3}$$

$$(2p_i-4q_i)u_{i+1}^{n+1}+(4+6q_i)u_i^{n+1}+(-p_i-2q_i)u_{i-1}^{n+1}=(4-2q_i)u_i^n+(p_i+2q_i)u_{i-1}^n \tag{4}$$

$$(p_i-2q_i)u_{i+1}^{n+1}+(4+6q_i)u_i^{n+1}+(-2p_i-4q_i)u_{i-1}^{n+1}=(-p_i+2q_i)u_{i+1}^n+(4-2q_i)u_i^n \tag{5}$$

$$(4+2q_i)u_i^{n+1}+(-2p_i-2q_i)u_{i-1}^{n+1}=(-2p_i+4q_i)u_{i+1}^n+(4-6q_i)u_i^n+2q_iu_{i-1}^n \tag{6}$$

$$-2q_iu_{i+1}^{n+1}+(4+6q_i)u_i^{n+1}+(-2p_i-4q_i)u_{i-1}^{n+1}=(-2p_i+2q_i)u_{i+1}^n+(4-2q_i)u_i^n \tag{7}$$

$$(4+2q_i)u_i^{n+1}+(-p_i-2q_i)u_{i-1}^{n+1}=(-2p_i+4q_i)u_{i+1}^n+(4-6q_i)u_i^n+(p_i+2q_i)u_{i-1}^n \tag{8}$$

$$(4+2q_i)u_i^{n+1}+(p_i-2q_i)u_{i+1}^{n+1}=(-p_i+2q_i)u_{i+1}^n+(4-6q_i)u_i^n+(p_i+4q_i)u_{i-1}^n \tag{9}$$

$$(2p_i-4q_i)u_{i+1}^{n+1}+(4+6q_i)u_i^{n+1}-2q_iu_{i-1}^{n+1}=(4-2q_i)u_i^n+(2p_i+2q_i)u_{i-1}^n \tag{10}$$

Based on (3)-(10), we present four basic computing groups as follows:

"G1" group: four grid points are involved, and (3)-(6) are used at each grid point respectively.

"G2" group: four grid points are involved, and (7)-(10) are used at each grid point respectively.

"GL" group: four grid points are involved, and (7)-(8) are used at each grid point respectively.

"GR" group: four grid points are involved, and (9)-(10) are used at each grid point respectively.

The purpose of the paper is to get the solution of the (n+1)-th and the (n+2)-th time level with the solution of the n-th time level known.

Let $m\text{-}1 = 4p$, here p is an integer, then the alternating group method will be presented as following:

First at the (n+1)-th time level, we will have p "G1" groups. (3), (4), (5), (6) are used in each group.

Second at the (n+2)-th time level, we will have (p+1) point groups. (7) and (8) are used to solve u_1^{n+2}, u_2^{n+2}. (9) and (10) are used to solve $u_{m-2}^{n+2}, u_{m-1}^{n+2}$, while the rest (4p-4) inner grid points are divided into (p-1) "G2" groups, and (7), (8), (9), (10) are used in each group.

Thus the alternating group method is established by alternating use of the schemes (3)-(10) in the two time levels. We notice the computation in each group can be finished independently.

Let $\boldsymbol{U}^n=(u_1^n,u_2^n,......,u_{m-1}^n)^T$, then we can denote the alternating group explicit method as below

$$\begin{cases}(\boldsymbol{I}+r\boldsymbol{G}_1)\boldsymbol{U}^{n+1}=(\boldsymbol{I}-r\boldsymbol{G}_2)\boldsymbol{U}^n+\boldsymbol{F}_1^n,\\(\boldsymbol{I}+r\boldsymbol{G}_2)\boldsymbol{U}^{n+2}=(\boldsymbol{I}-r\boldsymbol{G}_1)\boldsymbol{U}^{n+1}+\boldsymbol{F}_2^n.\end{cases} \tag{11}$$

Here

$$\boldsymbol{F}_1^n=((2p_1+4q_1)u_0^n,0,0......0,(-2p_{m-1}+4q_{m-1})u_m^n)^{\mathrm{T}}$$

$$\boldsymbol{F}_2^n=((2p_1+4q_1)u_0^{n+1},0,0......0,(-2p_{m-1}+4q_{m-1})u_m^{n+1})^{\mathrm{T}}$$

$$,\ \boldsymbol{G}_1=\begin{bmatrix}\boldsymbol{G}_{11}&&&&\\&\cdots&&&\\&&\boldsymbol{G}_{1i}&&\\&&&\cdots&\\&&&&\boldsymbol{G}_{1p}\end{bmatrix}_{(m-1)\times(m-1)}$$

$$\boldsymbol{G}_2=\begin{bmatrix}\boldsymbol{G}_{21}&&&&\\&\boldsymbol{G}_{22}&&&\\&&\cdots&&\\&&&\boldsymbol{G}_{2p}&\\&&&&\boldsymbol{G}_{2\times(p+1)}\end{bmatrix}_{(m-1)\times(m-1)}$$

$$\boldsymbol{G}_{1i}=\begin{bmatrix}\boldsymbol{G}_{1i1}&\boldsymbol{G}_{1i2}\\\boldsymbol{G}_{1i3}&\boldsymbol{G}_{1i4}\end{bmatrix},i=1...p$$

$$\boldsymbol{G}_{1i1}=\begin{bmatrix}2q_{(i-1)\times4+1}&2p_{(i-1)\times4+1}-2q_{(i-1)\times4+1}\\-p_{(i-1)\times4+2}-2q_{(i-1)\times4+2}&6q_{(i-1)\times4+2}\end{bmatrix}$$

