New Correlation Analysis Method for Nonstationary Signal

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Abstract: - This paper proposes a new correlation analysis method for nonstationary and energy-limited continual signals, which works out a formula similar to time correlation theory; that is to say when the operator, f, meets a certain condition, its auto-correlation property of Fourier transform can be presented by Fourier transform of its energy function, $|f(t)|^2$, revealing its correlation of the Fourier transform coefficients of frequency interval, ω. It proves that the Parseval identical equation is the special case happening when $\omega = 0$. The above conclusion explains the theory of image compressing and noise removing through wavelet transform.

Key-Words: - operator, energy function, wavelet transform, correlation function, image process

1. Introduction

Large amount of nonstationary signals exist in engineering applications, so much so that its analysis and process is under the attention of scholars home and abroad. The time dependence of its energy spectrum and power spectrum causes some difficulty in analysis. The current mainstream analysis methods are mostly based on time-frequency technology [1] [2].

Correlation is a basic method in signal process, being important in the fields of data fusion [3, 4, 5], automatic control [6], signal inspection [7], system recognition [8], image compression [9] and biomedicine [10], etc. Now, thanks to the rapid development of wavelet analysis technology, application of correlation technology has a broader space.

Considering Wiener-Khinchine Theory reflects only the statistic character of continual signals under stationary state, not applicable to energy-limited and nonstationary continual signals, literature [11] makes a correlation analysis to nonstationary, energy-limited and power-limited signals, revealing their relations with time correlation theory. Literature [12] [13] prove that wavelet transform can reduce correlation and enhance the convergence speed of noise-removing algorithm, but theoretical explanations are not given. This paper conducts mathematic deduction to reach a conclusion similar to time correlation theory, which has brand-new theoretical and physical significance. Through computer simulation, this paper explains the theory and phenomenon of image compression and noise removal. This theory is different from the method in literature [14], which uses singular maximum value to remove noises; therefore it has a promising prospect in application.

2. Definition of correlation function and mathematic basis of wavelet transform

Assuming that for each p, $1 \le p < +\infty$, $L^p(R)$ represents Lebesgue measurable functional folders on the real number folder, R. Let the operator $T \in L^p(R)$, Lebesgue integral $\int_{-\infty}^{+\infty} |T(x)|^p dx$ is limited, namely $-\infty < \int_{-\infty}^{+\infty} |T(x)|^p dx < +\infty$. Define p-order norm as $\|T(x)\|_p \overset{def}{=} \{\int_{-\infty}^{+\infty} |T(x)|^p dx\}^{\frac{1}{p}}$, $1 \le p < +\infty$. When $p = 1$, $\|T(x)\|_1 = \int_{-\infty}^{+\infty} |T(x)| dx$;

when $p = 2$, $\|T(x)\|_2 = \{\int_{-\infty}^{+\infty} |T(x)|^2 dx\}^{\frac{1}{2}}$.

2.1 Definition of auto-correlation function

The auto-correlation function of operator $T \in L^2(R)$ is defined as:

$$H(x) = \int_{-\infty}^{+\infty} T(x+y)\overline{T(y)} dy \qquad (1)$$

while $\overline{T}(y)$ represents the conjugacy of $T(y)$.

2.2 Definition of cross-correlation function [11]

The cross-correlation function of operator $S, T \in L^2(R)$ is defined as:

$$H(x) = \int_{-\infty}^{+\infty} S(x+y)\overline{T(y)} dy \qquad (2)$$

while $\overline{T}(y)$ represents the conjugacy of $T(y)$.

2.3 Mathematic basis of wavelet transform

For continual signal $f(t) \in L^2(R)$, the wavelet transform (namely the decomposition formula of signals) and wavelet inversion (namely the reconstruction of signals) of signal $f(t)$ are:

$$W_\psi(f)(a,b) = \frac{1}{\sqrt{a}} \int f(t) \overline{\psi(\frac{t-b}{a})} dt \qquad (3)$$

$$f(t) = \frac{1}{C_\psi} \int_0^{+\infty} \frac{da}{a^2} \int_{-\infty}^{+\infty} W_\psi(f)(a,b) \frac{1}{\sqrt{a}} \phi(\frac{t-\tau}{a}) d\tau \qquad (4)$$

while a is a contraction factor, b is a translation factor, $\frac{1}{\sqrt{a}}$ is the normalized coefficient of the wavelet base, ψ and ϕ are dual wavelet base, usually $a > 0$. When applied in practice, a and b are separated as $\{a_m, b_n | m, n \in z\}$ and discrete wavelet transform and wavelet inversion are:

$$W_\psi(f)(a_m, b_n) = \frac{1}{\sqrt{a_m}} \int f(t) \overline{\psi(\frac{t-b_n}{a_m})} dt \qquad (5)$$

and

$$f(t) = \sum_{m,n \in Z} W_\psi(f)(a_m, b_n) \overline{\phi(a_m, b_n, t)} \qquad (6)$$

