Conducting fuzzy division by using linear programming

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Abstract: Some approximation methods have been proposed for fuzzy multiplication and division in the literature. Instead of doing arithmetic operations using fuzzy membership functions for fuzzy numbers, parameterized representation of fuzzy numbers have been used in arithmetic operations. The most applied parameterized fuzzy numbers used in many of the research papers are symmetric and asymmetric triangular and trapezoidal fuzzy numbers. In this study, we propose a new approximation method based on linear programming for fuzzy division. In order to show the applicability of the proposed method, some examples are solved using the proposed method and the results are compared with those generated by other methods in the literature. The proposed method has produced better results than those generated by the others.

Key-words: Approximation method; Fuzzy arithmetic; Fuzzy division; Linear programming; Triangular fuzzy number

1 Introduction

Instead of using membership functions in fuzzy arithmetic, parameterized form of fuzzy numbers have been used for arithmetic operations. Despite of the fact that the addition and subtraction of the parameterized fuzzy numbers result in closed form, the same does not hold for multiplication and division. However, fuzzy multiplication in closed form can be done under the weakest T norm as an exception [4]. Therefore, some approximation methods are proposed for multiplication and division of the parameterized fuzzy numbers [1,2,3,4]. The most applied parameterized fuzzy numbers are symmetric and asymmetric triangular and trapezoidal fuzzy numbers since they are simple and easy to implement for many purposes [8]. Dubois and Prade [1] first introduced arithmetic operations on parameterized fuzzy numbers. Then, Giachetti and Young [2,3] proposed a new method for fuzzy multiplication and division. Another method for fuzzy arithmetic operation is the weakest T norm [4]. In these studies, researchers have focused on decreasing the fuzziness of the resulting fuzzy number.

It is assumed that two parameterized triangular fuzzy numbers are multiplied or divided. This computation can be done using one of the methods mentioned above [1,2,3,4]. However, the results obtained from these methods have different fuzzy end points or spread values except center value. Instead of using parameterized form of fuzzy numbers that consist of real numbers, the area and the proportions of center value to left and to right end points are used to do multiplication or division of two fuzzy numbers. We proposed a new method based on linear programming for parameterized fuzzy division. The parameterized fuzzy numbers are limited to symmetric and asymmetric and trapezoidal fuzzy numbers because of their simplicity and ease of use. In this paper, new fuzzy division method is applied to triangular fuzzy numbers. The essence of proposed method depends on utilizing geometric features and definition of triangular fuzzy numbers [5,7].

In the next section, some preliminary knowledge related to fuzzy set is given. Section 3 gives brief information about existing approaches available in the literature for fuzzy division. The new proposed method is introduced in Section 4. Section 5 contains the implementation of the proposed method and the comparisons with other available methods in the literature. Last section is discussion and conclusion.

2 Preliminaries

A fuzzy set [6] A of the real line R with membership function $\mu_A : R \to [0,1]$ is called a fuzzy number if

1. A is normal, namely, there exist an element x such that $\mu_A(x) = 1$
2. A is fuzzy convex, that is,
$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \mu_A(x_1) \wedge \mu_A(x_2)$
$\forall x_1, x_2 \in R,\ \forall \lambda \in [0,1]$
3. μ_A is semi continuous
4. supp A is bounded where supp $A = \{x \in R : \mu_A(x) > 0\}$

The α cut of a fuzzy set A is a non- fuzzy set defined as $A_\alpha = \{x \in R : \mu_A(x) \geq \alpha\}$.

According to the definition of a fuzzy number, every α cut of a fuzzy number is a closed interval. Thus, $A_\alpha = [A_L(\alpha), A_U(\alpha)]$ can be written, where

$$A_L(\alpha) = \inf\{x \in R : \mu_A(x) \geq \alpha\}$$

$$A_U(\alpha) = \sup\{x \in R : \mu_A(x) \geq \alpha\}$$

The membership function for triangular fuzzy number is [7] defined as

$$\mu_A(x) = \begin{cases} 0 & for\ x < a \\ \dfrac{x-a}{b-a} & for\ a \leq x \leq b \\ \dfrac{c-x}{c-b} & for\ b \leq x \leq c \\ 0 & for\ x > c \end{cases}$$

There are two ways of denoting parameterized fuzzy numbers. The first one denotes left end point, center value and the right end point. On the other hand, the second one shows left spread value, center value, and the right spread value. Throughout the paper, two ways of denoting fuzzy numbers are adopted in fuzzy arithmetic operations.