$$\boldsymbol{G}_{1i2}=\begin{bmatrix}0&0\\2p_{(i-1)\times4+2}-4q_{(i-1)\times4+2}&0\end{bmatrix}$$

$$\boldsymbol{G}_{1i3}=\begin{bmatrix}0&-2p_{(i-1)\times4+3}-4q_{(i-1)\times4+3}\\0&0\end{bmatrix}$$

$$\boldsymbol{G}_{1i4}=\begin{bmatrix}6q_{(i-1)\times4+3}&p_{(i-1)\times4+3}-2q_{(i-1)\times4+3}\\-2p_{(i-1)\times4+4}-2q_{(i-1)\times4+4}&2q_{(i-1)\times4+4}\end{bmatrix}$$

$$\boldsymbol{G}_{2\times(p+1)}=\begin{bmatrix}2q_{m-2}&p_{m-2}-2q_{m-2}\\-2q_{m-1}&6q_{m-1}\end{bmatrix}$$

$$\boldsymbol{G}_{21}=\begin{bmatrix}6q_1&-2q_1\\-p_2-2q_2&2q_2\end{bmatrix}$$

$$\boldsymbol{G}_{2i}=\begin{bmatrix}\boldsymbol{G}_{2i1}&\boldsymbol{G}_{2i2}\\\boldsymbol{G}_{2i3}&\boldsymbol{G}_{2i4}\end{bmatrix},i=2...p$$

$$\boldsymbol{G}_{2i1}=\begin{bmatrix}2q_{(i-2)\times4+3}&p_{(i-2)\times4+3}-2q_{(i-2)\times4+3}\\-2q_{(i-2)\times4+4}&6q_{(i-2)\times4+4}\end{bmatrix}$$

$$\boldsymbol{G}_{2i2}=\begin{bmatrix}0&0\\2p_{(i-2)\times4+4}-4q_{(i-2)\times4+4}&0\end{bmatrix}$$

$$\boldsymbol{G}_{2i3} = \begin{bmatrix} 0 & -2p_{(i-2)\times4+5} - 4q_{(i-2)\times4+5} \\ 0 & 0 \end{bmatrix}$$

$$\boldsymbol{G}_{2i4} = \begin{bmatrix} 6q_{(i-2)\times4+5} & -2q_{(i-2)\times4+5} \\ -p_{(i-2)\times4+6} - 2q_{(i-2)\times4+6} & 2q_{(i-2)\times4+6} \end{bmatrix}$$

The alternating use of the asymmetry schemes (3)-(10) can lead to partly counteracting of truncation error, and then can increase the numerical accuracy. On the other hand, grouping computation can be obviously obtained. Thus computing in the whole domain can be divided into many sub-domains. So the method has the obvious property of parallelism.

3 Analysis Of Stability

Lemma 1(Kellogg) [18] Assume $\theta>0$ and $(\boldsymbol{G}+\boldsymbol{G}^{\mathrm{T}})$ is non negative definite real matrix, then $(\theta\boldsymbol{I}+\boldsymbol{G})^{-1}$ exists, and the fallowing inequalities hold

$$\begin{cases} \|(\theta\boldsymbol{I}+\boldsymbol{G})^{-1}\|_2 \le \theta^{-1}, \\ \|(\theta\boldsymbol{I}-\boldsymbol{G})(\theta\boldsymbol{I}+\boldsymbol{G})^{-1}\|_2 \le 1. \end{cases} \tag{12}$$

Theorem 1 The alternating group method (11) is of absolute stability.

Proof: From the construction of the matrices $(\boldsymbol{G}_1+\boldsymbol{G}_1^{\mathrm{T}})$ and $(\boldsymbol{G}_2+\boldsymbol{G}_2^{\mathrm{T}})$, we can see they are non negative definite real matrices. Then we have $\|(\boldsymbol{I}-r\boldsymbol{G}_1)(\boldsymbol{I}+r\boldsymbol{G}_1)^{-1}\|_2 \le 1$, $\|(\boldsymbol{I}-r\boldsymbol{G}_2)(\boldsymbol{I}+r\boldsymbol{G}_2)^{-1}\|_2 \le 1$. Let n is an even number, then we have $\boldsymbol{U}^{n+2}=\boldsymbol{G}\boldsymbol{U}^n$ .Here $\boldsymbol{G}$ is growth matrix, $\boldsymbol{G}=(\boldsymbol{I}+r\boldsymbol{G}_2)^{-1}(\boldsymbol{I}-r\boldsymbol{G}_1)(\boldsymbol{I}+r\boldsymbol{G}_1)^{-1}(\boldsymbol{I}-r\boldsymbol{G}_2)$.

Let

$\bar{\boldsymbol{G}}=(\boldsymbol{I}+r\boldsymbol{G}_2)\boldsymbol{G}(\boldsymbol{I}+r\boldsymbol{G}_2)^{-1}=(\boldsymbol{I}-r\boldsymbol{G}_1)(\boldsymbol{I}+r\boldsymbol{G}_1)^{-1}(\boldsymbol{I}-r\boldsymbol{G}_2)(\boldsymbol{I}+r\boldsymbol{G}_2)^{-1}$.

By Lemma 1, we have $\rho(\boldsymbol{G})=\rho(\bar{\boldsymbol{G}})\le\|\bar{\boldsymbol{G}}\|_2\le 1$.

Therefore, the alternating group method (11) is of absolute stability.

