Hyperbolic Heat Equation as Mathematical Model for Steel Quenching of L-shape and T-shape Samples, Direct and Inverse Problems

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Abstract: - In this paper we develop mathematical model for 2D and 3D hyperbolic heat equation and construct an analytical solution of inverse problem for thin L-shape and T-shape samples. Solutions for both direct and inverse problems are obtained in closed analytical form as 2nd kind iterative integral equation, which is Fredholm integral equation with respect to space coordinate and Volterra integral equation with respect to time.

Key-Words: - Steel quenching, Hyperbolic heat equation, L-shape sample, T-shape sample, Direct problem, Inverse problem, Exact solution, Integral equation

1 Introduction

This is a continuation of our investigation [6]. In some of our previous papers we have considered problems of heat conduction for one element of the system with rectangular fin. In the papers [2]-[5], [7] we have investigated both steady-state problems and transient problems. Some ideas from these papers are used here for solving a mixed problem for an L-form sample.

Generally, in deriving the heat equation the heat flux vector $\mathbf{q}$ is assumed to satisfy Fourier's law

$$\mathbf{q} = -k\nabla T, \tag{1}$$

where the ***del*** operator, ∇, is defined in 3D as

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k},$$

$\mathbf{x}$ is an element from real coordinate space, t is time but $T(\mathbf{x},t)$ denotes the temperature in the body being considered, with thermal conductivity k.

Plugging equation (1) into the law of conservation of energy (the 1st law of thermodynamics)

$$\tilde{c}\rho\frac{\partial T}{\partial t} = -\nabla\cdot\mathbf{q} + q, \tag{2}$$

where $\tilde{c}$ is specific heat capacity, ρ is density of the body and $q(\mathbf{x},t)$ denotes an internal source or volumetric heat generation rate, one gets the well-known Fourier's heat conduction equation

$$\tilde{c}\rho\frac{\partial T}{\partial t} = \nabla\cdot(k\nabla T) + q. \tag{3}$$

This parabolic equation, although suitable in many situations, implies infinite thermal speed of propagation. One of such physical situations when Fourier's law is no longer valid is, for example, Intensive Quenching (IQ). When immersing the heated part into a quenchant, the initial heat flux tends to infinity but actually is finite ([11]). So Fourier's law at the initial time is no longer suited to describe heat propagation. In these situations Fourier's law (1) can be replaced by

$$\mathbf{q}(\mathbf{x}, t+\tau) = -k\nabla T,\ \tau > 0,$$

or, by its approximation (see, e.g., [14])

$$\tau\frac{\partial\mathbf{q}}{\partial t} + \mathbf{q} = -k\nabla T, \tag{4}$$

which is called the modified Fourier's equation or Cattaneo's equation. Here τ is a relaxation time. τ, k, $\tilde{c}$ and ρ are generally dependent on T and on the material But in the sequel we shall assume that these are constants.

Combining equation (4) with (2), we obtain hyperbolic heat conduction equation, which admits a finite speed of propagation for T

$$\tau\frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = a^2\nabla^2 T + \frac{\tau}{\tilde{c}\rho}\frac{\partial q}{\partial t} + \frac{q}{\tilde{c}\rho} \tag{5}$$

or

$$\frac{1}{C^2}\frac{\partial^2 T}{\partial t^2}+\frac{1}{a^2}\frac{\partial T}{\partial t}=\nabla^2 T+\frac{\tau}{k}\frac{\partial q}{\partial t}+\frac{q}{k}.$$

It is also known as Telegrapher's equation or a damped wave equation. Here C is the speed of heat propagation with $C^2=\frac{a^2}{\tau}$ and $a^2=\frac{k}{\tilde{c}\rho}$ is thermal diffusivity coefficient.

2 Mathematical Formulation of 3D Problem

In this section we are going to use equation (5) to describe IQ process and give full mathematical statement of 3D problem.