3 Review of Existing Approaches

The first way of doing parameterized fuzzy arithmetic was introduced by Dubois and Prade [1]. The resulting fuzzy number is just an approximation. However, the usage of them is easy and simple in applications. Let $\tilde{M}$ and $\tilde{N}$ be two triangular fuzzy numbers denoted by $\tilde{M} = (m, \alpha, \beta)$, $\tilde{N} = (n, \gamma, \delta)$. Dubois and Prade [1] introduced fuzzy division with the expression given below.

$$\tilde{X} = \frac{\tilde{M}}{\tilde{N}} = (\frac{m}{n}, \frac{m\delta + n\alpha}{n^2}, \frac{m\gamma + n\beta}{n^2})$$

In Table 1, three illustrations are given utilizing the expression given by [1].

Giachetti and Young [2,3] introduced a new approximation method for fuzzy multiplication and division. While a new approximation method for multiplication was introduced in [2], [3] included a new approach for fuzzy division. The motivation behind those papers is that the resulting fuzzy number should be obtained with less error which means that the fuzziness of the resulting fuzzy number is expected to grow less. the mathematical expressions for the new approximation method for fuzzy division [3] was given as follows:

$$Q_{N(L)} = D_L + G(\alpha, n)\tau_L(n, \bar{\lambda})(b-a)$$

$$Q_{N(R)} = D_R + G(\alpha, n)\tau_R(n, \bar{\rho})(c-b)$$

where D_L is the actual division using α -cut, $G(\alpha, n)$ is the correction term which is written in terms of α. The linear fit between the geometric mean of the spread ratios and the number included in division is denoted by $\tau_L(n, \bar{\lambda})$. $(b-a)$ is the left spread value. The same terms can be written for the right side of the resulting fuzzy number.

Herein, we are not going into the detailed explanations about how the method proposed by [2,3] is applied to triangular fuzzy numbers. The more detailed explanations can be found in [2,3].

4 The Proposed Method

Instead of using numbers given in parameterized form of the fuzzy numbers to perform fuzzy arithmetic, we propose a new method based on linear programming utilizing the geometric features and definition of fuzzy numbers.

It is assumed that two asymmetric fuzzy numbers are divided. The parameterized form of these fuzzy numbers are denoted by (l_1, c_1, r_1) and (l_2, c_2, r_2), where l_1 and l_2 , c_1 and c_2, and, r_1 and r_2 denote left end points, center points and, right end points, respectively.

Let $\tilde{A} = (l_1, c_1, r_1)$ and $\tilde{B} = (l_2, c_2, r_2)$ be two fuzzy numbers. The resulting fuzzy number obtained from the division of these fuzzy numbers can be written as follows.

$$\tilde{X} = \frac{\tilde{A}}{\tilde{B}} \tag{1}$$

where $\tilde{X} = (x_l, x_c, x_r)$ is a fuzzy number.

The constraints and the objective function of the linear programming problem for the fuzzy division given in (1) are defined as follows:

The first constraint is based on the area of fuzzy numbers. The area of the resulting fuzzy number $\tilde{X}$ should be equal to or less than the area obtained from the division of two areas of fuzzy numbers $\tilde{A}$ and $\tilde{B}$. Thus, the mathematical expression for the first constraint is written as follows:

$$\frac{(x_r - x_l)}{2} \leq \frac{\dfrac{(r_1 - l_1)}{2}}{\dfrac{(r_2 - l_2)}{2}} \tag{2}$$

where the areas of fuzzy number $\tilde{A}$ and $\tilde{B}$ are computed as $\frac{(r_1-l_1)}{2}$ and $\frac{(r_2-l_2)}{2}$ respectively and the area of fuzzy number $\tilde{X}$ is computed as $\frac{(x_r-x_l)}{2}$.