2.**4Mathematic basis of Fourier transform [16]**

Let $\hat{f} \in L^1(R)$ is the Fourier transform of the function $f \in L^1(R)$, $\hat{f}^{-1}$ representing Fourier inverted transforms, and then we get:

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt \qquad (7)$$

$$\hat{f}^{-1}(\hat{f}(\omega)) = f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{f}(\omega)e^{i\omega t}d\omega. \qquad (8)$$

3. Correlation analysis

Based on the correlation definition of the operator T, we can get the Fourier transform $\hat{f} \in L^1(R)$ correlation of $f \in L^1(R)$. To make things easier, assume functions f and $\hat{f}$ are tow forms of the operator T.

Conclusion 1: let $\hat{f}, \hat{g} \in L^1(R)$ be the Fourier transform of the function $f, g \in L^1(R)$, and if they meet the conditions: $\hat{f}, \hat{g} \in L^1(R) \cap L^2(R)$ and $f, g \in L^1(R) \cap L^2(R)$, and then it comes out as:

$$\begin{aligned} H(\omega) &= \int_{-\infty}^{+\infty}\hat{f}(\omega+\omega')\overrightarrow{\hat{g}(\omega')}d\omega' \\ &= 2\pi\int_{-\infty}^{+\infty}f(t)\overline{g(t)}e^{-i\omega t}dt \end{aligned} \qquad (9)$$

Proof: because $\hat{f}, \hat{g} \in L^1(R) \cap L^2(R)$, from the mathematic theory we know that in double integral operation the orders of integral operation can be changed.

$$\begin{aligned} H(\omega) &= \int_{-\infty}^{+\infty}\hat{f}(\omega+\omega')\overline{\hat{g}(\omega')}d\omega' \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}[f(t)e^{-i(\omega+\omega')t}]\overline{[g(x)e^{-i\omega' x}]}dtdxd\omega' \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(t)\overline{g(x)}e^{-i\omega t}[\int_{-\infty}^{+\infty}e^{i(x-t)\omega'}d\omega']dtdx \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(t)\overline{g(t)}e^{-i\omega t}[2\pi\delta(x-t)]dtdx \\ &= 2\pi\int_{-\infty}^{+\infty}f(t)[\int_{-\infty}^{+\infty}\overline{g(x)}\delta(x-t)dx]dt \\ &= 2\pi\int_{-\infty}^{+\infty}f(t)\overline{g(t)}e^{-i\omega t}dt \end{aligned} \qquad (10)$$

Above proof process mainly involves the nature of function δ, namely:

$$\int_{-\infty}^{+\infty}\delta(t)e^{-i\omega t}dt = 1, \int_{-\infty}^{+\infty}e^{i\omega t}d\omega = 2\pi\delta(t) \quad (11)$$

This conclusion is quite meaningful theoretically, by which the following deductions can be made.

Deduction 1: let $\hat{f}, \hat{g} \in L^1(R)$ be the Fourier transform of the function, $f, g \in L^1(R)$, and if $\hat{f}, \hat{g} \in L^1(R) \cap L^2(R)$ and $f, g \in L^1(R) \cap L^2(R)$, Parseval identical equation, $<\hat{f}, \hat{g}> = 2\pi < f, g>$, is the special case of Conclusion 1.