2.1 General Mathematical Statement of 3D Problem for L-shape Sample

Let's imagine an L-shape sample made up from two rectangular cuboids that are joined along the surface $x=\delta$. We'll call the vertical part the *base*. And it occupies the domain $\{x\in[0,\delta],y\in[0,1],z\in[0,\omega]\}$. But the horizontal part, where $\{x\in[\delta,\delta+l],y\in[0,b],z\in[0,\omega]\}$ and $b<1$, will be the *foot*. The sample is heated and then cooled rapidly (IQ process) in a suitable fluid, e.g., water or brine. As mentioned before, for describing intensive steel quenching of the sample we should use hyperbolic heat equation, which takes into account the finite speed of heat propagation. Since the sample consists of two parts, we are able to define IQ process for each part separately.

To state the boundary-value problem, we need to derive the equations of the functions describing IQ process in the sample, and then establish boundary and initial conditions. Let's assume that $V^0(x,y,z,t)$ denotes the temperature distribution in the base. Therefore the temperature distribution in the foot will be designed by $V(x,y,z,t)$. The equations of heat conduction therefore have the following form

$$\tau_{r,0}\frac{\partial^2 V^0}{\partial t^2}+\frac{\partial V^0}{\partial t}=a^2\left(\frac{\partial^2 V^0}{\partial x^2}+\frac{\partial^2 V^0}{\partial y^2}+\frac{\partial^2 V^0}{\partial z^2}\right)+\Psi_0(x,y,z,t), \quad (6)$$
$$x\in(0,\delta),y\in(0,1),z\in(0,\omega),t\in(0,T];$$

$$\tau_r\frac{\partial^2 V}{\partial t^2}+\frac{\partial V}{\partial t}=a^2\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}\right)+\Psi(x,y,z,t), \quad (7)$$
$$x\in(\delta,\delta+l),y\in(0,b),z\in(0,\omega),t\in(0,T],$$

where

$$\Psi_0=\frac{1}{\tilde{c}\rho}\left(\tau_{r,0}\frac{\partial q}{\partial t}(x,y,z,t)+q(x,y,z,t)\right),$$
$$\Psi=\frac{1}{\tilde{c}\rho}\left(\tau_r\frac{\partial q}{\partial t}(x,y,z,t)+q(x,y,z,t)\right).$$

When the heat generation is absent, $\Psi_0,\Psi=0$. The quantities a^2, k, $\tilde{c}$, ρ, τ_r, $\tau_{r,0}$ have the same meaning as in the preceding section.

On the surface of the sample a heat exchange takes place with the surrounding medium the temperature of which is $\Theta(x,y,z,t)$, so we have 3rd type boundary conditions. The boundary conditions for the base take the form

$$\left.\left(\frac{\partial V^0}{\partial x}-\beta V^0\right)\right|_{x=0}=-\beta\Theta|_{x=0},y\in[0,1],z\in[0,\omega], \quad (8)$$
$$\left.\left(\frac{\partial V^0}{\partial x}+\beta V^0\right)\right|_{x=\delta}=\beta\Theta|_{x=\delta},y\in[b,1],z\in[0,\omega], \quad (9)$$
$$\left.\left(\frac{\partial V^0}{\partial y}-\beta V^0\right)\right|_{y=0}=-\beta\Theta|_{y=0},x\in[0,\delta],z\in[0,\omega], \quad (10)$$
$$\left.\left(\frac{\partial V^0}{\partial y}+\beta V^0\right)\right|_{y=1}=\beta\Theta|_{y=1},x\in[0,\delta],z\in[0,\omega], \quad (11)$$
$$\left.\left(\frac{\partial V^0}{\partial z}-\beta V^0\right)\right|_{z=0}=-\beta\Theta|_{z=0},x\in[0,\delta],y\in[0,1], \quad (12)$$
$$\left.\left(\frac{\partial V^0}{\partial z}+\beta V^0\right)\right|_{z=\omega}=\beta\Theta|_{z=\omega},x\in[0,\delta],y\in[0,1], \quad (13)$$

where $\beta=\frac{h}{k}$, h is heat-transfer coefficient.