Expression (2) can be rewritten as follows:

$$x_r - x_l \le \frac{2(r_1-l_1)}{(r_2-l_2)} \tag{3}$$

The second and the third constraints are defined using the ratios of center values to left and right spreads respectively. The second constraint is constructed based on the left spreads. The left spread value over center value for $\tilde{X}$ should be equal to or less than the division of the same ratios calculated for $\tilde{A}$ and $\tilde{B}$. Thus, the mathematical expression of the second constraint is written as follows:

$$\frac{(x_c-x_l)}{x_c} \le \frac{\frac{(c_1-l_1)}{c_1}}{\frac{(c_2-l_2)}{c_2}} \tag{4}$$

The center value of $\tilde{X}$ is equal to the division of the center values of $\tilde{A}$ and $\tilde{B}$ since all the available methods in the literature [1,2,3] have used this notion. Thus, the expression given in (4) is rewritten as follows:

$$x_l \ge \frac{c_1}{c_2} - \frac{(c_1-l_1)}{(c_2-l_2)} \tag{5}$$

In a similar manner, the third constraint is constructed as follows:

$$x_r \le \frac{c_1}{c_2} + \frac{(r_1-c_1)}{(r_2-c_2)} \tag{6}$$

The fourth and fifth constraints can be defined based on the definition of fuzzy number. The left end point of a fuzzy number should be less than the center value. Then, the mathematical expression for the fourth constraint is given below.

$$x_l < \frac{c_1}{c_2} \tag{7}$$

The right end point of a fuzzy number should be greater than the center value. Then, the mathematical expression for the fifth constraint is given below.

$$x_r > \frac{c_1}{c_2} \tag{8}$$

When taking the second (5), third (6), fourth (7), and the fifth (8) constraints into account, the upper and lower bounds for x_l and x_r can be written as given below.

$$\frac{c_1}{c_2} - \frac{(c_1-l_1)}{(c_2-l_2)} \le x_l < \frac{c_1}{c_2}$$

$$\frac{c_1}{c_2} < x_r \le \frac{c_1}{c_2} + \frac{(r_1-c_1)}{(r_2-c_2)}$$

In the objective function, what we are looking for is a fuzzy number $\tilde{X}$ which contains the fuzziness resulted from dividing two fuzzy numbers. Therefore, the difference between the end points x_r and x_l should be maximized. Then, the objective function is given below.

$$f(x) = x_r - x_l \tag{9}$$

Instead of maximizing the objective function in (9), if the difference between x_r and x_l were tried to be minimized, the value of $f(x)$ would take zero. This means that the resulting fuzzy number becomes $\frac{c_1}{c_2}$ having zero spreads, that is, it is a degenerated fuzzy number.

Finally, the defined linear programming problem is given below.

$Max \quad f(x) = x_r - x_l$

Subject to

$$\begin{aligned} & x_r - x_l \le \frac{2(r_1-l_1)}{(r_2-l_2)} \\ & x_l \ge \frac{c_1}{c_2} - \frac{(c_1-l_1)}{(c_2-l_2)} \\ & x_r \le \frac{c_1}{c_2} + \frac{(r_1-c_1)}{(r_2-c_2)} \\ & x_l < \frac{c_1}{c_2} \\ & x_r > \frac{c_1}{c_2} \end{aligned} \tag{9}$$

In the formulation of the problem, there are two unknown variables x_r and x_l. In the result of solving the linear programming problem, the left and right end points of $\tilde{X}$, whose center value known as $\frac{c_1}{c_2}$ in advance, are obtained.

5 Implementation

Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy numbers which are denoted by $\tilde{A} = (l_1, c_1, r_1)$ and $\tilde{B} = (l_2, c_2, r_2)$ where l_1 and l_2, c_1 and c_2, and, r_1 and r_2 denote left end points, center points and, right end points, respectively.