4 Application Of AGE Method For 2D Diffusion Equations

Considering the initial boundary value problem of 2D diffusion equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial^2 y}, \\ 0 \le x \le 1, 0 \le y \le 1, 0 \le t \le T \\ u(x,y,0) = f(x,y) \\ u(0,y,t) = g_1(y,t), \\ u(1,y,t) = g_2(y,t) \\ u(x,0,t) = f_1(x,t), \\ u(x,1,t) = f_2(x,t) \end{cases} \tag{13}$$

The domain : [0, 1] × [0, 1] × [0, T] will be divided into (m × m × N) meshes with spatial step size $h=\Delta x=\Delta y=1/m$ in x, y direction and the time step size $\tau=T/N=\Delta t$. Grid points are denoted by (x_i, y_j, t_n) or (i, j, n), $x_i=ih(i=0,1......m)$, $y_j=jh(j=0,1......m)$, $t_n=n\tau(n=0,1......T/\tau)$.

The numerical solution of (20)-(21) is denoted by $u_{i,j}^n$, while the exact solution $u(x_i,y_j,t_n)$. $r=\tau/h^2$.

We present 16 basic asymmetry schemesby use of the second class of saul'yev schemes as follows (Figure 1-16):

$$(1+r)u_{i,j}^{n+1} - \frac{r}{2}u_{i+1,j}^{n+1} - \frac{r}{2}u_{i,j+1}^{n+1} = ru_{i-1,j}^{n} + ru_{i,j-1}^{n} + (1-3r)u_{i,j}^{n} + \frac{r}{2}u_{i,j+1}^{n} + \frac{r}{2}u_{i+1,j}^{n} \tag{14}$$

$$-\frac{r}{2}u_{i,j}^{n+1} + (1+2r)u_{i+1,j}^{n+1} - ru_{i+2,j}^{n+1} - \frac{r}{2}u_{i+1,j+1}^{n+1} = ru_{i+1,j-1}^{n} + \frac{r}{2}u_{i,j}^{n} + (1-2r)u_{i+1,j}^{n} + \frac{r}{2}u_{i+1,j+1}^{n} \tag{15}$$

$$-ru_{i+1,j}^{n+1} + (1+2r)u_{i+2,j}^{n+1} - \frac{r}{2}u_{i+3,j}^{n+1} - \frac{r}{2}u_{i+2,j+1}^{n+1} = ru_{i+2,j-1}^{n} + (1-2r)u_{i+2,j}^{n} + \frac{r}{2}u_{i+3,j}^{n} + \frac{r}{2}u_{i+2,j+1}^{n} \tag{16}$$

$$-\frac{r}{2}u_{i+2,j}^{n+1} + (1+r)u_{i+3,j}^{n+1} - \frac{r}{2}u_{i+3,j+1}^{n+1} = ru_{i+3,j-1}^{n} + (1-3r)u_{i+3,j}^{n} + ru_{i+4,j}^{n} + \frac{r}{2}u_{i+3,j+1}^{n} + \frac{r}{2}u_{i+2,j} \tag{17}$$

$$-\frac{r}{2}u^{n+1}_{i,j}+(1+2r)u^{n+1}_{i,j+1}-\frac{r}{2}u^{n+1}_{i+1,j+1}-ru^{n+1}_{i,j+2}$$

$$=ru^{n}_{i-1,j+1}+(1-2r)u^{n}_{i,j+1}+\frac{r}{2}u^{n}_{i,j}+\frac{r}{2}u^{n}_{i+1,j+1} \tag{18}$$

$$-\frac{r}{2}u^{n+1}_{i+1,j}+(1+3r)u^{n+1}_{i+1,j+1}-\frac{r}{2}u^{n+1}_{i,j+1}-ru^{n+1}_{i+2,j+1}$$

$$-ru^{n+1}_{i+1,j+2}=\frac{r}{2}u^{n}_{i+1,j}+\frac{r}{2}u^{n}_{i,j+1}+(1-r)u^{n}_{i+1,j+1} \tag{19}$$

$$-\frac{r}{2}u^{n+1}_{i+2,j}-ru^{n+1}_{i+1,j+1}+(1+3r)u^{n+1}_{i+2,j+1}-\frac{r}{2}u^{n+1}_{i+3,j+1}$$

$$-ru^{n+1}_{i+2,j+2}=\frac{r}{2}u^{n}_{i+2,j}+\frac{r}{2}u^{n}_{i+3,j+1}+(1-r)u^{n}_{i+2,j+1} \tag{20}$$

$$-\frac{r}{2}u^{n+1}_{i+3,j}-\frac{r}{2}u^{n+1}_{i+2,j+1}+(1+2r)u^{n+1}_{i+3,j+1}-ru^{n+1}_{i+3,j+2}$$

$$=\frac{r}{2}u^{n}_{i+3,j}+\frac{r}{2}u^{n}_{i+2,j+1}+(1-2r)u^{n}_{i+3,j+1}+ru^{n}_{i+4,j+1} \tag{21}$$

$$-ru^{n+1}_{i,j+1}+(1+2r)u^{n+1}_{i,j+2}-\frac{r}{2}u^{n+1}_{i+1,j+2}-\frac{r_2}{2}u^{n+1}_{i,j+3}$$

$$=ru^{n}_{i-1,j+2}+(1-2r)u^{n}_{i,j+2}+\frac{r}{2}u^{n}_{i+1.j+2}+\frac{r}{2}u^{n}_{i,j+3} \tag{22}$$

$$-ru^{n+1}_{i+1,j+1}-\frac{r}{2}u^{n+1}_{i,j+2}+(1+3r)u^{n+1}_{i+1,j+2}-ru^{n+1}_{i+2,j+2}$$

