The effects of suction and injection on a moving flat plate in a parallel stream with prescribed surface heat flux

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Abstract: - The effect of surface mass flux on a moving flat plate in a moving fluid with prescribed surface heat flux is studied. The governing partial differential boundary layer equations are first transformed into ordinary differential equations before being solved numerically by a finite difference method. The features of the flow and heat transfer characteristics for different values of the governing parameters are analyzed and discussed. It is found that dual solutions exist when the plate and the free stream move in the opposite directions. The results indicate that the range of known dual solutions increases with suction and decreases with injection and the rate of heat transfer increases with increasing heat flux exponent parameter.

Key-Words: - Heat transfer, Moving plate, Moving fluid, Boundary layer, Heat flux, Suction/injection, Dual solutions

1 Introduction

The classical boundary layer flow past a flat plate or the Blasius problem has attracted considerable interest of many researchers since introduced by Blasius [1]. Blasius considered the boundary layer flow on a fixed flat plate, without considering the heat transfer aspects. The study of the flow and heat transfer in an electrically conducting fluid has many practical applications in manufacturing process in industry. The thermal fluid flow problem have been extensively studied numerically, theoretically as well as experimentally (see [2-4]). The Blasius equation has never yielded to exact analytical solution, and Blasius himself gave matching inner and outer series solutions.

Different from Blasius [1], Sakiadis [5] considered the boundary-layer flow on a moving plate in a quiescent ambient fluid. He found exactly the same equation as Blasius, but the boundary conditions are different. Ishak et al. [6] extended the classical problems of Blasius [1] and Sakiadis [5], by considering a flat plate moving in the same or opposite directions to a parallel free stream, all with constant velocities. Similar problems with various boundary conditions and in different situations have been considered by Klemp and Acrivos [7], Merkin [8], Abdelhafez [9], Hussaini et al. [10], Afzal et al. [11], Bianchi and Viskanta [12], Lin and Huang [13], Chen [14], Afzal [15], Abraham and Sparrow [16], Sparrow and Abraham [17] and Weidman et al. [18]. However, the existence of dual solutions was reported only in the papers by Merkin [8], Afzal et al. [11], Afzal [15] and Weidman et al. [18]. Examples of practical applications include the aerodynamic extrusion of plastic sheets, the cooling of an infinite metallic plate in a cooling bath, the boundary layers along material handling conveyers and along a liquid film in condensation processes, glass blowing, continuous casting spinning of fibers, etc.

It is well known that the skin friction along a continuous moving flat surface in a quiescent fluid (Sakiadis problem) is about 30% higher than those along a static flat plate in a moving fluid (Blasius problem). As discussed by Abdelhafez [9], these two problems are physically different and can not be mathematically transformed into one another. This fact indicates that we cannot regard the velocity difference $|U_w - U_\infty|$ as a relative velocity in the sense of Galilei. Excellent descriptions of the problem of laminar fluid flow which results from the simultaneous motions of a free stream and its bounding surface in the same direction have been in detail examined by Abraham and Sparrow [16] and Sparrow and Abraham [17] using the relative-velocity model, which uses magnitude of the relative velocity in conjunction with the drag formula for the case in which only one of the participating media is in motion. They found that the results of exact solutions demonstrate that this model is flawed and under predicts the drag force, and thus the use of the relative-velocity model can lead to gross errors in the drag force. The extent of the error increases as the two participating velocities approach each other in magnitude. The solution depends not only on the velocity difference $|U_w - U_\infty|$ but also on the velocity ratio U_w / U_∞. Following Afzal et al. [11], in this paper we define the reference velocity U as $U = U_w + U_\infty$, and thus a single set of boundary conditions is formed irrespective of whether $U_w > U_\infty$ or $U_w < U_\infty$.

The effects of suction and injection on a moving flat plate opposite to the free stream in a power law fluid, into or out of the origin at uniform speed and in the same or opposite direction to the free stream were studied by Weidman et al. [18], Zheng et al. [19] and Ishak et al. [20], respectively. Continuous surface heat transfer problems have many practical applications in industrial manufacturing processes. Such processes are hot rolling, wire drawing and glass fiber production. Problems with variable surface heat flux has been introduced in many other studies [21-24]. The purpose of this investigation is to study the effects of suction and injection on a moving flat plate in a parallel stream with variable surface heat flux.