For example, let $\tilde{A}$ = (70,100,130) and $\tilde{B}$ = (4,10,16) be two asymmetric fuzzy numbers. The proposed method is used in order to divide these two fuzzy numbers. Then, the linear programming problem is given below.

$$Max\ f(x) = x_r - x_l$$

Subject to

$$x_r - x_l \le \frac{120}{8}$$

$$x_l \ge 10 - \frac{30}{6}$$

$$x_r \le 10 + \frac{30}{6}$$

$$x_l < 10$$

$$x_r > 10$$

If the problem is rewritten, the new formulation is given below.

$$Max\ f(x) = x_r - x_l$$

Subject to

$$x_r - x_l \le 15$$

$$x_l \ge 5$$

$$x_r \le 15$$

$$x_l < 10$$

$$x_r > 10$$

When the problem is solved, the solution gives $x_r = 15$ and $x_l = 5$. Thus, the result of division is $\tilde{X} = (5,10,15)$.

When the fuzzy numbers having different features such as greater center values or wide spread values are divided, results are given in order to show the applicability of the proposed method. For comparison, the results of the solved examples based on the proposed method and other methods available in the literature are summarized in Table 1. The real number under the resulting fuzzy number denotes the difference between the end points. For example, when $\tilde{A} = (1,5,7)$ is divided by $\tilde{B} = (5,20,32)$, using the proposed method gives $\tilde{X} = (-0.0167,0.25,0.4167)$. Then the difference between end points of this result is 0.4334. When the results are examined, the proposed method has resulting fuzzy number whose fuzziness is less than the fuzziness of the method proposed by Dubois and Prade [1]. The first two examples produce approximately the same fuzziness for the methods proposed by Giachetti and Young [3] and the proposed method. However, when the third example is exaimed, the proposed method has produced much narrower fuzziness than the method proposed by Giachetti and Young [3] since the parameters used in their method are out of the range specified by Giachetti and Young [3]. Therefore, the proposed method should be preferred to the Giachetti and Young's method since the proposed method does not impose any limitation on any parameters.

Table 1. The results of the different fuzzy division methods

	(1,5,7) / (5,20,32)	
	results	range
The proposed method	(0.0,0.25,0.4167)	0.4167
Dubois and Prade	(-0.05,0.25,0.5375)	0.5875
Giachetti and Young	(0.01,0.25,0.36)	0.3500
	(70,110,130) / (4,10,16)	
	results	range
The proposed method	(5,10,15)	10
Dubois and Prade	(1,10,19)	18
Giachetti and Young	(6.7,10,17.4)	10.70
	(18,147,178) / (3,42,87)	
	results	range
The proposed method	(0.192,3.500,4.002)	3.81
Dubois and Prade	(-3.33,3.5,7.49)	10.82
Giachetti and Young	(0.765,3.5,29.18)	28.42

It is a known fact that the variables in linear programming take positive values. However, the components of the resulting fuzzy number can take positive and negative values at the same time. In order to solve this problem in fuzzy division, the linear programming problem is modified. This modification enables the proposed method to generate negative end points just as doing arithmetic operations in real lines. Therefore, the limitation of the linear programming can be eliminated.

When the center value of the resulting fuzzy number nears to zero or the difference between left spread values of dividend fuzzy number and of divisor fuzzy number is large, the left end point of the resulting fuzzy number x_l can be a negative number. The constraint given below can point that x_l might have negative value.

