Analytical and numerical results for detecting attractors

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Abstract: - In this paper we present some qualitative and quantitative results for a particular type of *k*-order exchange rate models. These results concern the existence of attractors: the fixed point, its stability and its attraction domain, period-*p* cycles, limit cycles and chaotic attractors. Given the nonlinear nature, the dynamics of these systems cannot be detected using only analytical tools. For this reason, in the last section, we make numerical simulations and we present some examples. The algorithms implementation is made using *VBA* (*Visual Basic for Applications*) program in *Excel*, and the images of the figures in this paper are made using *Mathematica*.

Key-Words: - nonlinear system, attractors, numerical simulations, programming, *VBA* & *Excel*.

1 Introduction

According to [4] a general equation modeling the exchange rate evolution is given by:

(1) $$S_t = X_t E_t\left(S_{t+1}\right)^b$$

In the above equation, S_t is the exchange rate at the moment *t*; X_t describes the exogenous variables that drive the exchange rate at the moment *t*; $E_t\left(S_{t+1}\right)$ is the expectation held at the moment *t* in the market about the exchange rate at the moment *t+1*; *b* is the discount factor that speculators use to discount the future expected exchange rate *(0<b<1)*.

This model allows us to take into account two components for forecasting: a forecast made by the chartists $E_{ct}\left(S_{t+1}\right)$ and a forecast made by the fundamentalists $E_{ft}\left(S_{t+1}\right)$:

(2) $$E_t\left(S_{t+1}\right)/S_{t-1} = \left(E_{ct}\left(S_{t+1}\right)/S_{t-1}\right)^{m_t}\left(E_{ft}\left(S_{t+1}\right)/S_{t-1}\right)^{1-m_t}$$

where m_t is the weight given by the chartists and $1-m_t$ is the weight given by the fundamentalists at the moment *t*.

The fundamentalists assume the existence of an equilibrium exchange rate S^*. If at the moment *t-1* the exchange rate S_{t-1} is above, respectively below, the equilibrium rate S^*, the fundamentalists expect the future exchange rate S_{t+1} to go down, respectively increase, with the speed α. More precisely, if they observe a deviation today, then their forecasts is the following:

(3) $$\frac{E_{ft}\left(S_{t+1}\right)}{S_{t-1}} = \left(\frac{S^*}{S_{t-1}}\right)^{\alpha}, \quad \alpha > 0 \cdot$$

The chartists use the past values of the exchange rate to detect patterns that they extrapolate in the future. An equation which gives a general description of the different models used by chartists is the following:

(4) $$\frac{E_{ct}\left(S_{t+1}\right)}{S_{t-1}} = f\left(S_{t-1},\ldots,S_{t-N}\right).$$

According to [4] it is possible to specify such a rule, in general terms, as follows:

(5) $$\frac{E_{ct}\left(S_{t+1}\right)}{S_{t-1}} = \left(\frac{S_{t-1}}{S_{t-2}}\right)^{C_1}\left(\frac{S_{t-2}}{S_{t-3}}\right)^{C_2}\cdots\left(\frac{S_{t-N+1}}{S_{t-N}}\right)^{C_{N-1}}.$$

The exact nature of this rule is determined by the coefficients c_i. These can be positive, negative, or zero.

The weight m_t, in equation (2), given by chartists is

(6) $$m_t = \frac{1}{1+\beta\left(S_{t-1}-S^*\right)^2}, \quad \beta > 0.$$

The parameter β measures the precision degree of the fundamentalists' estimation. When the exchange rate is in the neighbourhood of the equilibrium rate, chartists' behavior dominates. When the exchange rate differs from the fundamental rate, then the expectation will be dominated by the fundamentalists.

In this paper we consider the case $X_t = 1$ (which means that $S^* = 1$) and for chartists we consider the expectation:

(7) $$\frac{E_{ct}\left(S_{t+1}\right)}{S_{t-1}} = \left(\frac{S_{t-1}}{S_{t-k}}\right)^c, \quad c > 1, \quad k \geq 2, \quad k \in N. \cdot$$

In equation (2) we will use the expectations given by equations (3) and (7). In equation (1) we will use the expectations given by equation (2). In this way, we obtain the following difference equation:

(8) $S_t = S_{t-1}^{\left(\frac{(2+\alpha)b}{1+\beta(S_{t-1}-1)^2}+(1-\alpha)b\right)} S_{t-k}^{\left(\frac{-2b}{1+\beta(S_{t-1}-1)^2}\right)}$.

If we denote $s_t = \ln S_t$, then equation (8) can be written in the form:

(9) $s_t = \left(\frac{(2+\alpha)b}{1+\beta\left(e^{s_{t-1}}-1\right)^2} + (1-\alpha)b\right)s_{t-1} + \left(\frac{-2b}{1+\beta\left(e^{s_{t-1}}-1\right)^2}\right)s_{t-k}$

with $s_t \in R$ and $t \in Z$. We can rewrite equation (9) in the following vectorial form:

(10) $(s_{t+2},..., s_{t+k+1}) = F(s_{t+1}, s_t, s_{t+k})$

where

$F: R^k \to R^k$, $F(x_1,..., x_k) = (F_1(x_1,..., x_k),..., F_k(x_1,..., x_k))$, is defined in the following way: $F_i(x_1,..., x_k) = x_{i+1}$, $i = \overline{1, k-1}$ and $F_k(x_1,..., x_k) = \varphi(x_k)x_k + \psi(x_k)x_1$, with $\varphi(x) = \frac{(2+\alpha)b}{1+\beta(e^x-1)^2} + (1-\alpha)\beta$ and $\psi(x) = \frac{-2b}{1+\beta(e^x-1)^2}$.

In *Sections 2* and *3* we will present some analytical results for system (10) and in *Section 4* we will present some numerical simulations.

2 Fixed point. Existence, unicity, stability and attraction domain

2.1. Steady-state existence, unicity and stability

A fixed point for system (10) is a point $(x^*,..., x^*) \in R^k$ for which $(x^*,..., x^*) = F(x^*,..., x^*)$.

We recall that a fixed point $(x^*,..., x^*)$ is stable if, for any sufficiently small neighbourhood $U \ni (x^*,..., x^*)$ there is a neighbourhood $V_U(\alpha, b, \beta, c) \ni (x^*,..., x^*)$ so that $F^t(x_1^{'},..., x_k^{'}) \in U$ for every point $(x_1^{'},..., x_k^{'}) \in V_U(\alpha, b, \beta)$ and all $t > 0$, where $F^t = \underbrace{F \circ ... \circ F}_{t-times}$.