2 Problem Formulation

Consider a two-dimensional boundary layer flow on a fixed or continuously moving permeable flat surface immersed in a viscous and incompressible fluid of constant temperature T_∞. It is assumed that the plate is subjected to a variable surface heat flux $q_w(x) = ax^n$, where a and n are constants and x is the distance from the slit where the plate is issued, and moves in the same or opposite direction to the free stream, both with constant velocities U_w and U_∞, respectively. Under these assumptions, the boundary layer equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}, \tag{3}$$

subject to the boundary conditions

$$\begin{aligned} &u = U_w, \quad v = 0, \quad \frac{\partial T}{\partial y} = -\frac{q_w}{k} \quad \text{at} \quad y = 0, \\ &u \to U_\infty, \quad T \to T_\infty \quad \text{as} \quad y \to \infty, \end{aligned} \tag{4}$$

where u and v are the velocity components in the x- and y- directions, respectively, ν, T, q_w, k, α and V_w are, respectively the kinematic viscosity, temperature of the fluid in the boundary layer, surface heat flux, thermal conductivity, thermal diffusivity and the mass transfer velocity through the surface of the plate.
In order to solve Eqs. (1) – (3) subject to the boundary conditions (4), we introduce the following similarity transformation:

$$\begin{aligned} \eta &= \left(\frac{U}{\nu x}\right)^{1/2} y, \\ f(\eta) &= \frac{\psi}{(\nu x U)^{1/2}}, \\ \theta(\eta) &= \frac{k(T - T_\infty)}{q_w}\left(\frac{U}{\nu x}\right)^{1/2}, \end{aligned} \tag{5}$$

where U is the composite velocity defined as $U = U_w + U_\infty$ (Afzal et al. [11]). Further, ψ is the stream function defined as $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$, which identically satisfies Eq. (1). Using (5) we obtain

$$\begin{aligned} u &= Uf'(\eta), \\ v &= \frac{1}{2}\left(\frac{\nu U}{x}\right)^{1/2}(\eta f' - f), \end{aligned} \tag{6}$$

where primes denote differentiation with respect to η. In order that similarity solutions of Eqs. (1) – (3) exist, we take

$$V_w(x) = -\frac{1}{2}\left(\frac{\nu U}{x}\right)^{1/2} f_w, \tag{7}$$

where $f_w = f(0)$ is a non-dimensional constant which determines the transpiration rate at the surface, with $f_w > 0$ for suction, $f_w < 0$ for injection, and $f_w = 0$ corresponds to an impermeable plate. By employing the similarity variables (5), Eqs. (2) and (3) reduce to the following ordinary differential equations:

$$f''' + \frac{1}{2} f f'' = 0, \tag{8}$$

$$\frac{1}{Pr}\theta'' + \frac{1}{2} f\theta' - \frac{1}{2}(2n+1) f'\theta = 0 . \tag{9}$$

The boundary conditions (4) now become

$$\begin{aligned} &f(0) = f_w, \quad f'(0) = \lambda, \quad \theta'(0) = -1, \\ &f'(\eta) \to 1-\lambda, \quad \theta(\eta) \to 0 \quad \text{as} \quad \eta \to \infty, \end{aligned} \tag{10}$$

where λ is the velocity ratio parameter defined by

$$\lambda = \frac{U_w}{U}. \tag{11}$$

with $\lambda > 0$ and $\lambda < 0$ correspond to the plate moving in the same and opposite directions to the free stream, respectively, while $\lambda = 0$ represents a fixed plate. We notice that for an impermeable plate $(f_w = 0)$, the flow problem under consideration reduces to that considered by Blasius [1] when $\lambda = 0$, and to that of Sakiadis [5] when $\lambda = 1$.

The physical quantities of interest are the skin friction coefficient C_f and the local Nusselt number Nu_x, which are defined as

$$\begin{aligned} C_f &= \frac{\tau_w}{\rho U^2/2}, \\ Nu_x &= \frac{x q_w}{k(T_w - T_\infty)}, \end{aligned} \tag{12}$$

where the wall shear stress τ_w and the wall heat flux q_w are given by

$$\begin{aligned} \tau_w &= \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}, \\ q_w &= -k\left(\frac{\partial T}{\partial y}\right)_{y=0}, \end{aligned} \tag{13}$$

with μ and k being the dynamic viscosity and thermal conductivity, respectively. Using the similarity variables (5), we obtain

$$\begin{aligned} &\frac{1}{2} C_f \operatorname{Re}_x^{1/2} = f''(0), \\ &Nu_x / \operatorname{Re}_x^{1/2} = \frac{1}{\theta(0)}, \end{aligned} \tag{14}$$

where $\operatorname{Re}_x = Ux/\nu$ is the local Reynolds number.