$$-\frac{r}{2}u^{n+1}_{i+1,j+3}=\frac{r}{2}u^{n}_{i,j+2}+(1-r)u^{n}_{i+1,j+2}+\frac{r}{2}u^{n}_{i+1,j+3} \tag{23}$$

$$-ru^{n+1}_{i+2,j+1}-ru^{n+1}_{i+1,j+2}+(1+3r)u^{n+1}_{i+2,j+2}-\frac{r_1}{2}u^{n+1}_{i+3,j+2}$$

$$-\frac{r}{2}u^{n+1}_{i+2,j+3}=(1-r)u^{n}_{i+2,j+2}+\frac{r}{2}u^{n}_{i+3.j+2}+\frac{r}{2}u^{n}_{i+2,j+3} \tag{24}$$

$$-ru^{n+1}_{i+3,j+1}-\frac{r}{2}u^{n+1}_{i+2,j+2}+(1+2r)u^{n+1}_{i+3,j+2}-\frac{r}{2}u^{n+1}_{i+3,j+3}$$

$$=\frac{r}{2}u^{n}_{i+2,j+2}+(1-2r)u^{n}_{i+3,j+2}+\frac{r}{2}u^{n}_{i+3,j+3}+ru^{n}_{i+4,j+2} \tag{25}$$

$$-\frac{r}{2}u^{n+1}_{i,j+2}+(1+r)u^{n+1}_{i,j+3}-\frac{r}{2}u^{n+1}_{i+1,j+3}=ru^{n}_{i-1,j+3}$$

$$+\frac{r}{2}u^{n}_{i,j+2}+(1-3r)u^{n}_{i,j+3}+\frac{r}{2}u^{n}_{i+1.j+3}+ru^{n}_{i,j+4} \tag{26}$$

$$-\frac{r}{2}u^{n+1}_{i+1,j+2}-\frac{r}{2}u^{n+1}_{i,j+3}+(1+2r)u^{n+1}_{i+1,j+3}-ru^{n+1}_{i+2,j+3}$$

$$=\frac{r}{2}u^{n}_{i+1,j+2}+\frac{r}{2}u^{n}_{i,j+3}+(1-2r)u^{n}_{i+1,j+3}+ru^{n}_{i+1,j+4} \tag{27}$$

$$-\frac{r}{2}u^{n+1}_{i+2,j+2}-ru^{n+1}_{i+1,j+3}+(1+2r)u^{n+1}_{i+2,j+3}-\frac{r}{2}u^{n+1}_{i+3,j+3}$$

$$=\frac{r}{2}u^{n}_{i+2,j+2}+(1-2r)u^{n}_{i+2,j+3}+\frac{r}{2}u^{n}_{i+3,j+3}+ru^{n}_{i+2,j+4} \tag{28}$$

$$-\frac{r}{2}u^{n!}_{i+3,j+2}-\frac{r}{2}u^{n+1}_{i+2,j+3}+(1+r)u^{n+1}_{i+3,j+3}=\frac{r}{2}u^{n}_{i+3,j+2}$$

$$+\frac{r}{2}u^{n}_{i+2,j+3}+(1-3r)u^{n}_{i+3,j+3}+ru^{n}_{i+4,j+3}+ru^{n}_{i+3,j+4} \tag{29}$$

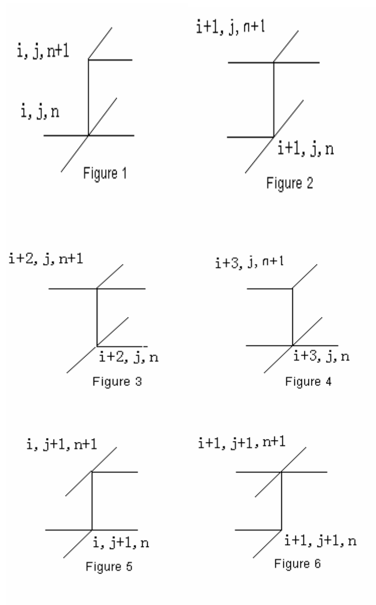

Figure 1 Figure 2 Figure 3 Figure 4 Figure 5 Figure 6

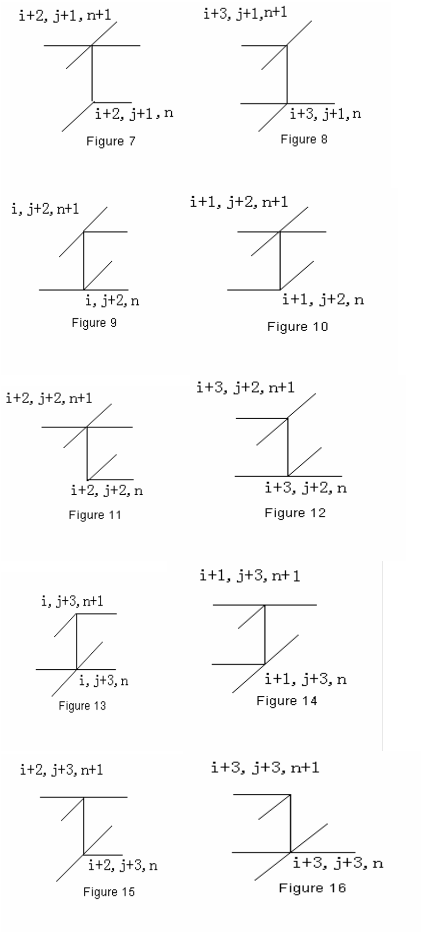

Figure 7 Figure 8 Figure 9 Figure 10 Figure 11 Figure 12 Figure 13 Figure 14 Figure 15 Figure 16