The boundary conditions imposed on the outer sides of the foot are of the same type

$$\left.\left(\frac{\partial V}{\partial x}+\beta V\right)\right|_{x=\delta+l}=\beta\Theta|_{x=\delta+l},y\in[0,b],z\in[0,\omega], \quad (14)$$
$$\left.\left(\frac{\partial V}{\partial y}-\beta V\right)\right|_{y=0}=-\beta\Theta|_{y=0},x\in[\delta,\delta+l],z\in[0,\omega], \quad (15)$$
$$\left.\left(\frac{\partial V}{\partial y}+\beta V\right)\right|_{y=b}=\beta\Theta|_{y=b},x\in[\delta,\delta+l],z\in[0,\omega], \quad (16)$$
$$\left.\left(\frac{\partial V}{\partial z}-\beta V\right)\right|_{z=0}=-\beta\Theta|_{z=0},x\in[\delta,\delta+l],y\in[0,b], \quad (17)$$
$$\left.\left(\frac{\partial V}{\partial z}+\beta V\right)\right|_{z=\omega}=\beta\Theta|_{z=\omega},x\in[\delta,\delta+l],y\in[0,b]. \quad (18)$$

As for the interface between the adjacent sides of the cuboids, we present necessary conditions to ensure the continuity of the temperatures and the heat fluxes (ideal thermal contact)

$$V^0\Big|_{x=\delta-0} = V\Big|_{x=\delta+0}, y \in [0,b], z \in [0,\omega], \quad (19)$$

$$\left.\frac{\partial V^0}{\partial x}\right|_{x=\delta-0} = \left.\frac{\partial V}{\partial x}\right|_{x=\delta+0}, y \in [0,b], z \in [0,\omega]. \quad (20)$$

The solutions of equations (6) and (7) must also satisfy the initial conditions

$$V^0\Big|_{t=0} = V_0^0(x,y,z), \quad \left.\frac{\partial V^0}{\partial t}\right|_{t=0} = W_0^0(x,y,z), \quad (21)$$

$$V\Big|_{t=0} = V_0(x,y,z), \quad \left.\frac{\partial V}{\partial t}\right|_{t=0} = W_0(x,y,z), \quad (22)$$

where $V_0^0(x,y,z)$, $V_0(x,y,z)$, $W_0^0(x,y,z)$ and $W_0(x,y,z)$ are given functions.

From an experimental point of view conditions (21) (22) are unrealistic because the initial heat flux densities are not really known. But they must be calculated for their comparison with critical heat flux densities to predict heat transfer modes at steel quenching. During IQ it is required for initial heat flux density to be less then the first critical heat flux density $\mathbf{q}_{cr1}$ to eliminate film boiling (see [11]). For determination of initial heat flux densities, we can assume that the temperature distributions and the distribution of heat fluxes are given at the end of the process

$$V^0\Big|_{t=T} = V_T^0(x,y,z), \quad \left.\frac{\partial V^0}{\partial t}\right|_{t=T} = W_T^0(x,y,z), \quad (23)$$

$$V\Big|_{t=T} = V_T(x,y,z), \quad \left.\frac{\partial V}{\partial t}\right|_{t=T} = W_T(x,y,z). \quad (24)$$

2.2 General Mathematical Statement of 3D Problem for T-shape Sample

In this case we have a sample of T-form. It consists of two L-shape samples which are each other's mirror image. One of them is the sample described in the preceding subsection, and these two are joined along the side $x = 0$.

The boundary-value problems for determining the temperature of this sample are written in the same form as in the preceding subsection except the boundary condition at the dividing plane. Along this plane the insulation (symmetry) condition

$$\left.\frac{\partial V^0}{\partial x}\right|_{x=0} = 0, y \in [0,1], z \in [0,\omega] \quad (25)$$

must be applied.

In such situation both L-shape samples have the same temperature distribution.

3 Direct Problem for L-shape Sample

Here we find a solution for direct problem and briefly explain the idea how we can modify the well known Green's function method to obtain a closed-form Green's function for so called regular non-canonical domain. The main idea is to represent the original domain as a finite union of canonical sub-domains with appropriate boundary conditions along the lines (planes) connecting two neighbours. In the case of L-shape sample we have already divided it into two rectangles (rectangular cuboids).