If there is a neighborhood $V_U(\alpha, b, \beta, c) \ni (x^*,..., x^*)$ so that $F^t(x_1^{'},..., x_k^{'}) \to (x^*,..., x^*)$, when $t \to \infty$, for every point $(x_1^{'},..., x_k^{'}) \in V_U(\alpha, b, \beta)$, then the fixed point is asymptotically stable (attracting fixed point).

Proposition 1. In the case in which $c>1$, $b \in (0,1)$, $\alpha > 0$ and $\beta > 0$ system (10) has a unique fixed point and this point is $(0,...,0) \in R^k$.

Observation 1. The Jacobian matrix of function F (defined in relation (10)) for $(0,...,0) \in R^k$, is the matrix

$$\begin{pmatrix} 0 & 1 & ... & 0 \\ ... & & & \\ 0 & 0 & ... & 1 \\ -cb & 0 & ... & (c+1)b \end{pmatrix}$$

and it has the determinant $(-1)^k \lambda^k + (-1)^{k-1}(1+c)b\lambda^{k-1} + (-1)^k bc$. Calculating the eigenvalues of the Jacobian matrix, we can establish when the fixed point is stable or unstable.

We do not give a mathematical solution for this problem, but we can use the computer, like in the examples from *Table 1*, where we make *c=2*:

System order	The fixed point is stable for	The fixed point is unstable for
k=2	$b \in (0, 0.5)$	$b \in (0.5, 1)$
k=3	$b \in (0, 0.3165)$	$b \in (0.3165, 1)$
k=4	$b \in (0, 0.2654)$	$b \in (0.2654, 1)$
k=5	$b \in (0, 0.2428)$	$b \in (0, 0.2428)$
k=6	$b \in (0, 0.2306)$	$b \in (0.2306, 1)$
k=10	$b \in (0, 0.2119)$	$b \in (0.2119, 1)$

Table 1: Fixed point stability

2.2. Attraction domain for the fixed point

In the case in which the fixed point is stable, it is important to study its attraction domain. In order to make such a study, now, we give the following result:

Proposition 2. Under the assumption: $c>1$, $b \in \left(0, \frac{1}{2c+1}\right]$, $\alpha \in (0,1)$, $\beta > 0$, $s_t \in R$, $t \in Z$, the following relations are verified:

c.1. For $s_{t+1}, s_{t+k} < 0$:

c.1.1. if $s_{t+1} \in \left(-\infty, \frac{(c+1)+(1-\alpha)\beta\left(e^{s_{t+k}}-1\right)^2}{c}s_{t+k}\right)$ then $-s_{t+1} > s_{t+k+1} > 0 > s_{t+k} > s_{t+1}$

c.1.2. if $s_{t+1} = \frac{(c+1)+(1-\alpha)\beta\left(e^{s_{t+k}}-1\right)^2}{c}s_{t+k}$ then $s_{t+k+1} = 0 > s_{t+k} > s_{t+1}$

c.1.3. if $s_{t+1} \in \left(\frac{(c+1)+(1-\alpha)\beta\left(e^{s_{t+k}}-1\right)^2}{c}s_{t+k}, \frac{(c+1)b+(1-\alpha)b\beta\left(e^{s_{t+k}}-1\right)^2}{1+cb+\beta\left(e^{s_{t+k}}-1\right)^2}s_{t+k}\right)$ then $0 > s_{t+k+1} > s_{t+k}$, $0 > s_{t+k+1} > s_{t+1}$

c.1.4. if $s_{t+1} = \frac{(c+1)b+(1-\alpha)b\beta\left(e^{s_{t+k}}-1\right)^2}{1+cb+\beta\left(e^{s_{t+k}}-1\right)^2}s_{t+k}$ then $0 > s_{t+k+1} = s_{t+1} > s_{t+k}$

c.1.5. if $s_{t+1} \in \left(\frac{(c+1)b + (1-\alpha)b\beta\left(e^{s_{t+k}} - 1\right)^2}{1 + cb + \beta\left(e^{s_{t+k}} - 1\right)^2} s_{t+k}, 0 \right)$ then $0 > s_{t+1} > s_{t+k+1} > s_{t+k}$

c.2. For $s_{t+1}, s_{t+k} > 0$ **:**

c.2.1. if $s_{t+1} \in \left(0, \frac{(c+1)b + (1-\alpha)b\beta\left(e^{s_{t+k}} - 1\right)^2}{1 + cb + \beta\left(e^{s_{t+k}} - 1\right)^2} s_{t+k} \right)$ then $s_{t+k} > s_{t+k+1} > s_{t+1} > 0$

c.2.2. if $s_{t+1} = \frac{(c+1)b + (1-\alpha)b\beta\left(e^{s_{t+k}} - 1\right)^2}{1 + cb + \beta\left(e^{s_{t+k}} - 1\right)^2} s_{t+k}$ then $s_{t+k} > s_{t+k+1} = s_{t+1} > 0$

c.2.3. if $s_{t+1} \in \left(\frac{(c+1)b + (1-\alpha)b\beta\left(e^{s_{t+k}} - 1\right)^2}{1 + cb + \beta\left(e^{s_{t+k}} - 1\right)^2} s_{t+k}, \frac{(c+1) + (1-\alpha)\beta\left(e^{s_{t+k}} - 1\right)^2}{c} s_{t+k} \right)$

then $s_{t+k} > s_{t+k+1} > 0$, $s_{t+1} > s_{t+k+1} > 0$

c.2.4. if $s_{t+1} = \frac{(c+1) + (1-\alpha)\beta\left(e^{s_{t+k}} - 1\right)^2}{c} s_{t+k}$ then $s_{t+1} > s_{t+k} > s_{t+k+1} = 0$

c.2.5. if $s_{t+1} \in \left(\frac{(c+1) + (1-\alpha)\beta\left(e^{s_{t+k}} - 1\right)^2}{c} s_{t+k}, \infty \right)$ then $s_{t+1} > s_{t+k} > 0 > s_{t+k+1} > -s_{t+1}$

c.3. For $s_{t+1} < 0, s_{t+k} > 0$ **:**

c.3.1. if $s_{t+1} \in \left(-\infty, \frac{(c+1)b - 1 + [(1-\alpha)b - 1]\beta\left(e^{s_{t+k}} - 1\right)^2}{cb} s_{t+k} \right)$