$$x_l \ge \frac{c_1}{c_2} - \frac{(c_1 - l_1)}{(c_2 - l_2)}$$

Thus, when the condition

$$\frac{c_1}{c_2} < \frac{(c_1 - l_1)}{(c_2 - l_2)} \tag{10}$$

is realized, the domain of x_l contains negative values. This is the mathematical expression of the verbal explanation given above. However, the results obtained from solving the linear programming problem for fuzzy division can not be negative. In order to overcome this issue, the variable x_l can be redefined as follows:

$$x_l = x_{l1} - x_{l2}$$

where x_{l1} and x_{l2} are positive numbers. When the new expression given above is written in the problem constructed for fuzzy division, the new linear programming problem is obtained as follows:

$$Max \quad f(x) = x_r - x_{l1} + x_{l2}$$

Subject to

$$x_r - x_{l1} + x_{l2} \le \frac{2(r_1 - l_1)}{(r_2 - l_2)}$$

$$x_{l1} - x_{l2} \ge \frac{c_1}{c_2} - \frac{(c_1 - l_1)}{(c_2 - l_2)}$$

$$x_r \le \frac{c_1}{c_2} + \frac{(r_1 - c_1)}{(r_2 - c_2)} \tag{11}$$

$$x_{l1} - x_{l2} < \frac{c_1}{c_2}$$

$$x_r > \frac{c_1}{c_2}$$

The formulation constructed above should be solved in order to obtain the parameters of the resulting fuzzy number for fuzzy division. There are three unknown variables x_r, x_{l2} and x_{l1}.Then, the left and right end points of $\tilde{X}$, whose center value is known as $\frac{c_1}{c_2}$ in advance, are obtained. The left end point of the resulting fuzzy number is obtained by subtracting x_{l2} from x_{l1}.

For example, let $\tilde{A} = (1,5,7)$ and $\tilde{B} = (5,20,32)$ be two asymmetric fuzzy numbers. When the values are written as in (10), the inequality given below is obtained.

$$\frac{5}{20} < \frac{(5-1)}{(20-5)}$$

Then

$$0.250 < 0.267$$

is obtained.

It is seen that the condition is satisfied so the left end point of the resulting fuzzy number x_l will be a negative number. Therefore, the proposed method is used in order to divide these two fuzzy numbers. Then, the modified linear programming problem is given below.

$$Max \quad f(x) = x_r - x_{l1} + x_{l2}$$

Subject to

$$x_r - x_{l1} + x_{l2} \le 0.444$$

$$x_{l1} - x_{l2} \ge -0.017$$

$$x_r \le 0.417$$

$$x_{l1} - x_{l2} < 0.250$$

$$x_r > 0.250$$

When the problem is solved, the solution gives $x_r = -0.017$ and $x_l = 0.417$. Thus, the result of division is $\tilde{X} = (-0.017, 0.250, 0.417)$.

It should be noted that checking the condition given in (9) is not necessary when fuzzy division is conducted since the modified version of linear programming formulation eliminates finding just positive numbers. Even though the condition given in (9) is not satisfied, results obtained from the solution of the formulation (9) are same as those obtained from the formulation (11). Therefore, the formulation (11) can be used every time without checking the condition given in (9). In short, the formulation given in (9) is a special type of the formulation given in (11) that can produce positive and negative end points.

6 Conclusion

There have been various methods available in the literature in order to do fuzzy parameterized arithmetic since using membership functions in fuzzy arithmetic is a complex process. However, utilizing parameterized fuzzy arithmetic just produces approximated results and using parameterized fuzzy arithmetic repeatedly causes to increase the fuzziness of the resulting fuzzy number, which is obtained at end of the fuzzy operations.

In this paper, we proposed a new method based on linear programming for fuzzy division of triangular fuzzy numbers. When the constraints of the linear programming problem are constructed, we exploit the geometric features and definition of a fuzzy number. The proposed method calculates the center value of the resulting fuzzy number just as the other methods [1,2,3] available in the literature. Thus, the left and right end points of the resulting fuzzy number are unknowns. Then, these values are called decision variables for the linear programming problem. The resulting fuzzy number is obtained by solving this linear programming problem consisting of five constraints.

In order to show the applicability of the proposed method, various examples are solved. The results of the proposed method are compared with the results of the other methods in the literature [1,2,3]. It is clearly seen that the proposed method produces much narrower fuzziness than those produced by the other methods. Another advantage of the proposed method is no dependencies for any parameters. For instance, Giachetti and Young [2,3] claimed that their method works well for parameters in specified ranges. They called these parameters as left and right side ratios.

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