Let $m-1=4s+2$, s is an integer. Then we construct the alternating group schemes at the two adherent time levels as in Figure 17-18:

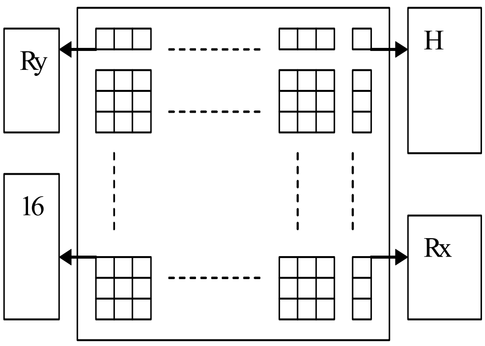

Figure 17 the grouping at n+1 time level

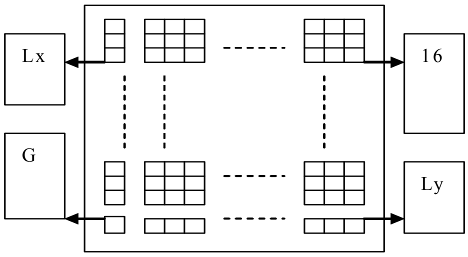

Figure 18 the grouping at n+2 time level

We present several basic computing groups as below: ("16 point" group) Let $\overline{u}_{i,j}^{n}=(\mathrm{u}_j^{\mathrm{n}},u_{j+1}^n,u_{j+2}^n,u_{j+3}^n)^T$, $u_{j+k}^n=(u_{i,j+k}^n,u_{i+1,j+k}^n,u_{i+2,j+k}^n,u_{i+3,j+k}^n)^T,k=0,1,2,3$

$$\overline{F}_{i,j}^{n}=(F_j^n,F_{j+1}^n,F_{j+2}^n,F_{j+3}^n)^T,$$

$$F_j^n=(ru_{i-1,j}^n+ru_{i,j-1}^n,ru_{i+1,j-1}^n,$$
$$ru_{i+2,j-1}^n,ru_{i+4.j}^n+ru_{i+3,j-1}^n)^T$$

$$F_{j+1}^n=(ru_{i-1,j+1}^n,0,0,ru_{i+4,j+1}^n)^T,$$

$$F_{j+2}^n=(ru_{i-1,j+2}^n,0,0,ru_{i+4,j+2}^n)^T,$$

$$F_{j+3}^n=(ru_{i-1,j+3}^n+ru_{i,j+4}^n,ru_{i+1,j+4}^n,$$
$$ru_{i+2,j+4}^n,ru_{i+3,j+4}^n+ru_{i+4,j+3}^n)^T$$

Then we debote the " 16 point" group as follows:

$$(I+rA_1)\overline{u}_{i,j}^{n+1}=(I-rB_1)\overline{u}_{i,j}^{n}+\overline{F}_{i,j}^{n} \tag{30}$$

Let

$$C_{11}=\begin{bmatrix}1 & -1/2\\ -1/2 & 2\end{bmatrix},C_{12}=\begin{bmatrix}2 & -1/2\\ -1/2 & 1\end{bmatrix},C_{13}=\begin{bmatrix}0 & -1\\ 0 & 0\end{bmatrix},$$

$$C_{14}=\begin{bmatrix}-1/2 & 0\\ 0 & -1/2\end{bmatrix},C_{15}=\begin{bmatrix}-1 & 0\end{bmatrix},$$

then it follows

$$A_1=\begin{bmatrix}A_{11} & A_{12} & & \\ A_{12} & A_{13} & A_{14} & \\ & A_{14} & A_{13} & A_{12}\\ & & A_{12} & A_{11}\end{bmatrix}_{16\times16},$$

$$B_1=\begin{bmatrix}B_{11} & B_{12} & & \\ B_{12} & B_{13} & & \\ & & B_{13} & B_{12}\\ & & B_{12} & B_{11}\end{bmatrix}_{16\times16},$$

$$A_{11}=\begin{bmatrix}C_{11} & C_{13}^{\ T}\\ C_{13} & C_{12}\end{bmatrix},\ A_{12}=\begin{bmatrix}C_{14} & O\\ O & C_{14}\end{bmatrix},$$

$$\begin{cases}A_{13}=I+A_{11}\\ A_{14}=2A_{12}\end{cases},\ B_{11}=I+\begin{bmatrix}C_{12} & O\\ O & C_{11}\end{bmatrix},$$

$$B_{12}=\begin{bmatrix} C_{14} & O \\ O & C_{14} \end{bmatrix},\ B_{13}=B_{11}-I.$$

("Lx" group)

According to the principle of row first and column second, (16), (17), (20), (21), (24), (25), (28), (29) are used to get the solution of the eight grid points as shown in figure 17-18.

Let $\overline{v}_{i,j}^{n+1}=(v_j^{n+1},v_{j+1}^{n+1},v_{j+2}^{n+1},v_{j+3}^{n+1})^T$,

$$v_{j+k}^{n+1}=(v_{1,j+k}^{n+1},v_{2,j+k}^{n+1})^T, k=0,1,2,3$$

$$\overline{w}_{i,j}^{n}=(w_j^n,w_{j+1}^n,w_{j+2}^n,w_{j+3}^n)^T,$$

$$w_j^n=(ru_{1,j-1}^n+ru_{0,j}^{n+1},ru_{2,j-1}^n+ru_{3,j}^n)^T,$$

$$w_{j+1}^n=(ru_{0,j+1}^{n+1},ru_{3,j+1}^n)^T,$$

$$w_{j+2}^n=(ru_{0,j+2}^{n+1},ru_{3,j+2}^n)^T,$$

$$w_{j+3}^n=(ru_{0,j+3}^{n+1}+ru_{1,j+4}^n,ru_{2,j+4}^n+ru_{3,j+3}^n)^T.$$

then we denote the "Lx" group as follows:

$$(I+rA_2)\overline{v}_{i,j}^{n+1}=(I-rB_2)\overline{v}_{i,j}^{n}+\overline{w}_{i,j}^{n} \quad (31)$$

$$A_2=\begin{bmatrix} A_{21} & A_{22} \\ A_{23} & A_{24} \end{bmatrix}_{8\times8},\ B_2=\begin{bmatrix} B_{21} & O \\ O & B_{22} \end{bmatrix}_{8\times8},$$

$$B_{21}=\begin{bmatrix} I+C_{11} & C_{14} \\ C_{14} & C_{11} \end{bmatrix},\ B_{22}=\begin{bmatrix} C_{11} & C_{14} \\ C_{14} & I+C_{11} \end{bmatrix}$$

$$A_{21}=\begin{bmatrix} C_{12} & C_{14} \\ C_{14} & I+C_{12} \end{bmatrix},\ A_{23}=A_{22}{}^T,$$

$$A_{24}=\begin{bmatrix} I+C_{12} & C_{14} \\ C_{14} & C_{12} \end{bmatrix},$$

$$A_{22}=\begin{bmatrix} O & O \\ 2C_{14} & O \end{bmatrix}.$$

("Ly" group)

According to the principle of row first and column second, (22), (26), (23), (27), (24), (28), (25), (29) are used to get the solution of the eight grid points as shown in figure 17-18.

("Rx"group)

According to the principle of row first and column second, (24), (25), (28), (29), (22), (23), (26), (27) are used to get the solution of the eight grid points as shown in figure 17-18.

("Ry"group)

According to the principle of column first and row second, (24), (28), (25), (29), (26), (30), (27), (31) are used to get the solution of the eight grid points as shown in figure 17-18.

("G"group)

(24), (25), (28), (29) are used to get the solution of the four grid points $(1,1),(2,1),(1,2),(2,2)$.

("H"group)

(24), (25), (28), (29) are used to get the solution of the four grid points $(m-2,m-2),(m-2,m-1),$
$(m-1,m-2),(m-1,m-1)$

The construction of "Ly" group, "Ry" group, "Rx" group is similar to "Lx" group.

According to the groups shownin Figure 17-18 , if we let

$$\overline{u}^n=(\overline{u}_1^n,\overline{u}_5^n,\overline{u}_9^n......\overline{u}_{m-6}^n,\overline{u}_{m-2}^n)^T,$$

$$\overline{u}_j^n=(\overline{u}_{1,j}^n,\overline{u}_{5,j}^n,\overline{u}_{9,j}^n,......\overline{u}_{m-6,j}^n,\overline{u}_{m-2,j}^n)^T,\ j=1,5,9......m-6,$$

$$\overline{u}_{m-2,j}^n=(u_{m-2,j}^n,u_{m-1,j}^n,u_{m-2,j+1}^n,u_{m-1,j+1}^n,u_{m-2,j+2}^n,u_{m-1,j+2}^n,u_{m-2,j+3}^n,u_{m-1,j+3}^n)^T,$$

$$\overline{u}_{m-2}^n=(\overline{u}_{1,m-2}^n,\overline{u}_{5,m-2}^n,\overline{u}_{9,m-2}^n.....\overline{u}_{m-6,m-2}^n,\overline{u}_{m-2,m-2}^n)^T$$

$$\overline{u}_{i,m-2}^n=(u_{i,m-2}^n,u_{i,m-1}^n,u_{i+1,m-2}^n,u_{i+1,m-1}^n,u_{i+2,m-2}^n,u_{i+2,m-1}^n,u_{i+3,m-2}^n,u_{i+3,m-1}^n)^T,$$

$i=1,5,9......m-6$

$$\overline{u}_{m-2,m-2}^n=(u_{m-2,m-2}^n,u_{m-2,m-1}^n,u_{m-1,m-2}^n,u_{m-1,m-1}^n)^T,$$

Then we denote the alternating group explicit method as follows:

$$\begin{cases}(I+rH_1)\overline{u}_{n+1}=(I-rH_2)\overline{u}_n \\ (I+rH_2)\overline{u}_{n+2}=(I-rH_1)\overline{u}_{n+1}\end{cases} \quad (32)$$

Let $(m-1)^2=a$, $2(m-3)+4=b$, $4(m-3)+8=c$, then

$$H_1=\begin{bmatrix} H_{11} &&&& \\ & H_{11} &&& \\ && .... && \\ &&& H_{11} & \\ &&&& H_{12} \end{bmatrix}_{a\times a},$$

$$H_{11}=\begin{bmatrix} A_1 &&&& \\ & A_1 &&& \\ && .... && \\ &&& A_1 & \\ &&&& A_3 \end{bmatrix}_{c\times c},$$

$$H_{12}=\begin{bmatrix} A_3 &&&& \\ & A_3 &&& \\ && .... && \\ &&& A_3 & \\ &&&& A_4 \end{bmatrix}_{b\times b}$$

$$A_3=\begin{bmatrix} A_{31} & A_{32} \\ A_{33} & A_{34} \end{bmatrix}_{8\times8},\ \begin{cases}A_{31}=B_{22}\\ A_{33}=A_{22}{}^T\end{cases},$$

$$\begin{cases}A_{32}=A_{22}\\ A_{34}=B_{21}\end{cases},\ \begin{cases}A_4=B_{22}\\ B_4=A_{24}\end{cases}$$