then $-s_{t+1} > s_{t+k+1} > s_{t+k} > 0 > s_{t+1}$

c.3.2. if $s_{t+1} = \frac{(c+1)b - 1 + [(1-\alpha)b - 1]\beta\left(e^{s_{t+k}} - 1\right)^2}{cb} s_{t+k}$

then $s_{t+k+1} = s_{t+k} > 0 > s_{t+1}$

c.3.3. if $s_{t+1} \in \left(\frac{(c+1)b - 1 + [(1-\alpha)b - 1]\beta\left(e^{s_{t+k}} - 1\right)^2}{cb} s_{t+k}, 0 \right)$ then $s_{t+k} > s_{t+k+1} > 0 > s_{t+1}$

c.4. For $s_{t+1} > 0, s_{t+k} < 0$ **:**

c.4.1. if $s_{t+1} \in \left(0, \frac{(c+1)b - 1 + [(1-\alpha)b - 1]\beta\left(e^{s_{t+k}} - 1\right)^2}{cb} s_{t+k} \right)$ then $s_{t+1} > 0 > s_{t+k+1} > s_{t+k}$

c.4.2. if $s_{t+1} = \frac{(c+1)b - 1 + [(1-\alpha)b - 1]\beta\left(e^{s_{t+k}} - 1\right)^2}{cb} s_{t+k}$

then $s_{t+1} > 0 > s_{t+k+1} = s_{t+k}$

c.4.3. if $s_{t+1} \in \left(\frac{(c+1)b - 1 + [(1-\alpha)b - 1]\beta\left(e^{s_{t+k}} - 1\right)^2}{cb} s_{t+k}, \infty \right)$

then $s_{t+1} > 0 > s_{t+k} > s_{t+k+1} > -s_{t+1}$

c.5. if $s_{t+1} < 0$ şi $s_{t+k} = 0$ then $0 < s_{t+k+1} < -s_{t+1}$

c.6. if $s_{t+1} > 0$ şi then $-s_{t+1} < s_{t+k+1} < 0$

c.7. if $s_{t+1} = 0$ şi $s_{t+k} < 0$ then $s_{t+k} < s_{t+k+1} < 0$

c.8. if $s_{t+1} = 0$ şi $s_{t+k} > 0$ then $0 < s_{t+k+1} < s_{t+k}$

Remark 1. For $c > 1, \alpha \in (0,1), \beta > 0$ and $b \in \left(0, \frac{1}{2c+1} \right]$ we find that $|s_{t+k+1}| \le \max\{|s_{t+1}|, |s_{t+k}|\}, \forall s_t \in R, t \in Z$. This relation implies that $\max\{|s_{t+jk+1}|, ..., |s_{t+(j+1)k}|\} \le \max\{|s_{t+(j-1)k+1}|, ..., |s_{t+jk}|\}, \forall t \in Z, j \in Z$. If we define: $s^*_{t+jk} = s_{t+(j-1)k+i}$ for $|s_{t+(j-1)k+i}| = \max\{|s_{t+(j-1)k+1}|, ..., |s_{t+jk}|\}$, where $j \in Z, i = \overline{1,k}$, then the sequence $\{|s^*_{t+jk}|\}_{t \in Z}$ is monotonously decreasing and positive. This means that the sequence $\{|s^*_{t+jk}|\}_{t \in Z}$ is convergent. If $p = \lim_{t \to \infty} |s^*_{t+jk}|$, then:

1) $\lim_{t \to \infty} s^*_{t+jk} = p$ or

2) $\lim_{t \to \infty} s^*_{t+jk} = -p$ or

3) $\lim_{t \to \infty} s^*_{t+jk} = p$ for $t + jk \in T$, $\lim_{t \to \infty} s^*_{t+jk} = -p$ for $t + jk \in T'$, where $T, T' \subset Z$ with $T \cap T' = \emptyset$, $T \cup T' = Z$ and T, T' are infinite.

Using *Proposition 2* and *Remark 1*, we provide the following result:

Proposition 3. For $c > 1, \alpha \in (0,1), \beta > 0$ and $b \in \left(0, \frac{1}{2c+1} \right]$ and any initial condition of system (10), the limit p is 0. This implies that the fixed point $(0,...,0) \in R^k$ is globally attractive.

Proposition 4. The fixed point is stable for $b \in (0, y(c,k))$, where $\frac{1}{2c+1} \le y(c,k) < 1$.

3 Period-two cycles for system (10)

A period-2 point of system (10) is a solution of the equation $(s_1, ..., s_k) = F^2(s_1, ..., s_k)$ where $(s_1, ..., s_k) \ne F(s_1, ..., s_k)$. The relations $(s_2, ..., s_{k+1}) = F(s_1, ..., s_k)$, $(s_3, ..., s_{k+2}) = F(s_2, ..., s_{k+1})$ and

$(s_3,\dots,s_{k+2})=(s_1,\dots,s_k)$ imply that

1) for k, an odd number, a period – two cycle has the form: $\{(s_2,s_1,\dots,s_1,s_2),(s_1,s_2,\dots,s_2,s_1)\}$, where the vectors are from R^k, $(s_2,s_1,\dots,s_1,s_2)=F(s_1,s_2,\dots,s_2,s_1)$ and $(s_1,s_2,\dots,s_2,s_1)=F(s_2,s_1,\dots,s_1,s_2)$.

2) for k, an even number, a period – two cycle has the form: $\{(s_2,s_1,\dots,s_2,s_1),(s_1,s_2,\dots,s_1,s_2)\}$, where the vectors are from R^k, $(s_2,s_1,\dots,s_2,s_1)=F(s_1,s_2,\dots,s_1,s_2)$ and $(s_1,s_2,\dots,s_1,s_2)=F(s_2,s_1,\dots,s_2,s_1)$.

3.1. The case when *k* is an even number

Now, we consider that k is an even number and we get the following poposition:

Proposition 5. For $c>1$ and $b\in(0,1)$, if $\alpha\in\left(0,\ 1+\frac{1}{b}\right]$ or $\alpha\in\left(1+\frac{1}{b},\infty\right)$ and $\beta\in\left(0,\frac{(c+1)b^2(\alpha-1)+1+cb}{(\alpha-1)^2b^2-1}\right]$, then system (10) has no cycles of period two.