3.1 Direct Problem for 3D L-shape Sample

We can transform equations (6) and (7) into more common form by introducing the well known substitutions

$$V^0(x,y,z,t) = \exp\left(-\frac{t}{2\tau_{r,0}}\right)U^0(x,y,z,t), \quad (26)$$

$$V(x,y,z,t) = \exp\left(-\frac{t}{2\tau_r}\right)U(x,y,z,t). \quad (27)$$

Plugging these expressions into equations (6) and (7) (and in all the conditions as well), we get new equations without first time derivatives. These are called Klein-Gordon equations

$$\frac{\partial^2 U^0}{\partial t^2} = a_{\tau,0}^2\left(\frac{\partial^2 U^0}{\partial x^2} + \frac{\partial^2 U^0}{\partial y^2} + \frac{\partial^2 U^0}{\partial z^2}\right) + \frac{1}{4\tau_{r,0}^2}U^0 + \frac{1}{\tau_{r,0}}\exp\left(\frac{t}{2\tau_{r,0}}\right)\Psi_0(x,y,z,t), \quad (28)$$

$$x \in (0,\delta), y \in (0,1), z \in (0,\omega), t \in (0,T];$$

$$\frac{\partial^2 U}{\partial t^2} = a_\tau^2\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right) + \frac{1}{4\tau_r^2}U + \frac{1}{\tau_r}\exp\left(\frac{t}{2\tau_r}\right)\Psi(x,y,z,t), \quad (29)$$

$$x \in (\delta,\delta+l), y \in (0,b), z \in (0,\omega), t \in (0,T],$$

where $a_{\tau,0}^2 = \dfrac{a^2}{\tau_{r,0}}$, $a_\tau^2 = \dfrac{a^2}{\tau_r}$.

The new boundary conditions for these equations can be written as before in (8) – (18) using functions U^0, U instead of V^0 and V, and $\exp\left(\dfrac{t}{2\tau_{r,0}}\right)\Theta$

(for the base) or $\exp\left(\frac{t}{2\tau_r}\right)\Theta$ (for the foot) instead of Θ. But the continuity conditions (19), (20) transform into a new form that shows discontinuity in the temperature field and the heat fluxes at the connecting surface $x=\delta$

$$U^0\Big|_{x=\delta-0} = \exp\left(\frac{t}{2\tau_{r,0}} - \frac{t}{2\tau_r}\right) U\Big|_{x=\delta+0}, \quad (30)$$

$y \in [0,b], z \in [0,\omega]$,

$$\left.\frac{\partial U^0}{\partial x}\right|_{x=\delta-0} = \exp\left(\frac{t}{2\tau_{r,0}} - \frac{t}{2\tau_r}\right) \left.\frac{\partial U}{\partial x}\right|_{x=\delta+0}, \quad (31)$$

$y \in [0,b], z \in [0,\omega]$.

We have corresponding initial conditions for the direct problem

$$U^0\Big|_{t=0} = V_0^0(x,y,z),$$
$$\left.\frac{\partial U^0}{\partial t}\right|_{t=0} = W_0^0(x,y,z) + \frac{V_0^0(x,y,z)}{2\tau_{r,0}}, \quad (32)$$

$$U\Big|_{t=0} = V_0(x,y,z),$$
$$\left.\frac{\partial U}{\partial t}\right|_{t=0} = W_0(x,y,z) + \frac{V_0(x,y,z)}{2\tau_r}, \quad (33)$$

or additional conditions at final moment for the inverse problem

$$U^0\Big|_{t=T} = \exp\left(\frac{T}{2\tau_{r,0}}\right) V_T^0(x,y,z), \quad (34)$$

$$\left.\frac{\partial U^0}{\partial t}\right|_{t=T} = \exp\left(\frac{T}{2\tau_{r,0}}\right)\left[W_T^0(x,y,z) + \frac{V_T^0(x,y,z)}{2\tau_{r,0}}\right],$$

$$U\Big|_{t=T} = \exp\left(\frac{T}{2\tau_r}\right) V_T(x,y,z), \quad (35)$$

$$\left.\frac{\partial U}{\partial t}\right|_{t=T} = \exp\left(\frac{T}{2\tau_r}\right)\left[W_T(x,y,z) + \frac{V_T(x,y,z)}{2\tau_r}\right].$$

We'll consider the inverse problem later.