The nonlinear ordinary differential equations (8) and (9) subjected to (10) are solved numerically by a finite-difference scheme known as the Keller-box method, which is very familiar to the present authors (see Bachok et al. [25], Bachok and Ishak [26] and Ishak et al. [27, 28]).

3 Solution Procedure

3.1 Finite-difference method

To solve the transformed differential Eqs. (8) and (9) subjected to the boundary conditions (10), Eqs. (8) and (9) are first converted into a system of five first-order equations, and the difference equations are then expressed using central differences. For this purpose, we introduce new dependent variables $p(\eta)$, $q(\eta)$, $s(\eta) = \theta(\eta)$ and $t(\eta)$ so that Eqs. (8) and (9) can be written as

$$f' = p, \tag{15}$$

$$p' = q, \tag{16}$$

$$s' = t, \tag{17}$$

$$q' + \frac{1}{2} f q = 0, \tag{18}$$

$$\frac{1}{Pr} t' + \frac{1}{2} f t - \frac{1}{2}(2n+1) p s = 0. \tag{19}$$

In terms of the new dependent variables, the boundary conditions (10) are given by

$$\begin{aligned} &f(0) = f_w, \quad p(0) = \lambda, \quad t(0) = -1, \\ &p(\eta) \to 1-\lambda, \quad s(\eta) \to 0 \quad \text{as} \quad \eta \to \infty, \end{aligned} \tag{20}$$

We now consider the segment $\eta_{j-1}\eta_j$, with $\eta_{j-1/2}$ as the midpoint, which is defined as below:

$$\eta_0 = 0, \ \ \eta_j = \eta_{j-1} + h_j, \ \ \eta_J = \eta_\infty, \tag{21}$$

where h_j is the $\Delta\eta -$spacing and $j = 1, 2, ..., J$ is a sequence number that indicates the coordinate location. The finite-difference approximations to the ordinary differential equations (15)-(19) are written for the midpoint $\eta_{j-1/2}$ of the segment $\eta_{j-1}\eta_j$. This procedure gives

$$\frac{f_j - f_{j-1}}{h_j} = \frac{p_j + p_{j-1}}{2} = p_{j-1/2}, \tag{22}$$

$$\frac{p_j - p_{j-1}}{h_j} = \frac{q_j + q_{j-1}}{2} = q_{j-1/2}, \tag{23}$$

$$\frac{s_j - s_{j-1}}{h_j} = \frac{t_j + t_{j-1}}{2} = t_{j-1/2}, \tag{24}$$

$$\frac{q_j - q_{j-1}}{h_j} + \frac{1}{2}(fq)_{j-1/2} = 0, \tag{25}$$

$$\frac{1}{\mathrm{Pr}}\frac{t_j - t_{j-1}}{h_j} + \frac{1}{2}(ft)_{j-1/2} - \frac{1}{2}(2n+1)(ps)_{j-1/2} = 0. \tag{26}$$

Rearranging of expressions (22)-(26) gives

$$f_j - f_{j-1} - \frac{1}{2}h_j\left(p_j + p_{j-1}\right) = 0, \tag{27}$$

$$p_j - p_{j-1} - \frac{1}{2}h_j\left(q_j + q_{j-1}\right) = 0, \tag{28}$$

$$s_j - s_{j-1} - \frac{1}{2}h_j\left(t_j + t_{j-1}\right) = 0, \tag{29}$$

$$q_j - q_{j-1} + \frac{1}{8}h_j\left(f_j + f_{j-1}\right)\left(q_j + q_{j-1}\right) = 0, \tag{30}$$

$$\begin{aligned}&\frac{1}{Pr}\left(t_j - t_{j-1}\right) + \frac{1}{8}h_j\left(f_j + f_{j-1}\right)\left(t_j + t_{j-1}\right) - \\ &\qquad \frac{1}{2}h_j\left(2n+1\right)\left(p_j + p_{j-1}\right)\left(s_j + s_{j-1}\right) = 0.\end{aligned} \tag{31}$$

Equations (27)-(31) are imposed for $j = 1, 2, 3, \ldots, J$, and the transformed boundary layer thickness η_J is to be sufficiently large so that it is beyond the edge of the boundary layer. The boundary conditions are

$$\begin{aligned}&f_0 = f_w, \quad p_0 = \lambda, \quad t_0 = -1, \\ &p_J = 1 - \lambda, \quad s_J = 0.\end{aligned} \tag{32}$$