If $\alpha\in\left(1+\frac{1}{b},\infty\right)$ and $\beta\in\left(\frac{(c+1)b^2(\alpha-1)+1+cb}{\left[(\alpha-1)^2b^2-1\right]},\infty\right)$ then system (10) has an unique cycle of period two.

This cycle is $\{(s_1,s_2,\dots,s_1,s_2),(s_2,s_1,\dots,s_2,s_1)\}$ where s_1 and s_2 are solutions of the equation:

$$\frac{\varphi\left(\frac{\varphi(x)}{1-\psi(x)}x\right)}{1-\psi\left(\frac{\varphi(x)}{1-\psi(x)}x\right)}\frac{\varphi(x)}{1-\psi(x)}x=1,$$

which means that s_1 and s_2 are solutions of equation:

$$\frac{(c+1)b+(1-\alpha)b\beta\left(e^{\frac{(c+1)b+(1-\alpha)b\beta(e^x-1)^2}{1+cb+\beta(e^x-1)^2}x}-1\right)^2}{1+cb+\beta\left(e^{\frac{(c+1)b+(1-\alpha)b\beta(e^x-1)^2}{1+cb+\beta(e^x-1)^2}x}-1\right)^2}\frac{(c+1)b+(1-\alpha)b\beta(e^x-1)^2}{1+cb+\beta(e^x-1)^2}=1$$

The numbers s_1 and s_2 verify the relation $s_1s_2<0$. Let $s_1>0$ be the positive number.

If $\beta\in\left(\frac{(c+1)b^2(\alpha-1)+1+cb}{\left[(\alpha-1)^2b^2-1\right]},\frac{(1+2c)b+1}{\left[(\alpha-1)b-1\right]}\right)$, then we find that $s_1>\ln\left(1+\sqrt{\frac{1}{\beta}\frac{1+(2c+1)b}{\left[(\alpha-1)b-1\right]}}\right)$ and $s_2<-\ln\left(1+\sqrt{\frac{1}{\beta}\frac{1+(1+2c)b}{\left[(\alpha-1b-1)\right]}}\right)$.

If $\beta\in\left[\frac{(2c+1)b+1}{\left[(\alpha-1)b-1\right]},\infty\right)$, then we find that

$s_1\in\left(\ln\left(1+\sqrt{\frac{1}{\beta}\frac{(2c+1)b+1}{\left[(\alpha-1)b-1\right]}}\right),-\ln\left(1-\sqrt{\frac{1}{\beta}\frac{(2c+1)b+1}{\left[(\alpha-1)b-1\right]}}\right)\right)$ and

$s_2\in\left(\ln\left(1-\sqrt{\frac{1}{\beta}\frac{(2c+1)b+1}{\left[(\alpha-1)b-1\right]}}\right),-\ln\left(1+\sqrt{\frac{1}{\beta}\frac{(2c+1)b+1}{\left[(\alpha-1)b-1\right]}}\right)\right)$

From *Propositions 4* and *5* we get the following result:

Proposition 6. If $c>1$, $b\in\left(0,\frac{1}{2c+1}\right)$, $\alpha\in\left(1+\frac{1}{b},\infty\right)$ and $\beta\in\left(\frac{(c+1)b^2(\alpha-1)+1+cb}{\left[(\alpha-1)^2b^2-1\right]},\infty\right)$, then the fixed point $(0,\dots,0)\in R^k$ of the system is locally attractive.

3.2. The case when *k* is an odd number

Now, we consider that k is an odd number and we get the following poposition:

Proposition 7. For $c>1$ and $b\in(0,1)$, if $\alpha\in\left(0,\ 1+\frac{1}{b}\right]$ or $\alpha\in\left(1+\frac{1}{b},\infty\right)$ and $\beta\in\left(0,\frac{b^2(\alpha-1)+1}{(\alpha-1)^2b^2-1}\right]$, then system (10) has no cycles of period two.

If $\alpha\in\left(1+\frac{1}{b},\infty\right)$ and $\beta\in\left(\frac{b^2(\alpha-1)+1}{\left[(\alpha-1)^2b^2-1\right]},\infty\right)$, then system (10) has an unique period-2 cycle.

This cycle is $\{(s_1,s_2,\dots,s_2,s_1),(s_2,s_1,\dots,s_1,s_2)\}$ where s_1 and s_2 are solutions of the equation:

$(\varphi((\varphi(x)+\psi(x))x)+\psi((\varphi(x)+\psi(x))x))(\varphi(x)+\psi(x))=1$

which means that s_1 and s_2 are solutions of equation:

$$\left(\frac{\alpha b}{1+\beta\left(e^{\left(\frac{\alpha b}{1+\beta(e^x-1)^2}+(1-\alpha b)\right)x}-1\right)^2}+(1-\alpha)b\right)\left(\frac{\alpha b}{1+\beta(e^x-1)^2}+(1-\alpha b)\right)=1$$

Numbers s_1 and s_2 verify the relation $s_1s_2<0$. Let $s_1>0$ be the positive number.

If $\beta\in\left(\frac{b^2(\alpha-1)+1}{\left[(\alpha-1)^2b^2-1\right]},\frac{b+1}{\left[(\alpha-1)b-1\right]}\right)$ then

$s_1 > \ln\left(1+\sqrt{\frac{1}{\beta}\frac{1+b}{[(\alpha-1)b-1]}}\right)$ and $s_2 < -\ln\left(1+\sqrt{\frac{1}{\beta}\frac{1+b}{[(\alpha-1b-1)]}}\right)$.

If $\beta \in \left[\frac{b+1}{[(\alpha-1)b-1]},\infty\right)$ then

$$s_1 \in \left(\ln\left(1+\sqrt{\frac{1}{\beta}\frac{b+1}{[(\alpha-1)b-1]}}\right),-\ln\left(1-\sqrt{\frac{1}{\beta}\frac{b+1}{[(\alpha-1)b-1]}}\right)\right) \text{ and}$$

$$s_2 \in \left(\ln\left(1-\sqrt{\frac{1}{\beta}\frac{b+1}{[(\alpha-1)b-1]}}\right),-\ln\left(1+\sqrt{\frac{1}{\beta}\frac{b+1}{[(\alpha-1)b-1]}}\right)\right)$$

From *Propositions 4* and *7* we get the following result:

Proposition 8. If $c>1$, $b \in \left(0,\frac{1}{2c+1}\right)$, $\alpha \in \left(1+\frac{1}{b},\infty\right)$ and $\beta \in \left(\frac{b^2(\alpha-1)+1}{[(\alpha-1)^2 b^2-1]},\infty\right)$, then the fixed point $(0,...,0) \in R^k$ of the system is locally attractive.