We can find solutions to the original 3D problems (6) and (7) by solving the problems for equations (28), (29), and using transformations (26), (27). Since finding solution of 3D direct problem is quite similar as in 2D case, we will consider only the latter.

3.2 Direct Problem for Thin L-shape Sample

In this case we have a sample that is thin in the $z-$direction ($\omega << l, \omega << b, \omega << \delta$). To obtain mathematical formulation for 2D problem from the problem considered before, we can introduce averaged values in the $z-$direction of all the functions used before. For instance,

$$u^0(x,y,t) = \omega^{-1}\int_0^\omega U^0(x,y,z,t)dz, \quad (36)$$

$$u(x,y,t) = \omega^{-1}\int_0^\omega U(x,y,z,t)dz. \quad (37)$$

By using these approximations and applying the boundary conditions which are obtained from (12), (13), and (17), (18), two 2D differential equations with source terms are obtained

$$\frac{\partial^2 u^0}{\partial t^2} = a_{\tau,0}^2\left(\frac{\partial^2 u^0}{\partial x^2} + \frac{\partial^2 u^0}{\partial y^2}\right) - c_0 u^0 + \psi_0^*(x,y,t), \quad (38)$$

$x \in (0,\delta), y \in (0,1), t \in (0,T]$;

$$\frac{\partial^2 u}{\partial t^2} = a_\tau^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - cu + \psi^*(x,y,t), \quad (39)$$

$x \in (\delta, \delta+l), y \in (0,b), t \in (0,T]$,

where

$$c_0 = \frac{2\beta}{\omega} - \frac{1}{4\tau_{r,0}^2},$$

$$\psi_0^* = \exp\left(\frac{t}{2\tau_{r,0}}\right)\left(\frac{2\beta}{\omega}\theta(x,y,t) + \frac{1}{\tau_{r,0}}\psi_0(x,y,t)\right),$$

$$c = \frac{2\beta}{\omega} - \frac{1}{4\tau_r^2},$$

$$\psi^* = \exp\left(\frac{t}{2\tau_r}\right)\left(\frac{2\beta}{\omega}\theta(x,y,t) + \frac{1}{\tau_r}\psi(x,y,t)\right).$$

It is possible to use more accurate function approximation. In that case the new expressions for coefficients c, c_0 and functions ψ_0^*, ψ^* will differ from those written above.

The same type of boundary conditions as in 3D case are used here

$$\left.\left(\frac{\partial u^0}{\partial x} - \beta u^0\right)\right|_{x=0} = -\theta_0^*\Big|_{x=0}, y \in [0,1], \quad (40)$$

$$\left.\left(\frac{\partial u^0}{\partial x} + \beta u^0\right)\right|_{x=\delta} = \theta_0^*\Big|_{x=\delta}, y \in [b,1], \quad (41)$$

$$\left.\left(\frac{\partial u^0}{\partial y} - \beta u^0\right)\right|_{y=0} = -\theta_0^*\Big|_{y=0}, x \in [0,\delta], \quad (42)$$

$$\left(\frac{\partial u^0}{\partial y}+\beta u^0\right)\Bigg|_{y=1}=\theta_0^*\Big|_{y=1},\, x\in[0,\delta], \tag{43}$$

$$\left(\frac{\partial u}{\partial x}+\beta u\right)\Bigg|_{x=\delta+l}=\theta^*\Big|_{x=\delta+l},\, y\in[0,b], \tag{44}$$

$$\left(\frac{\partial u}{\partial y}-\beta u\right)\Bigg|_{y=0}=-\theta^*\Big|_{y=0},\, x\in[\delta,\delta+l], \tag{45}$$

$$\left(\frac{\partial u}{\partial y}+\beta u\right)\Bigg|_{y=b}=\theta^*\Big|_{y=b},\, x\in[\delta,\delta+l], \tag{46}$$

where

$$\theta_0^*(x,y,t)=\beta\exp\left(\frac{t}{2\tau_{r,0}}\right)\theta(x,y,t),$$

$$\theta^*(x,y,t)=\beta\exp\left(\frac{t}{2\tau_r}\right)\theta(x,y,t).$$

The initial conditions assume the following form

$$u^0\big|_{t=0}=v_0^0(x,y),\ \frac{\partial u^0}{\partial t}\bigg|_{t=0}=\overline{w}_0^0(x,y), \tag{47}$$