Proof: let ω=0 in Conclusion 1, and then

$$H(0) = \int_{-\infty}^{+\infty}\hat{f}(\omega')\overline{\hat{g}(\omega')}d\omega' = 2\pi\int_{-\infty}^{+\infty}f(t)\overline{g(t)}dt$$

namely

$$<\hat{f}, \hat{g}> = 2\pi < f, g> \quad (12)$$

Above method puts forward a brand-new way to prove Parseval identical equation different from literature [11], which is simple and effective. From Deduction 1 we know that Parseval identical equation is a special case in Fourier transforms occurring when the frequency of cross-correlation function, ω=0. This conclusion can again deduce the following conclusion on the nature of auto-correlation function:

Conclusion 2: let $\hat{f} \in L^1(R)$ is the Fourier transform of function $f \in L^1(R)$, and then if

$\hat{f} \in L^1(R) \cap L^2(R)$ and $f \in L^1(R) \cap L^2(R)$, there is:

$$H(\omega) = \int_{-\infty}^{+\infty} \hat{f}(\omega + \omega') \overline{\hat{f}(\omega')} d\omega' = 2\pi \int_{-\infty}^{+\infty} |f(t)|^2 e^{-i\omega t} dt \quad (13)$$

while $|f(t)|^2 = f(t)\overline{f(t)}$.

Proof: from Conclusion 1 we know that if let $g(t) = f(t)$, there is

$$H(\omega) = \int_{-\infty}^{+\infty} \hat{f}(\omega + \omega') \bar{\hat{f}}(\omega') d\omega' = 2\pi \int_{-\infty}^{+\infty} f(t)\overline{f(t)} e^{-i\omega t} dt = 2\pi \int_{-\infty}^{+\infty} |f(t)|^2 e^{-i\omega t} dt \quad (14)$$

Conclusion 2 is significant theoretically and physically, from which we know that as long as the operator f meets some condition, the auto-correlation nature of its Fourier transform can be represented by the Fourier transform of its energy function, $|f(t)|^2$, reflecting the correlation of Fourier transform coefficients of frequency interval ω. From Conclusion 2 we can further get the following deduction:

Deduction 2: let $\hat{f} \in L^1(R)$ be the Fourier transform of the function $f \in L^1(R)$, and if $\hat{f} \in L^1(R) \cap L^2(R)$ and $f \in L^1(R) \cap L^2(R)$, then the Parseval identical equation, $< \hat{f}, \hat{f} >= 2\pi < f, f >$, is the special case of Conclusion 2, namely

$$\left\| \hat{f} \right\|_2^2 = 2\pi \| f \|_2^2 . \quad (15)$$

Proof: let ω=0 in Conclusion 2, and then

$$H(0) = \int_{-\infty}^{+\infty} \hat{f}(\omega') \overline{\hat{f}(\omega')} d\omega' = 2\pi \int_{-\infty}^{+\infty} f(t)\overline{f(t)} dt = 2\pi \int_{-\infty}^{+\infty} |f(t)|^2 dt \quad (16)$$

namely

$$\left\| \hat{f} \right\|_2^2 = 2\pi \| f \|_2^2 \quad (17)$$

It is widely known that the energy-limited continual signals under nonstationary state are different from those under stationary state, whose statistic character is the function of time, *t*, and Fourier transforms are not real numbers, which complicates the physical explanation of its spectrum. Therefore, the Fourier transform of the character parameter, $R(\tau)$, of *t*-related nonstationary signals' correlation function, namely the power density spectrum, does not exist. That is to say, the auto-correlation function describing the time domain statistic rules of stationary continual random signals and the power density spectrum of frequency domain statistic rules form a Fourier transform pair, namely Wiener-Khinchine Theorem or Wiener-Khinchine Formula is not applicable to energy-limited continual signals under nonstationary state. For these signals, we can only analyze with time domain analysis method instead of frequency domain. However, for Fourier transforms conducted after wavelet or wavelet packet transforms, a frequency domain analysis method different from Wiener-Khinchine

Theorem can be applied if only the energy distribution of the frequency domain is known. The following conclusion explains who to apply this method.

Conclusion 3: assume the energy-limited continual signal is $f \in L^1(R)$ and let $W_\psi(f)(a,b) \in L^1(R)$ be the wavelet transform of the function f, and $\hat{W}_\psi(f)(a,b) \in L^1(R)$ be the Fourier transform of $W_\psi(f)(a,b)$, and when $\hat{W}_\psi(f)(a,b) \in L^1(R) \cap L^2(R)$, there is:

$$\begin{aligned} H(x) &= \int_{-\infty}^{+\infty} \hat{W}_\psi(f)(a,b)(x+y)\overline{\hat{W}_\psi(f)(a,b)(y)}dy \\ &= 2\pi\int_{-\infty}^{+\infty} \left|W_\psi(f)(a,b)(\omega)\right|^2 e^{-i\omega x} d\omega \end{aligned} \quad (18)$$

while

$$\left|W_{(\psi}(f)(a,b)(\omega)\right|^2 = W_\psi(f)(a,b)(\omega)\overline{W_\psi(f)(a,b)(\omega)}$$

Proof: the same as Conclusion 2.