4 Numerical simulations

4.1. Attractors

We now recall some notions which will be used in this section. We say that a set *A* is an attracting set with the fundamental neighbourhood *U*, if it verifies the following properties (see [14]):

1) attractivity: for every open set $V \supset A$, $F^t U \subset V$ for all sufficiently large *t*.
2) invariance: $F^t(A) = A$, for all *t*.
3) *A* is minimal: there is no proper subset of *A* that satisfies conditions 1 and 2.

The basin of attraction is the set of initial points *x* so that $F^t(x)$ is close to *A* when $t \to \infty$.

It is possible to classify the different attractors: attracting fixed point, attracting *p*-cycle, quasiperiodic attractor and strange attractor. An attractor, as an experimental object, gives a global description of the asymptotic behavior of a dynamical system.

When a deterministic mechanism presents complex behavior with intermittence, we can conclude that the series evinces chaos under certain conditions.

The sensitive dependence on initial conditions is one of the most essential aspects in identifying the chaos. We recall that the sensitive dependence on initial conditions means that two trajectories starting very close together will rapidly diverge from each other.

The strange attractor is associated with a chaotic state of time evolution and is characterized by the sensitive dependence on initial conditions.

A measure of the average rate of exponential divergence exhibited by a chaotic system is given by the Lyapunov exponents of the system; the positivity of one from these exponents can suggest the presence of chaos.

The Lyapunov exponents λ_1, $\lambda_2 \ldots$ λ_k are given by (11)

$$\left\{e^{\lambda_1},e^{\lambda_2},...,e^{\lambda_k}\right\} = \lim_{n\to\infty}\left\{eigenvalues\ of\left(\prod_{t=0}^{n-1} J(F(s_t,s_{t+1},...,s_{t+k}))\right)^{\frac{1}{n}}\right\}$$

where $J(F(s_t,s_{t+1},...,s_{t+k}))$ represents the Jacobian matrix of the function *F*. For a period-*p* point the Lyapunov exponents λ_1, λ_2 …. λ_k are given by

$$(12)\ \left\{e^{\lambda_1},e^{\lambda_2},...,e^{\lambda_k}\right\} = \left\{eigenvalues\ of\left(\prod_{t=0}^{p-1} J(F(s_t,s_{t+1},...,s_{t+k}))\right)^{\frac{1}{p}}\right\}$$

We recall now that for an attracting period-*p* cycle the Lyapunov exponents are negative; in case of a bifurcation point, at least one Lyapunov exponent is zero; for a limit cycle one Lyapunov exponent is zero and the others are negative and for a chaotic behavior the highest Lyapunov exponent is positive while the sum of all Lyapunov exponents is negative.

In order to compute the Lyapunov exponents, when system (10) displays a chaotic behavior, we use the method proposed in [2], based on the Householder QR factorization and the implementation method proposed in [19].

We have made many numerical simulations and we have found many situations in which the system displays this types of attractors. In order to illustrate these, now, we give some examples. The implementation of the algorithms is made using *VBA* (Visual Basic for Applications) program in *Excel*, and the images from the figures are made using *Mathematica*.

Example of chaotic attractors:

In the case *k=2, c=2, b=0.95,* $\alpha=2$, $\beta=600$ and $(s_1,s_2)=(0.02,-0.02)$, the trajectory tends towards the chaotic attractor presented in *Figure 1*. The Lyapunov exponents are: $(\lambda_1,\lambda_2)=(0.4029,-1.2158)$.

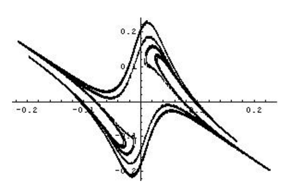

Figure 1: Chaotic behavior for *k=2*, the space (s_t, s_{t+1})

In the case *k=3, c=2, b=0.95,* $\alpha=2$, $\beta=600$ and $(s_1,s_2,s_3)=(0.02,-0.02,-0.0794)$, the trajectory tends towards the chaotic attractor presented in *Figure 2*. The

Lyapunov exponents are: $(\lambda_1, \lambda_2, \lambda_3) = (0.2328, -0.2468, -0.4297)$.

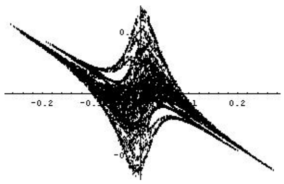

Figure 2: Chaotic behavior for *k=3*, the space (s_t, s_{t+1})

Example of limit cycle (quasiperiodic attractor)

In the case *k=2, c=2, b=0.95,* $\alpha = 0.3$, $\beta = 600$ and $(s_1, s_2) = (0.02, -0.02)$, the trajectory tends towards the limit cycle presented in *Figure 3*. The Lyapunov exponents are: $(\lambda_1, \lambda_2) = (0, -0.3449)$.

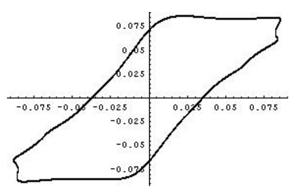

Figure 3: Limit cycle for *k=2*, the space (s_t, s_{t+1})

Attracting period- p cycles

β	s_1^*	s_2^*	period	λ_1	λ_2
40.5528	0.22816	-0.10586	1024	-0.0012	-0.6448
40.5513	-0.49251	0.23373	512	-0.0063	-0.6387
40.52	-0.49253	0.2336	128	-0.0177	-0.6272
40.51	-0.48868	0.228887	128	-0.0486	-0.5963
40.38	-0.49377	0.24238	64	-0.0568	-0.5882
40.3	-0.49496	0.23532	32	-0.0242	-0.6208
39	-0.50179	0.23606	16	-0.0062	-0.6398
36	-0.52696	0.24804	8	-0.0920	-0.5538
32	-0.5617	0.26258	4	-0.2447	-0.4125
28.3853	-0.58727	0.26821	4	-0.0002	-0.6963

Table 2: A sequence of period-doubling bifurcations (the first period is 4)

In *Table* 2, for the case *k=2, c=2, b=0.95,* $\alpha = 2$ and $(s_1, s_2) = (0.02, -0.02)$, we present some situations in which the trajectory tends towards the period-*p* attracting cycles. In this case, we have found a sequence of period-doubling bifurcations.