$$H_2=\begin{bmatrix} P & M^T &&&& \\ M & P & M^T &&& \\ & .... & .... & .... && \\ && M & P & M^T & \\ &&& M & P & N^T \\ &&&& N & Q \end{bmatrix}_{a\times a},$$

$$P=\begin{bmatrix} P_1 & E_1^T & & & & \\ E_1 & P_1 & E_1^T & & & \\ & \dots & \dots & \dots & & \\ & & E_1 & P_1 & E_1^T & \\ & & & E_1 & P_1 & F_1^T \\ & & & & F_1 & P_2 \end{bmatrix}_{c\times c},$$

$$Q=\begin{bmatrix} Q_1 & E_2^T & & & & \\ E_2 & Q_1 & E_2^T & & & \\ & \dots & \dots & \dots & & \\ & & E_2 & Q_1 & E_2^T & \\ & & & E_2 & Q_1 & F_2^T \\ & & & & F_2 & Q_2 \end{bmatrix}_{b\times b}.$$

$$P_1=\begin{bmatrix} P_{11} & P_{12} & O & O \\ P_{12} & P_{13} & O & O \\ O & O & P_{13} & P_{12} \\ O & O & P_{12} & P_{11} \end{bmatrix}_{16\times 16},$$

$$P_2=\begin{bmatrix} P_{21} & O \\ O & P_{22} \end{bmatrix},\ P_{11}=\begin{bmatrix} I+C_{12} & O \\ O & I+C_{11} \end{bmatrix},$$

$$P_{12}=\begin{bmatrix} C_{14} & O \\ O & C_{14} \end{bmatrix},\ P_{13}=\begin{bmatrix} C_{1,2} & O \\ O & C_{1,1} \end{bmatrix},$$

$$\begin{cases} P_{2,1} = A_{2,4} \\ P_{2,2} = A_{2,1} \end{cases}.$$

$$E_1=\begin{bmatrix} E_{1,1} & & & \\ & E_{1,1} & & \\ & & E_{1,1} & \\ & & & E_{1,1} \end{bmatrix}_{16\times 16},$$

$$F_1=\begin{bmatrix} F_{1,1} & & & \\ & F_{1,1} & & \\ & & F_{1,1} & \\ & & & F_{1,1} \end{bmatrix}_{8\times 8},$$

$$E_{1,1}=\begin{bmatrix} O & C_{1,3} \\ O & O \end{bmatrix}_{8\times 8},\ F_{1,1}=\begin{bmatrix} O & C_{13} \end{bmatrix}_{2\times 4},$$

$$E_2=\begin{bmatrix} O & 2C_{1,4} \\ O & O \end{bmatrix}_{8\times 8},\ F_2=\begin{bmatrix} O & 2C_{1,4} \\ O & O \end{bmatrix}_{4\times 8},$$

$$Q_1=P_2,\ Q_2=A_{2,4}.$$

$$M=\begin{bmatrix} M_1 & & & \\ & \dots & & \\ & & M_1 & \\ & & & M_2 \end{bmatrix}_{c\times c},$$

$$\begin{cases} M_1=\begin{bmatrix} O & M_{1,1} \\ O & O \end{bmatrix}_{16\times 16} \\ M_{1,1}=\begin{bmatrix} 2C_{1,4} & O \\ O & 2C_{1,4} \end{bmatrix}_{4\times 4} \end{cases},$$

$$M_2=\begin{bmatrix} O & 2C_{1,4} \\ O & O \end{bmatrix}_{8\times 8},$$

$$N=\begin{bmatrix} N_1 & & & \\ & \dots & & \\ & & N_1 & \\ & & & N_2 \end{bmatrix}_{c\times b},$$

$$\begin{cases} N_1=\begin{bmatrix} O & N_{11} \end{bmatrix}_{8\times 16} \\ N_2=\begin{bmatrix} O & N_{21} \end{bmatrix}_{4\times 8} \end{cases},$$

$$N_{2,1}=\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}^T,$$

$$N_{1,1}=\begin{bmatrix} C_{15} & & & \\ & C_{15} & & \\ & & C_{15} & \\ & & & C_{15} \end{bmatrix}^T_{8\times 4}.$$

From the construction of the matrices H^1 and H^2, we can see they are non negative definite real matrices. Then we have

$\|(I+rH_1)^{-1}\|_2\le 1$, $\|(I-rH_1)(I+rH_1)^{-1}\|_2\le 1$,

$\|(I+rH_2)^{-1}\|_2\le 1$, $\|(I-rH_2)(I+rH_2)^{-1}\|_2\le 1$.

Let n is an even number, then we have $\overline{u}^n = T\overline{u}^{n-2}$. Here $T=(I+rH_2)^{-1}(I-rH_1)(I+rH_1)^{-1}(I-rH_2)$ is the growth matrix, .

Let $\overline{H}=(I+rH_2)T(I+rH_2)^{-1}=(I-rH_1)$. Then by Lemma 1, we have $\rho(T)=\rho(\overline{H})\le\|\overline{H}\|_2\le 1$.

Then we have the following theorem:

Theorem 2 The alternating group explicit method denoted by (32) is unconditionally stable.