3.2 Newton's method

To linearize the nonlinear system (27)-(31), we use Newton's method, by introducing the following expressions:

$$\begin{aligned}&f_j^{(k+1)} = f_j^{(k)} + \delta f_j^{(k)}, \; p_j^{(k+1)} = p_j^{(k)} + \delta p_j^{(k)}, \\ &q_j^{(k+1)} = q_j^{(k)} + \delta q_j^{(k)}, \; s_j^{(k+1)} = s_j^{(k)} + \delta s_j^{(k)}, \\ &t_j^{(k+1)} = t_j^{(k)} + \delta t_j^{(k)},\end{aligned} \tag{33}$$

where $k = 0, 1, 2, \ldots$. We then insert the left-hand side expressions in place of f_j, p_j, q_j, s_j and t_j into Eqs. (27)-(31) and drop the terms that are quadratic in $\delta f^{(k)}$, $\delta p^{(k)}$, $\delta q^{(k)}$, $\delta s^{(k)}$ and $\delta t^{(k)}$. This procedure yields the following linear system (the superscript k is dropped for simplicity):

$$\delta f_j - \delta f_{j-1} - \frac{h_j}{2}\left(\delta p_j + \delta p_{j-1}\right) = \left(r_1\right)_{j-1/2}, \tag{34}$$

$$\delta p_j - \delta p_{j-1} - \frac{h_j}{2}\left(\delta q_j + \delta q_{j-1}\right) = \left(r_2\right)_{j-1'2}, \tag{35}$$

$$\delta s_j - \delta s_{j-1} - \frac{h_j}{2}\left(\delta t_j + \delta t_{j-1}\right) = \left(r_3\right)_{j-1/2}, \tag{36}$$

$$\begin{aligned}&\left(a_1\right)\delta q_j + \left(a_2\right)\delta q_{j-1} + \left(a_3\right)\delta f_j + \\ &\qquad \left(a_4\right)\delta f_{j-1} = \left(r_4\right)_{j-1/2},\end{aligned} \tag{37}$$

$$\begin{aligned}&\left(b_1\right)\delta t_j + \left(b_2\right)\delta t_{j-1} + \left(b_3\right)\delta f_j + \left(b_4\right)\delta f_{j-1} + \\ &\left(b_5\right)\delta p_j + \left(b_6\right)\delta p_{j-1} + \left(b_7\right)\delta s_j + \\ &\qquad \left(b_8\right)\delta s_{j-1} = \left(r_5\right)_{j-1/2},\end{aligned} \tag{38}$$

where

$$\begin{aligned}&\left(a_1\right)_j = 1 + \frac{1}{4}h_j f_{j-1/2}, \left(a_2\right)_j = \left(a_1\right)_j - 2, \\ &\left(a_3\right)_j = \frac{1}{4}h_j q_{j-1/2}, \quad \left(a_4\right)_j = \left(a_3\right)_j, \\ &\left(b_1\right)_j = 1 + \frac{1}{4}Pr\, h_j f_{j-1/2}, \left(b_2\right)_j = \left(b_1\right)_j - 2, \\ &\left(b_3\right)_j = \frac{1}{4}Pr\, h_j t_{j-1/2}, \quad \left(b_4\right)_j = \left(b_3\right)_j, \\ &\left(b_5\right)_j = -\frac{1}{4}Pr\left(2n+1\right)h_j s_{j-1/2}, \left(b_6\right)_j = \left(b_5\right)_j, \\ &\left(b_7\right)_j = -\frac{1}{4}Pr\left(2n+1\right)h_j p_{j-1/2}, \left(b_8\right)_j = \left(b_7\right)_j,\end{aligned} \tag{39}$$

and

$$\begin{aligned}
(r_1)_{j-1/2} &= -f_j + f_{j-1} + h_j p_{j-1/2}, \\
(r_2)_{j-1/2} &= -p_j + p_{j-1} + h_j q_{j-1/2}, \\
(r_3)_{j-1/2} &= -s_j + s_{j-1} + h_j t_{j-1/2}, \\
(r_4)_{j-1/2} &= -(q_j - q_{j-1}) - \frac{1}{2} h_j (fq)_{j-1/2}, \\
(r_5)_{j-1/2} &= -(t_j - t_{j-1}) - \frac{1}{2} Pr\, h_j (ft)_{j-1/2} + \\
&\quad \frac{1}{2} h_j (2n+1)(ps)_{j-1/2}.
\end{aligned} \tag{40}$$

The boundary conditions (32) become

$$\begin{aligned}
&\delta f_0 = 0, \quad \delta p_0 = 0, \quad \delta t_0 = 0 \\
&\delta p_J = 0, \quad \delta s_J = 0,
\end{aligned} \tag{41}$$

which just express the requirement for the boundary conditions to remain constant during the iteration process.