$$u\big|_{t=0}=v_0(x,y),\quad \frac{\partial u}{\partial t}\bigg|_{t=0}=\overline{w}_0(x,y), \tag{48}$$

where

$$\overline{w}_0^0(x,y)=w_0^0(x,y)+\frac{v_0^0(x,y)}{2\tau_{r,0}},$$

$$\overline{w}_0(x,y)=w_0(x,y)+\frac{v_0(x,y)}{2\tau_r}.$$

All the right hand sides of all these conditions are obtained applying the averaging procedure to the conditions from 3D problem.

In order to simplify solutions of these 2D problems, we shall confine ourselves to one and the same relaxation time τ_r ($\tau_{r,0}=\tau_r$) for both parts of the sample. Therefore coefficients c_0 and c used in (38), (39) are equal, and the continuity conditions for the temperatures and the heat fluxes at the boundary between the base and the foot are automatically assured

$$u^0\big|_{x=\delta-0}=u\big|_{x=\delta+0},\, y\in[0,b], \tag{49}$$

$$\frac{\partial u^0}{\partial x}\bigg|_{x=\delta-0}=\frac{\partial u}{\partial x}\bigg|_{x=\delta+0},\, y\in[0,b]. \tag{50}$$

Using continuity conditions (49), (50), the boundary condition for the right hand side border of the base can be written in the form

$$\left(\frac{\partial u^0}{\partial x}+\beta u^0\right)\Bigg|_{x=\delta-0}=F^0(y,t),$$

$$F^0(y,t)=\begin{cases}\theta^*\big|_{x=\delta} & y\in(b,1]\\ \left(\dfrac{\partial u}{\partial x}+\beta u\right)\Bigg|_{x=\delta+0} & y\in[0,b]\end{cases}. \tag{51}$$

But the boundary condition for the left-hand side border of the foot is in the form

$$\left(\frac{\partial u}{\partial x}-\beta u\right)\Bigg|_{x=\delta+0}=F(y,t),$$

$$F(y,t)=\left(\frac{\partial u^0}{\partial x}-\beta u^0\right)\Bigg|_{x=\delta-0}\quad y\in[0,b]. \tag{52}$$

Now when the boundary-value problems are stated we can find solution for each part of the sample using Green's function method.

3.2.1 Solution for the Base

It is well known from the literature that the equation (38), satisfying the boundary conditions (40) – (43), (51) and the initial conditions (47) has a solution (see, e.g., [13])

$$\begin{aligned}
u^0(x,y,t)=&\\
=&\int_0^\delta d\zeta\int_0^1 \overline{w}_0^0(\zeta,\upsilon)G^0(x,y,\zeta,\upsilon,t)d\upsilon\\
&+\int_0^\delta d\zeta\int_0^1 v_0^0(\zeta,\upsilon)\frac{\partial}{\partial t}G^0(x,y,\zeta,\upsilon,t)d\upsilon\\
&+a_\tau^2\int_0^t d\iota\int_0^1\theta^*(0,\upsilon,\iota)G^0(x,y,0,\upsilon,t-\iota)d\upsilon\\
&+a_\tau^2\int_0^t d\iota\int_0^1 F^0(\upsilon,\iota)G^0(x,y,\delta,\upsilon,t-\iota)d\upsilon\\
&+a_\tau^2\int_0^t d\iota\int_0^\delta\theta^*(\zeta,0,\iota)G^0(x,y,\zeta,0,t-\iota)d\zeta\\
&+a_\tau^2\int_0^t d\iota\int_0^\delta\theta^*(\zeta,1,\iota)G^0(x,y,\zeta,1,t-\iota)d\zeta\\
&+\int_0^t d\iota\int_0^\delta d\zeta\int_0^1\psi^*(\zeta,\upsilon,\iota)G^0(x,y,\zeta,\upsilon,t-\iota)d\upsilon.
\end{aligned} \tag{53}$$