5 Numerical Examples

Example 1: Consider the initial-boundary problem of convection -diffusion equation as below

$$\begin{cases} \dfrac{\partial u}{\partial t}+a\dfrac{\partial u}{\partial x}=b\dfrac{\partial^2 u}{\partial x^2}, 0\le x\le 1, 0\le t\le T, \\ u(0,t)=0, u(1,t)=1, \\ u(x,0)=0. \end{cases} \tag{33}$$

The exact solution [6] is

$$u(x,t)=\frac{e^{ax/b}-1}{e^{ab}-1}+$$
$$\sum_{n=1}^{\infty}\frac{(-1)^n n\pi}{(n\pi)^2+(a/2b)^2}e^{a(x-1)/2b}\sin(n\pi x)e^{-[(n\pi)^2 b+(a^2/4b)]t}$$

Let $A.E$ and $P.E$ denote absolute error and relevant error respectively. We compare the numerical results of

(11) with the Crank-Nicholson scheme in Table 1 and Table 2.

Table 1: Partly results of comparisons
$m=25, \tau=10^{-4}, t=1000\tau, a=10, b=10$

$A.E$	5.150×10^{-9}
$A.E$ (C-N)	5.086×10^{-9}
$P.E$	5.361×10^{-2}
$P.E$ (C-N)	5.249×10^{-2}

Table 2: Partly results of comparisons
$m=29, \tau=10^{-4}, t=1000\tau, a=5, b=5$

$A.E$	4.776×10^{-6}
$A.E$ (C-N)	4.638×10^{-6}
$P.E$	3.100×10^{-2}
$P.E$ (C-N)	3.012×10^{-2}

The results in Table 1-2 show the method (11) is of nearly the same accuracy as the implicit C-N scheme

Example 2:
Considering the following problem:

$$\begin{cases} \dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial^2 y}, \\ 0\le x\le 1, 0\le y\le 1, 0\le t\le T \\ u(x,y,0)=\sin(\pi x)\sin(\pi y) \\ u(0,y,t)=0, u(1,y,t)=0 \\ u(x,0,t)=0, u(x,1,t)=0 \end{cases}$$

The exact solution[9] of the problem is
$u(x,y,t)=e^{-2\pi^2 t}\sin(\pi x)\sin(\pi y)$.

We compare the alternating group method (32) with the method in [9] and the exact solution.

In Table 3-4, we present part of the comparisons with
$r=0.4, m=19, h=1/19, \tau=r/361, t=390\tau, y=16h$

Table 3: Comparisons with [9] with
$r=0.4, m=19, h=1/19, \tau=r/361, t=390\tau, y=16h$

x	6h	7h	8h
$A.E$	1.185×10^{-7}	0.808×10^{-7}	1.05×10^{-7}
$P.E$	1.506×10^{-1}	9.385×10^{-2}	1.152×10^{-2}
$A.E$ [9]	6.677×10^{-7}	6.488×10^{-7}	7.800×10^{-7}
$P.E$ [9]	8.487×10^{-1}	7.539×10^{-1}	8.562×10^{-1}
Exact	7.867×10^{-5}	8.607×10^{-5}	9.111×10^{-5}

Table 4: Comparisons with [9] with
$r=0.4, m=19, h=1/19, \tau=r/361, t=390\tau, y=16h$

x	10h	11h	12h
$A.E$	0.948×10^{-7}	1.022×10^{-7}	0.882×10^{-7}
$P.E$	1.102×10^{-1}	1.121×10^{-1}	1.025×10^{-1}
$A.E$ [9]	7.651×10^{-7}	7.099×10^{-7}	6.520×10^{-7}
$P.E$ [9]	8.168×10^{-1}	7.793×10^{-1}	7.576×10^{-1}
Exact	9.366×10^{-5}	9.111×10^{-5}	8.607×10^{-5}

The results in Table 3-4 show the present method (32) is of higher accuracy than the method in [9].

In Figure 19 we present the comparison between the numerical results of the present method and the exact solution with
$r=0.4, m=23, h=1/23, \tau=r/361, t=390\tau, y=16h$.

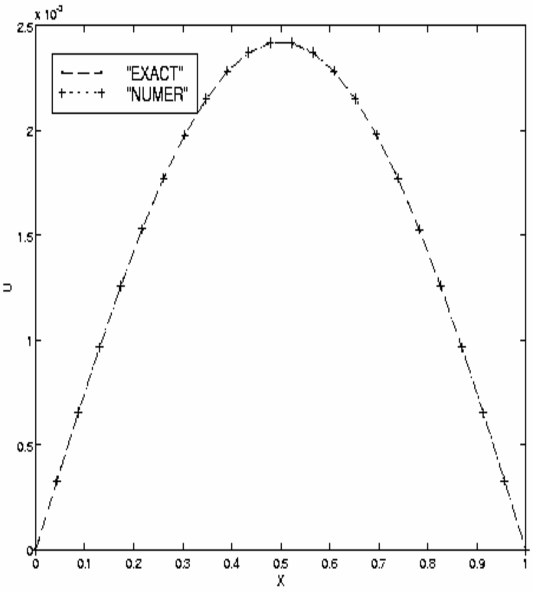

Figure 19 Comparison between numerical results and exact results with
$r = 0.4, m = 23, h = 1/23, \tau = r/361, t = 390\tau, y = 16h$

From Figure 19 we can see the numerical results is of high accuracy.

6 Conclusions

In this paper, based on the concept of domain decomposition, we present a class of parallel alternating group explicit method for convection-diffusion equations, which is verified to be unconditionally stable. From the results of Table 1 and Table 2 we can see that the numerical solution for the method is of nearly the same accuracy as the implicit Crank-Nicholson scheme. Furthermore, we constrcut another alternating group explicit method for 2D diffusion equations. The method is also unconditionally stable. The results in Table 3-4 show the method is superior to the method in [9]. Both of the two methods are suitable for parallel computing, and the computing in the whole domain can be divided into many independent sub domains. So the two alternating group methods are effective methods in solving large system of equations.

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