3.3 Block-elimination method

The linearized difference equations (34)-(38) can be solved by the block-elimination method as outlined by Na [29] and Cebeci and Bradshaw [30], since the system has block-tridiagonal structure. Commonly, the block-tridiagonal structure consists of variables or constants, but here an interesting feature can be observed that it consists of block matrices. In a matrix-vector form, Eqs. (34)-(38) can be written as

$$\boldsymbol{A\delta} = \boldsymbol{r} \tag{42}$$

where

$$\boldsymbol{A} = \begin{bmatrix}
[A_1] & [C_1] & & & & \\
[B_2] & [A_2] & [C_2] & & & \\
 & & \ddots & & & \\
 & & \ddots & & & \\
 & & \ddots & & & \\
 & & & [B_{J-1}] & [A_{J-1}] & [C_{J-1}] \\
 & & & & [B_J] & [A_J]
\end{bmatrix}$$

$$\boldsymbol{\delta} = \begin{bmatrix} [\delta_1] \\ [\delta_2] \\ \vdots \\ [\delta_{J-1}] \\ [\delta_J] \end{bmatrix} \quad \text{and} \quad \boldsymbol{r} = \begin{bmatrix} [r_1] \\ [r_2] \\ \vdots \\ [r_{J-1}] \\ [r_J] \end{bmatrix}.$$

The elements of the matrices are as follows:

$$[A_1] = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
-\frac{1}{2}h_1 & 0 & 0 & -\frac{1}{2}h_1 & 0 \\
0 & -1 & 0 & 0 & -\frac{1}{2}h_1 \\
(a_2)_1 & 0 & (a_3)_1 & (a_1)_1 & 0 \\
0 & (b_8)_1 & (b_3)_1 & 0 & (b_1)_1
\end{bmatrix} \tag{43}$$

$2 \le j \le J$,

$$[A_j] = \begin{bmatrix}
-\frac{1}{2}h_j & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & -\frac{1}{2}h_j & 0 \\
0 & -1 & 0 & 0 & -\frac{1}{2}h_j \\
0 & 0 & (a_3)_j & (a_1)_j & 0 \\
(b_6)_j & (b_8)_j & (b_3)_j & 0 & (b_1)_j
\end{bmatrix} \tag{44}$$

$$[B_j] = \begin{bmatrix}
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2}h_j & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{2}h_j \\
0 & 0 & (a_4)_j & (a_2)_j & 0 \\
0 & 0 & (b_4)_j & 0 & (b_2)_j
\end{bmatrix} \tag{45}$$

$1 \le j \le J-1$,

$$[C_j] = \begin{bmatrix}
-\frac{1}{2}h_j & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
(b_5)_j & (b_7)_j & 0 & 0 & 0
\end{bmatrix} \tag{46}$$

$2 \le j \le J$,

$$[\delta_1] = \begin{bmatrix} \delta q_0 \\ \delta s_0 \\ \delta f_1 \\ \delta q_1 \\ \delta t_1 \end{bmatrix}, \qquad [\delta_j] = \begin{bmatrix} \delta p_{j-1} \\ \delta s_{j-1} \\ \delta f_j \\ \delta q_j \\ \delta t_j \end{bmatrix} \tag{47}$$

and $1 \le j \le J$,

$$[r_j] = \begin{bmatrix} (r_1)_{j-1/2} \\ (r_2)_{j-1/2} \\ (r_3)_{j-1/2} \\ (r_4)_{j-1/2} \\ (r_5)_{j-1/2} \end{bmatrix} \tag{48}$$