Deduction 3: assume the energy-limited continual signal is $f \in L^1(R)$, and let $W_\psi(f)(a,b) \in L^1(R)$ be the wavelet transform of the function f and $\hat{W}_\psi(f)(a,b) \in L^1(R)$ be the Fourier transform of $W_\psi(f)(a,b)$, and when $W_\psi(f)(a,b) \in L^1(R) \cap L^2(R)$ and $\hat{W}_\psi(f)(a,b) \in L^1(R) \cap L^2(R)$, then

$$\begin{aligned} H(0) &= \int_{-\infty}^{+\infty} \left|\hat{W}_\psi(f)(a,b)(y)\right|^2 dy \\ &= 2\pi\int_{-\infty}^{+\infty} \left|W_\psi(f)(a,b)(\omega)\right|^2 d\omega \end{aligned} \quad (19)$$

while

$$\left|W_{(\psi}(f)(a,b)(\omega)\right|^2 = W_\psi(f)(a,b)(\omega)\overline{W_\psi(f)(a,b)(\omega)}$$
.

Proof: this deduction is evident from Conclusion 3.

Conclusion 3 and Deduction 3 show that energy-limited continual signals are the coefficients of the wavelet transform of $f \in L^1(R)$, $W_\psi(f)(a,b) \in L^1(R)$, and the correlation of different frequency intervals can be described by the energy of the wavelet transform, $W_\psi(f)(a,b)$, which concentrates mainly on the low-frequency section. This is because the distribution and change of the coefficient of wavelet transform, $W_\psi(f)(a,b)$, can be reflected by its Fourier transforms; energy distribution at different frequency intervals is that of the wavelet transform under different dimensions, and wavelet transforms cause energy to concentrate on the low-frequency section, especially the wavelet packet transforms. We can say that wavelet and wavelet packet transforms can reduce the auto-correlation and cross-correlation of signals, so as to improve the convergence rate of noise removal and image compression by iterative algorithm. Since noise is featured in high frequency and high energy, therefore if only the high-frequency energy can be removed, noise removal and image compression can be realized. Now this conclusion will be proved through calculating Fourier transforms of noises' wavelet transforms.

Conclusion 4: assume the statistic character of noise $s(t)$ is: average value $E(s(t))=0$ and variance $E(s^2(t))=\sigma^2$. Its energy after wavelet transforms mainly distributes in high-frequency section, proportionately, so the average value of its energy is a constant.

Proof: (1) The character of high frequency of the noise after wavelet transforms is apparent.

(2) Its average energy is a constant because:

From the theories of mathematic analysis and functional analysis we know that the essence of Fourier transform and wavelet transform calculation is to work out the integral. Therefore the operation order of average value and wavelet and Fourier transforms can be exchanged. From Conclusion 3 we can get the average energy of S (t) as:

$$
\begin{aligned}
& E(|W_\psi(S(t))(a,b)(\omega)|^2) \\
&= \frac{1}{2\pi} E\{\int_{-\infty}^{+\infty}[\int_{-\infty}^{+\infty}\hat{W}_\psi(S(t))(a,b)(x+y)\overline{\hat{W}_\psi(S(t))(a,b)(y)}dy\,]e^{j\omega x}dx\} \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}[\int_{-\infty}^{+\infty}W_\psi(\hat{E}(S^2(t))(x+y)))(a,b)\overline{W_\psi(\hat{E}(S^2(t)(y)))(a,b)}dy\,]e^{j\omega x}dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}[\int_{-\infty}^{+\infty}W_\psi(\hat{\sigma}^2)(a,b)\overline{W_\psi(\hat{\sigma}^2)(a,b)}dy\,]e^{j\omega x}dx \\
&= (2\pi)^{-3}\sigma^4\int_{-\infty}^{+\infty}[\int_{-\infty}^{+\infty}W_\psi(\delta(-y))(a,b)\overline{W_\psi(\delta(-y))(a,b)}dy\,]e^{j\omega x}dx \\
&= (2\pi)^{-3}\sigma^4\int_{-\infty}^{+\infty}[\int_{-\infty}^{+\infty}W_\psi(\delta(-y))(a,b)]\overline{W_\psi(\delta(-y))(a,b)}dy\,]e^{j\omega x}dx \\
&= (2\pi)^{-3}\sigma^4\int_{-\infty}^{+\infty}|W_\psi(1)(a,b)|^2 e^{j\omega x}dx \\
&= (2\pi)^{-3}\sigma^4\left|W_\psi(\int_{-\infty}^{+\infty}e^{j\omega x}dx)(a,b)\right|^2 \\
&= \frac{\sigma^4}{2\pi}|W_\psi(\delta(\omega))(a,b)|^2 = \frac{\sigma^4}{2\pi}\left|\frac{1}{\sqrt{a}}\int_{-\infty}^{+\infty}\delta(\omega)\psi(\frac{\omega-b}{a})\right|^2 \\
&= \frac{\sigma^4}{2\pi a}\left|\psi(\frac{0-b}{a})\right|^2 = \frac{\sigma^4}{2\pi a}\left|\psi(-\frac{b}{a})\right|^2 \qquad (20)
\end{aligned}
$$

Conclusion 4 proves that the energy of noise in high-frequency section is proportionately distributed, dependent on the choice of wavelet bases and their scale factors and reflected by the zero-point energy value of wavelet bases. It can be used as a reference when selecting threshold in compression algorithm so as to compress the images.

4. Experiment on image compression

In the image compression experiment, 9/7 Serial Daubeches two-dimension wavelet packet transforms are applied to decompose the image into multi-layer low-frequency components and high-frequency components, as shown in Picture 1; Picture 2 is the original image, Picture 3 is resulted from two layers of wavelet packet transforms and Picture 4, three layers. In these pictures, white sections represent high-energy sections and black, low energy.

It is clear in these pictures that the fringes of images are high-frequency and high-energy sections, however which can only give clear-cut outlines but blurred image; while the low-frequency section can almost reveal the original image. The section with highest frequency, namely the right corner, is most liable to be polluted by noises; even free of noise, it is the part that carries minor information. As its energy is distributed proportionately, this part of information can be removed when compressing images, so that the goal to compress image and remove noise is accomplished.

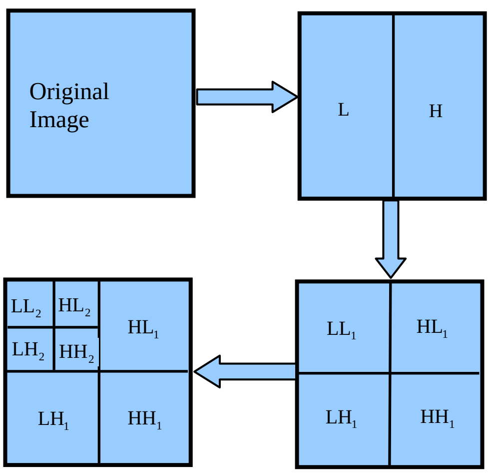

fig. 1 decomposition process

fig.2 original image

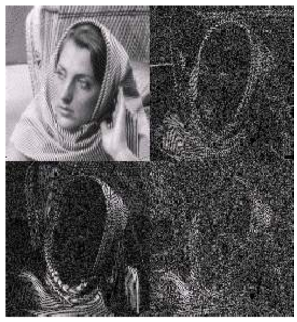

fig.3 Image after two-layer wavelet packet transforms

fig.4 Image after three-layer wavelet packet transforms

5. Conclusion

The following conclusions can be drawn from the above theory and experiment:

1） Energy of the image is mainly concentrated on low-frequency section, which can basically give the original picture;

2） Low-frequency section realizes image compression;

3） High-frequency and high-energy sections realize noise removal;

4） Middle-and high-frequency section realizes fringe pick-up;

5） Wavelet transform coefficients are apparently irrelevant between low-frequency and high-frequency sections and partly related in the two middle-and high-frequency sections, which is the part showing the fringe of the image.

6. Acknowledgements

This work is financed by National Basic Research Program of China (Program 973), No: **2007CB311005-01**

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