Here (s_1^*, s_2^*) is a point from the period-*p* cycle towards which tends the trajectory starting from point (0.02, -0.02).

4.2. Coexistence of attractors and weak chaotic attractors

In the cases presented in *Section 4.1.*, the fixed point of the system (10) is unstable.

We can find a complex behavior (period-*p* cycles, limit cycles, chaotic attractors) even in the case in which the fixed point is stable, like in the following examples:

Now, we consider the case *k=2, c=2 b = 0.44,* β *= 10,* s_1 *= 0.4602,* s_2 = -0.1533 and we make α variable, with $\alpha \in [2.83286, \ 4.8]$. We find a sequence of period-doubling bifurcations with the first period is 4 (see *Table 3*). In *Table 4* we present the Lyapunov exponents for each case from *Table 3*.

α	s_1^*	s_2^*	period
2.83286	0.46022	-0.155345	4
3.975	0.394045	-0.2284	8
4.49	0.39225	-0.28636	16
4.6	0.32111	-0.156965	32
4.615	-0.35799	0.10207	64
4.62	0.32661	-0.169586	128
4.6206	0.30506	-0.12812	256
4.620852	0.38884	-0.2976	512
4.620857	0.32868	-0.17382	1024

Table 3: A sequence of period-doubling bifurcations (the first period is 4)

α	λ_1	λ_2
2.83286	-0.00050254	-0.77408597
3.975	-0.0060483	-0.623160892
4.49	-0.39225	-0.28636
4.6	-0.0424086	-0.57590344
4.615	-0.00371722	-0.61293307
4.62	-0.018376058	-0.597758746
4.6206	-0.000931098	-0.57529
4.620852	-0.000359043	-0.574733605
4.620857	-0.000209248	-0.574584033

Table 4: The Lyapunov exponents for the periodic-*p* points from the sequence of period-doubling bifurcations from *Table 3*

For $\alpha = 4.8$ we find a chaotic attractor, which is presented in *Figure 4*.

In *Figure 4*, we can observe different parts which form the attractor, but the values of the Lyapunov exponents - λ_1=0.08907 and λ_2=-0.69458 confirm the existence a chaotic behaivor.

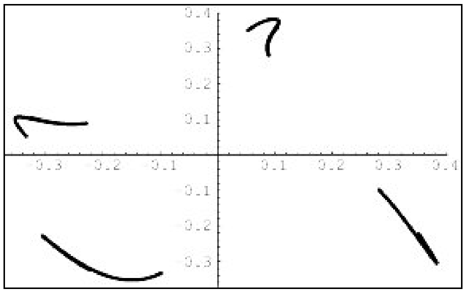

Figure 4: A weak chaotic attractor

4.3. Chaotic attractors and sequence of period-doubling bifurcation

In the case in which the fixed point is unstable, the behavior system (10) is more complex.

For the particular case where *k=2, c=2,* α*=2, b=0.95* and the initial condition $(s_1, s_2) = (0.02, -0.02)$, we investigate the ranges of parameter β for which system (10) presents a chaotic or a non chaotic behavior.

We observe different intervals of values for β for which system (10), in general, displays a chaotic behavior. These intervals are [46, ∞), [16,28], [6.2,12], [4,4.6], [2.9,3.8], [1.7,1.8]. Every two intervals presented here are separated by an interval of values of β which characterizes a sequence of period-doubling bifurcations for system (10). We present in *Figure 5* the strange attractors which characterize different types of intervals of values for β for which the system displays a chaotic behavior.

β	λ_1	λ_2
$6*10^{12}$	0.3728	-1.2735
600	0.4029	-1.2158
46	0.2685	-1.1734
12	0.3299	-1.341
6.2	0.2636	-1.9012
4.6	0.2244	-1.4617
2.2	0.2033	-1.5614
1	0.09489	-1.424
0.8	0.143	-1.5599
0.4	0.1054	-3.3219
0.2	0.0787	-2.5733
0.13	0.0132	-4.1387

Table 5: Lyapunov exponents in the case *k=2, c=2* , $(s_1, s_2) = (0.02, -0.02)$, *b =0.95,* α =2

In *Table 5* we give the values of Lyapunov exponents in the case of the strange attractors present in *Figure 5*.

The images from *Figure 5* seem to represent the same attractor which increases and is deformed, but probably that is not so clear. The transformation image cannot be obtained statically. In order to obtain an animation we can use different applications (e.g., in *Flash*) Our animation is presented on the page: http://www.catrinelvoicu.home.ro/chaos.html.