To solve Eq. (42), we assume that $\boldsymbol{A}$ is nonsingular and it can be factorized as

$$\boldsymbol{A} = \boldsymbol{LU}, \tag{49}$$

where

$$\boldsymbol{L} = \begin{bmatrix} [\alpha_1] & & & & \\ [B_2] & [\alpha_2] & & & \\ & & \ddots & & \\ & & \ddots & [\alpha_{J-1}] & \\ & & & [B_J] & [\alpha_J] \end{bmatrix}$$

and

$$\boldsymbol{U} = \begin{bmatrix} [I] & [\Gamma_1] & & & \\ & [I] & & & \\ & & \ddots & & \\ & & \ddots & [I] & [\Gamma_{J-1}] \\ & & & & [I] \end{bmatrix}$$

where $[I]$ is a 5×5 identity matrix, while $[\alpha_i]$ and $[\Gamma_i]$ are 5×5 matrices in which elements are determined by the following equations:

$$[\alpha_i] = [A_1], \tag{50}$$

$$[A_1][\Gamma_1] = [C_1], \tag{51}$$

$$[\alpha_i] = [A_1] - [B_j][\Gamma_{J-1}], \quad j = 2,3,\ldots,J, \tag{52}$$

$$[\alpha_i][\Gamma_j] = [C_j], \quad j = 2,3,\ldots, J-1. \tag{53}$$

Substituting Eq. (49) into Eq. (42), we obtain

$$\boldsymbol{LU\delta} = \boldsymbol{r}. \tag{54}$$

If we define

$$\boldsymbol{U\delta} = \boldsymbol{W}, \tag{55}$$

Eq. (54) becomes

$$\boldsymbol{LW} = \boldsymbol{r}, \tag{56}$$

where

$$\boldsymbol{W} = \begin{bmatrix} [W_1] \\ [W_2] \\ \vdots \\ [W_{J-1}] \\ [W_J] \end{bmatrix},$$

and $[W_j]$ are 5×1 column matrices. The elements of $\boldsymbol{W}$ can be determined from Eq. (55) by the following relations:

$$[\alpha_1][W_1] = [r_1], \tag{57}$$

$$[\alpha_j][W_j] = [r_j] - [B_j][W_{j-1}], \quad 2 \le j \le J. \tag{58}$$

When the elements of $\boldsymbol{W}$ have been found, Eq. (55) gives the solution for $\boldsymbol{\delta}$ in which the elements are found from the following relations:

$$[\delta_J] = [W_J], \tag{59}$$

$$[\delta_j] = [W_j] - [\Gamma_j][\delta_{j+1}], \quad 1 \le j \le J-1. \tag{60}$$

Once the elements of $\boldsymbol{\delta}$ are found, Eqs. (34)-(38) can be used to find the $(k+1)$th iteration in Eq. (33). These calculations are repeated until the convergence criterion is satisfied. In laminar boundary layer calculation, the wall shear stress parameter $q(0)$ is commonly used as the convergence criterion (Cebeci and Bradshaw [31]).

This is probably because in boundary layer calculations, it is found that the greatest error usually appears in the wall shear stress parameter. Thus, this convergence criterion is used in the present study. Calculations are stopped when

$$\left| \delta q_0^{(k)} \right| < \epsilon_1, \tag{61}$$

where ϵ_1 is a small prescribed value. In this study, $\epsilon_1 = 0.00001$ is used, which gives about four decimal places accuracy for most of the predicted quantities as suggested by Bachok et al. [25] and Ali et al. [32].

The present method has a second-order accuracy, unconditionally stable and is easy to be programmed, thus making it highly attractive for production use. The only disadvantage is the large amount of once-and-for-all algebra needed to write the difference equations and to set up their solutions.

4 Results and Discussion

The step size $\Delta\eta$ in η, and the position of the edge of the boundary-layer η_∞ have to be adjusted for different values of the parameters to maintain the necessary accuracy. In this study, the values of $\Delta\eta$ between 0.001 and 0.1 were used, depending on the values of the parameters considered, in order that the numerical values obtained are independent of $\Delta\eta$ chosen, at least to four decimal places. However, a uniform grid of $\Delta\eta = 0.01$ was found to be satisfactory for a convergence criterion of 10^{-5} which gives accuracy to four decimal places, in nearly all cases. On the other hand, the boundary-layer thickness η_∞ between 4 and 50 was chosen where the infinity boundary condition is achieved. To assess the accuracy of the present method, comparison with the previously reported data available in the open literature is made.