The Green's function has a form (see [13])

$$G^0(x,y,\zeta,\upsilon,t)=$$
$$=\sum_{i=1}^{\infty}\sum_{j=1}^{p_i-1}\frac{\varphi_i(x)\varphi_i(\zeta)\phi_j(y)\phi_j(\upsilon)\sinh\left(t\sqrt{|f_{i,j}|}\right)}{\|\varphi_i\|^2\|\phi_j\|^2\sqrt{|f_{i,j}|}}$$

$$+\sum_{i=1}^{\infty}\sum_{j=p_i}^{\infty}\frac{\varphi_i(x)\varphi_i(\zeta)\phi_j(y)\phi_j(\upsilon)\sin\left(t\sqrt{f_{i,j}}\right)}{\|\varphi_i\|^2\|\phi_j\|^2\sqrt{f_{i,j}}}, \qquad (54)$$

$$f_{i,j}=a_\tau^2(\lambda_i^2+k_j^2)+c.$$

The natural number p_i that appears in formula (54) can be obtained from the inequalities

$$f_{i,j}<0 \text{ for } j=\overline{1,p_i-1},$$

$$f_{i,j}>0 \text{ for } j=\overline{p_i,\infty}.$$

The eigenfunctions have the following expressions

$$\varphi_i(x)=\sin(\lambda_i x+d_i),\quad d_i=\arctan\left(\frac{\lambda_i}{\delta}\right),$$

$$\phi_j(y)=\sin(k_j y+e_j),\quad e_j=\arctan(k_j),$$

$$\|\varphi_i\|^2=\frac{\delta}{2}+\frac{\beta}{\lambda_i^2+\beta^2},\ \|\phi_j\|^2=\frac{1}{2}+\frac{\beta}{k_j^2+\beta^2}.$$

The eigenvalues are roots of these transcendental equations

$$\cot(\lambda_i\delta)=\frac{\lambda_i^2-\beta^2}{2\lambda_i\beta},\ \cot(k_j)=\frac{k_j^2-\beta^2}{2k_j\beta}.$$

If we have a T-shape sample then condition (40) has a form

$$\left.\frac{\partial u^0}{\partial x}\right|_{x=0}=0,\ y\in[0,1].$$

And it follows that

$$\|\varphi_i\|^2=\frac{\delta}{2}+\frac{\beta}{2\left(\lambda_i^2+\beta^2\right)},\quad \cot(\lambda_i\delta)=\frac{\lambda_i}{\beta}.$$

As for solution (53), $\theta^*(0,\upsilon,\iota)$ has to be replaced by zero in this case.

3.2.2 Solution for the Foot

The solution for the second rectangle is similar to (53)

$$u(x,y,t)=$$

$$=\int_\delta^{\delta+l}d\xi\int_0^b\overline{w}_0(\xi,\eta)G(x,y,\xi,\eta,t)d\eta$$

$$+\int_\delta^{\delta+l}d\xi\int_0^b v_0(\xi,\eta)\frac{\partial}{\partial t}G(x,y,\xi,\eta,t)d\eta$$

$$-a_\tau^2\int_0^t d\tau\int_0^b F(\eta,\tau)G(x,y,\delta,\eta,t-\tau)d\eta$$

$$+a_\tau^2\int_0^t d\tau\int_0^b\theta^*(\delta+l,\eta,\tau)G(x,y,\delta+l,\eta,t-\tau)d\eta$$

$$+a_\tau^2\int_0^t d\tau\int_\delta^{\delta+l}\theta^*(\xi,0,\tau)G(x,y,\xi,0,t-\tau)d\xi$$

$$+a_\tau^2\int_0^t d\tau\int_\delta^{\delta+l}\theta^*(\xi,b,\tau)G(x,y,\xi,b,t-\tau)d\xi$$

$$+\int_0^t d\tau\int_\delta^{\delta+l}d\xi\int_0^b\psi^*(\xi,\eta,\tau)G(x,y,\xi,\eta,t-\tau)d\eta. \qquad (55)$$