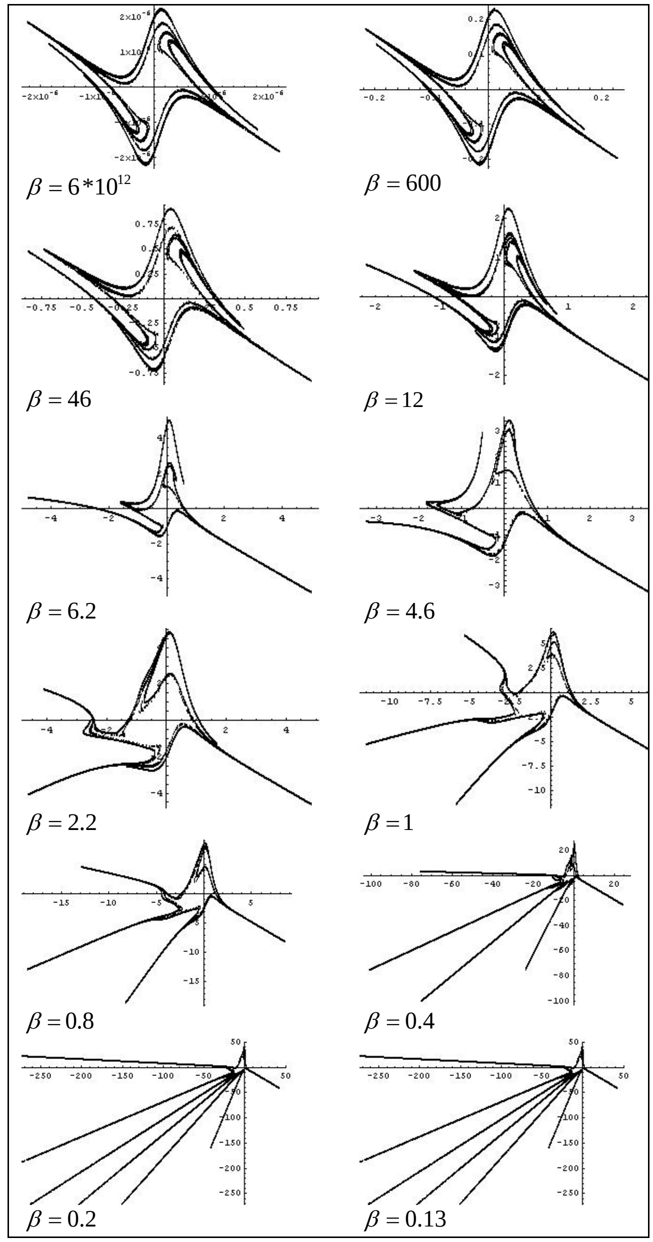

Figure 5: Chaotic attractors in the case *k=2, c=2* , $(s_1, s_2) = (0.02, -0.02)$, *b =0.95,* α =2, space (s_t, s_{t+1})

In *Table 2*, we have presented an example of a sequence of period-doubling bifurcations, displayed by system (10). We use the same initial conditions (0.02, -0.02). Here (s_1^*, s_2^*) means a periodical point from the periodical cycle towards which tends the trajectory starting from point (0.02, -0.02).

In the case 12.794 < β < 16 we find a sequence of period-doubling bifurcations, where the first period is 3 (see *Table 6*).

β	s_1^*	s_2^*	Period	λ_1	λ_2
15.33	-0.1777	-1.12504	192	-0.0007	-0.93342
15.324	-0.09388	-1.17613	96	-0.03141	-0.90265
15.32	-1.10702	0.562637	48	-0.01272	-0.9213
15.2	-0.18904	-1.11325	24	-0.09209	-0.84057
15	-0.11191	-1.1906	12	-0.02246	-0.90799
14.9	-1.13816	0.57627	6	-0.07288	-0.85699
13	0.61569	-0.12797	3	-0.43928	-0.43928
12.7941	-0.10965	-1.2832	3	-0.0068	-0.84512
12.79403	0.61151	-0.10934	3	-0.00099	-0.85053

Table 6: A sequence of period-doubling bifurcations (the first period is 3), *k=2, c=2* , $(s_1, s_2) = (0.02, -0.02)$, *b=0.95,* α =2

For β =12.794, the system (10) has a chaotic behavior.
For 4.8 < β < 16 we find a sequence of period-doubling bifurcations, where the first period is 3 (see *Table 7*).

β	s_1^*	s_2^*	Period	λ_1	λ_2
5.813097	2.9566	-2.80345	768	-0.00138	-3.68963
5.81309	5.1735	-4.9147	384	-0.00348	-3.68755
5.81305	-5.1226	0.46087	192	-0.01194	-3.6792
5.8128	-3.4724	0.17945	96	-0.01854	-3.63072
5.811	-3.21658	0.119	48	-0.00123	-3.6957
5.81	0.520483	2.9082	24	-0.01238	-3.68744
5.8	-2.82577	0.01044	12	-0.01699	-3.71189
5.7	0.133997	6.28767	6	-0.85609	-3.23492
4.82	10.91966	-10.373	3	-0.31398	-7.61383
4.80707	10.38804	-9.86864	3	-0.30781	-7.62973
4.807069	-0.48226	10.388	3	-0.30781	-7.26295

Table 7: A sequence of period-doubling bifurcations (the first period is 3), *k=2, c=2* , $(s_1, s_2) = (0.02, -0.02)$, *b=0.95,* α =2

For β =4.807069 and the initial condition $(s_1, s_2) = (0.02, -0.02)$, the trajectory tends towards a period-3 cycle. One point from this cycle is (-0.48226, 10.388).
For β =4.8 and the initial condition $(s_1, s_2) = (0.02, -0.02)$, the trajectory tends towards a chaotic attractor.
For β = 4.8 and the initial condition (-0.48226, 10.388) , the trajectory tends towards a period-3 cycle, and the Lyapunov exponents are $\lambda_1 = -0.304$ and $\lambda_2 = -7.08329$. In this case we find a coexistence of attractors. One attractor is chaotic and another is period-3 cycle.
From *Table 7* we can observe that the highest Lyapunov exponent do not tends towards 0.
In the interval (4.8, 4.807069), it exists a bifurcation (between a period-3 cycle and a chaotic attractor). In this case we remark that the highest Lyapunov exponent is not continuous function of β . From [14] it is known that, generally, the Lyapunov exponents are not continuous function of parameter. This fact has been in our numerical simulations the signal for coexistence of attractors. This implies that calculating the Lyapunos exponents we can detect that dynamic nature, but also we can observe the coexistence of attractors.

β	s_1^*	s_2^*	Period	λ_1	λ_2
3.954019	3.8413	-3.6475	2560	-0.00028	-1.67832
3.954018	-1.427	0.15598	1280	-0.00022	-1.67838
3.95401	-1.0675	0.87782	640	-0.00817	-1.67044
3.95395	-3.51987	-0.9793	320	-0.00024	-1.6784
3.9537	0.15576	3.70335	160	-0.00061	-1.67812
3.953	-1.94548	0.15465	80	-0.02232	-1.65673
3.949	2.10343	-1.95024	40	-0.0318	-1.64874
3.93	-1.0076	2.13413	20	-0.04899	-1.64159
3.899	0.91512	-0.30064	10	-0.07767	-1.61627
3.8803	0.07354	3.90653	10	-0.01076	-1.66638
3.880236	3.90424	-3.70742	10	-0.00056	-1.67502

Table 8: A sequence of period-doubling bifurcations (the first period is 10), *k=2, c=2* , $(s_1, s_2) = (0.02, -0.02)$, *b =0.95,* α =2

For 3.88< β < 3.96, we find a sequence of period-doubling bifurcations, where the first period is 10 (see *Table 8*). For β =3.88, the system (10) has a chaotic behavior.

For 2.27 < β <2.67 there it exists a sequence of period-doubling bifurcations, where the first period is 7 (see *Table 9*).

For β =2.2758, the system (10) has a chaotic behavior.
For 1.88 < β < 1.93 there it exists a sequence of period-doubling bifurcations, where the first period is 11 (see *Table 10*).