The variations of the skin friction coefficient $f''(0)$ with λ are shown in Fig. 1, while the corresponding local Nusselt number $1/\theta(0)$ are shown in Fig. 2, for some values of f_w and n. It is seen that the solution is unique when $\lambda \geq 0$, while dual solutions are found to exist when $\lambda < 0$, i.e. when the plate and the free stream move in the opposite directions. The values of $f''(0)$ are positive when $\lambda < 0.5$, and they become negative when the value of λ exceeds 0.5, for all values of the suction/injection parameter f_w.

Figs. 1 and 2 show that for a particular value of f_w, the solution exists up to certain critical value of λ (say λ_c). Beyond this value, the boundary-layer approximations breakdown, and thus the numerical solution cannot be obtained. The boundary-layer separated from the surface at $\lambda = \lambda_c$, where λ_c denotes a critical value of λ. Based on our computations, $\lambda_c = -0.3953, -0.5482$ and -0.7323 for $f_w = -0.2$, 0 and 0.2, respectively. This value of λ_c is in agreement with those reported by Merkin [8], Afzal et al. [11], Weidman et al. [18], Ishak et al. [6, 20] and Hussaini et al. [10]. From this observation, it can be concluded that suction $(f_w > 0)$ delays the boundary-layer separation, while injection $(f_w < 0)$ accelerates it. In contrast to the classical boundary-layer theory, the separation occurs when the skin friction coefficient $f''(0) > 0$, and not at the point of vanishing wall shear stress.

The samples of velocity and temperature profiles for some values of parameters are presented in Figs. 3, 4 and 5, respectively. These profiles satisfy the far field boundary conditions (10) asymptotically, which support the numerical results, besides supporting the dual nature of the solutions presented in Figs. 1 and 2.

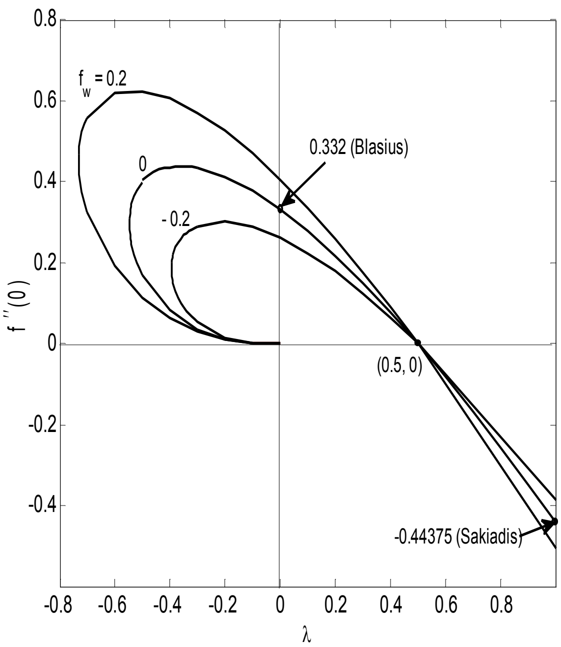

Fig. 1 Variation of the skin friction coefficient $f''(0)$ with λ for various values of f_w.

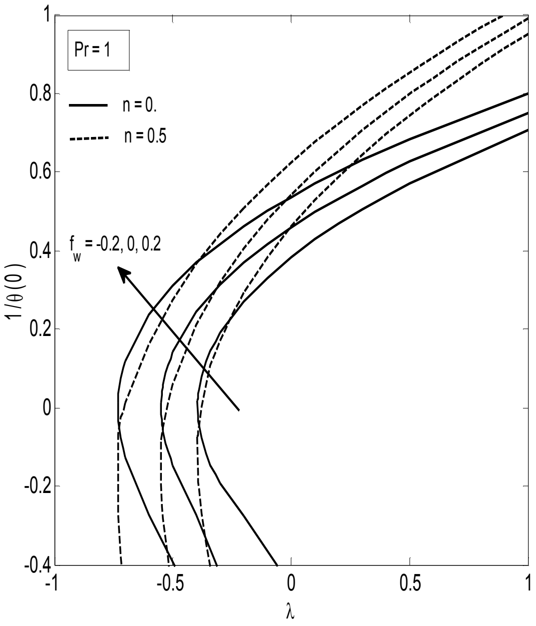

Fig.2 Variation of the local Nusselt number $1/\theta(0)$ with λ for various values of f_w and n when $\Pr = 1$.