β	s_1^*	s_2^*	Period	λ_1	λ_2
2.65598	1.71410	-1.42150	896	-0.0006	-1.7829
2.65590	0.27540	3.66750	448	-0.0042	-1.7794
2.65550	-0.02927	4.16840	224	-0.0040	-2.7799
2.65320	-1.51025	-2.06020	112	-0.0018	-1.7836
2.65000	1.71571	-1.42270	56	-0.0152	-1.7461
2.60000	-2.10960	0.28619	28	-0.0043	-1.8170
2.58000	-2.22529	0.03830	14	-0.0165	-1.8207
2.50000	1.63129	-1.27678	14	-0.0777	-1.8290
2.46100	-3.17703	1.73945	14	-0.0003	-1.9436
2.46000	-3.18038	1.73989	7	-0.0010	-1.9435
2.28000	1.75116	-1.40029	7	-0.0522	-1.9418
2.27590	-2.44856	0.00776	7	-0.0007	-1.9648

Table 9: A sequence of period-doubling bifurcations (the first period is 7), *k=2, c=2* , $(s_1, s_2) = (0.02, -0.02)$, *b =0.95,* α =2

β	s_1^*	s_2^*	Period	λ_1	λ_2
1.9165	-5.7301	1.5166	352	-0.00023	-2.75946
1.9162	-2.0355	-2.9517	176	-0.00963	-1.75102
1.916	-5.5935	1.4706	88	-0.01803	-1.74318
1.915	0.10727	5.7807	44	-0.04378	-1.71992
1.91	-0.8733	2.31299	22	-0.10864	-2.67673
1.89	5.9389	-0.56419	11	-0.11127	-1.67971
1.8877	1.4837	-0.66759	11	-0.0004	-1.76312

Table 10: A sequence of period-doubling bifurcations (the first period is *11*), *k=2, c=2* , $(s_1, s_2) = (0.02, -0.02)$, *b =0.95,* α =2

For β =1.88, we find a chaotic behavior.

β	s_1^*	s_2^*	Period	λ_1	λ_2
1.663575	-0.04218	7.0231	512	-0.000566	-2.92015
1.66357	-0.04104	7.03197	256	-0.005781	-2.91495
1.6635	0.90984	3.46749	128	-0.00505	-2.91687
1.6632	0.98291	2.91895	64	-0.00943	-2.91757
1.662	-2.76382	-3.9077	32	-0.033262	-2.91447
1.66	-3.92869	0.01714	16	-0.031718	-3.8934
1.65	0.94558	3.546296	8	-0.053025	-4.12819
1.455	-131.694	-6.57886	4	-0.00255	-49.1924

Table 11: A sequence of period-doubling bifurcations (the first period is *4*), *k=2, c=2* , $(s_1, s_2) = (0.02, -0.02)$, *b=0.95,* α =2

For 1.45 < β < 1.67 we find a sequence of period-doubling bifurcations, where the first period is 4 (see *Table 11*).

For $\beta \in \{0.4, 0.2, 0.13\}$ the system (10) has a chaotic evolution.

From these examples we can observe that in the case in which the fixed point is unstable we find many chaotic attractors. In our numerical study we fix the initial condition, value of parameters and we make variable only one parameter.

We consider now the case k=2 , c >1, b=0.95, β =6.2, α =2, $s_0 = 0.02$, $s_1 = -0.02$

For 1 < c <1.053 (which implies that $b \in \left(0, \frac{1}{c}\right)$, i.e. the fixed point is stable) we observe that the trajectory tend towards fixed point (0,0). For 1.053 < c < 1.26, generally, the trajectory tends towards a limit cycle or towards period-p cycle. For 1.26 < c < 10, the system has a chaotic evolution.

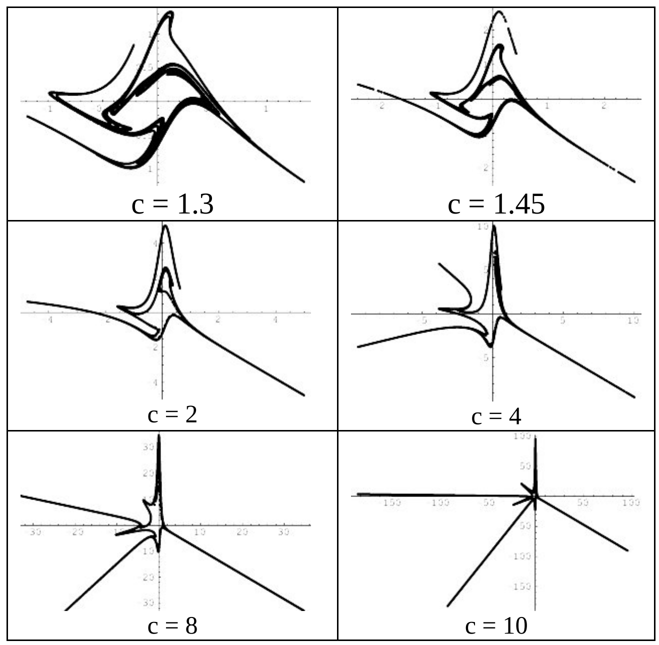

Figure 6: Chaotic attractors in the case *k=2,* β =6.2, $(s_1, s_2) = (0.02, -0.02)$, *b =0.95,* α =2, space (s_t, s_{t+1})

c	λ_1	λ_2
1.3	0.2284	-0.7987
1.45	0.2575	-1.2057
2	0.2646	-1.9035
4	0.1978	-1.701
8	0.1816	-4.4715
10	0.1304	-15.624

Table 12: Lyapunov exponents in the case *k=2,* β =6.2, $(s_1, s_2) = (0.02, -0.02)$, *b =0.95,* α =2

For c >10, when *t* increases, $|s_t|$ increases also.

5 Conclusion

The studies from *Section 1* and *2* lead to the conclusion that there are similarities between the dynamics of the studied systems. These results are interesting from a mathematical viewpoint and also, these results lead to economic interpretations. Given the nonlinear nature, the dynamics of these systems cannot be detected using only analytical tools. For this reason, in the last section, we make numerical simulations. The algorithms implementation is made using *VBA* (*Visual Basic for Applications*) program in *Excel*, and the images of the figures in this paper are made using *Mathematica*.

Our implementation methods are used for detecting the dynamics of a nonlinear system (see other studies [10], [11], [5], [7] and [16]) and for the calculation of the Lyapunov exponents (see also [6]). These methods concern the way in which we can obtain a very large number of observations with the computer, in a very short time, which, at the same time, is conclusive for the obtained results (see [19]).

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