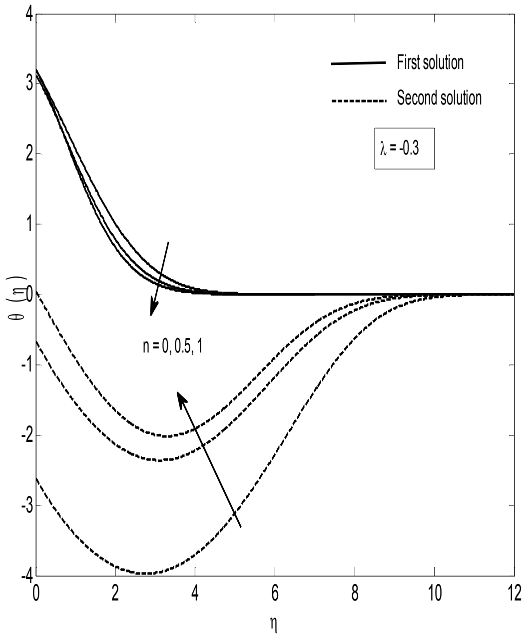

Fig. 4 Temperature profiles $\theta(\eta)$ for various values of n when $\lambda = -0.3$.

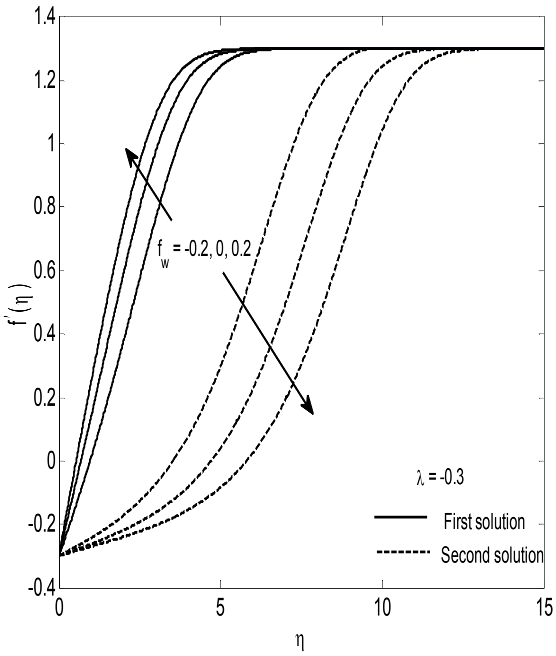

Fig. 3 Velocity profiles $f'(\eta)$ for various values of f_w when $\lambda = -0.3$.

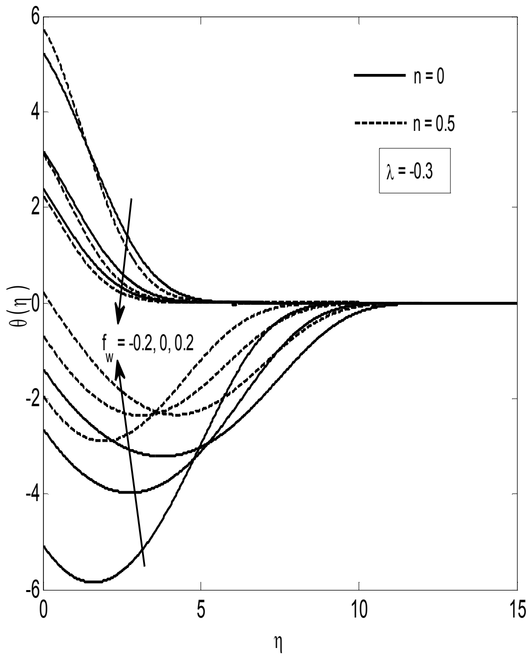

Fig. 5 Temperature profiles $\theta(\eta)$ for various values of f_w and n when $\lambda = -0.3$.

5 Conclusion

The development of the boundary layer on a fixed or moving surface parallel to a uniform free stream with variable surface heat flux has been investigated. The classical Blasius and Sakiadis problems are two particular cases of the present problem. Discussions for the effects of the heat flux exponent parameter, suction or injection parameter f_w and the velocity ratio parameter λ on the skin friction coefficient $f''(0)$ and the local Nusselt number $1/\theta(0)$ for $\Pr = 1$ have been done. From the present investigation, it may be concluded that:

- Dual solutions are obtained when the plate and the external stream move in the opposite directions $(\lambda < 0)$, and the solution exists up to a critical point λ_c (< 0).
- Suction $(f_w > 0)$ delays the boundary layer separation, while injection $(f_w < 0)$ accelerates it.
- Larger value of the heat flux exponent parameter increases the rate of heat transfer.

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