But it's Green's function has a form similar to (54)

$$G(x,y,\xi,\eta,t)=$$

$$=\sum_{m=1}^{\infty}\sum_{n=1}^{q_m-1}\frac{\varphi_m(x)\varphi_m(\xi)\phi_n(y)\phi_n(\eta)\sinh\left(t\sqrt{|g_{m,n}|}\right)}{\|\varphi_m\|^2\|\phi_n\|^2\sqrt{|g_{m,n}|}}$$

$$+\sum_{m=1}^{\infty}\sum_{n=q_m}^{\infty}\frac{\varphi_m(x)\varphi_m(\xi)\phi_n(y)\phi_n(\eta)\sin\left(t\sqrt{g_{m,n}}\right)}{\|\varphi_m\|^2\|\phi_n\|^2\sqrt{g_{m,n}}}, \qquad (56)$$

$$g_{m,n}=a_\tau^2(\mu_m^2+\nu_n^2)+c.$$

The natural number q_m is given by the following inequalities

$$g_{m,n}<0 \text{ for } n=\overline{1,q_m-1},$$

$$g_{m,n}>0 \text{ for } n=\overline{q_m,\infty}.$$

The eigenvalues and the eigenfunctions for the Green's function (56) are given by these expressions

$$\cot(\mu_m l)=\frac{\mu_m^2-\beta^2}{2\mu_m\beta},\ \cot(\nu_n b)=\frac{\nu_n^2-\beta^2}{2\nu_n\beta},$$

$$\varphi_m(x)=\sin[\mu_m(x-\delta)+d_m],\ d_m=\arctan\left(\frac{\mu_m}{l}\right),$$

$$\phi_n(y)=\sin(\nu_n y+e_n),\qquad e_n=\arctan\left(\frac{\nu_n}{b}\right),$$

$$\|\varphi_m\|^2=\frac{l}{2}+\frac{\beta}{\mu_m^2+\beta^2},\ \|\phi_n\|^2=\frac{b}{2}+\frac{\beta}{\nu_n^2+\beta^2}.$$

3.2.3 Junction of Both Solutions

If we substitute (53) into formula (52), we get

$$F(y,t)=$$

$$=\int_0^\delta d\zeta\int_0^1\overline{w}_0^0(\zeta,\upsilon)\Gamma^0(\delta,y,\zeta,\upsilon,t)d\upsilon$$

$$+\int_0^\delta d\zeta\int_0^1 v_0^0(\zeta,\upsilon)\frac{\partial}{\partial t}\Gamma^0(\delta,y,\zeta,\upsilon,t)d\upsilon$$

$$+a_\tau^2\int_0^t d\iota\int_0^1\theta^*(0,\upsilon,\iota)\Gamma^0(\delta,y,0,\upsilon,t-\iota)d\upsilon$$

$$+a_\tau^2\int_0^t d\iota\int_0^1 F^0(\upsilon,\iota)\Gamma^0(\delta,y,\delta,\upsilon,t-\iota)d\upsilon$$

$$+a_\tau^2\int_0^t d\iota\int_0^\delta \theta^*(\zeta,0,\iota)\Gamma^0(\delta,y,\zeta,0,t-\iota)d\zeta$$
$$+a_\tau^2\int_0^t d\iota\int_0^\delta \theta^*(\zeta,1,\iota)\Gamma^0(\delta,y,\zeta,1,t-\iota)d\zeta$$
$$+\int_0^t d\iota\int_0^\delta d\zeta\int_0^1 \psi^*(\zeta,\upsilon,\iota)\Gamma^0(\delta,y,\zeta,\upsilon,t-\iota)d\upsilon. \quad (57)$$

In a similar way, i.e., combining (55) with (51), we obtain representation for the combination of the solution for the foot and its derivative at the border $y\in[0,b]$ between both parts

$$F^0(y,t)=$$
$$=\int_\delta^{\delta+l} d\xi\int_0^b \overline{w}_0(\xi,\eta)\Gamma(\delta,y,\xi,\eta,t)d\eta$$
$$+\int_\delta^{\delta+l} d\xi\int_0^b v_0(\xi,\eta)\frac{\partial}{\partial t}\Gamma(\delta,y,\xi,\eta,t)d\eta$$
$$-a_\tau^2\int_0^t d\tau\int_0^b F(\eta,\tau)\Gamma(\delta,y,\delta,\eta,t-\tau)d\eta$$
$$+a_\tau^2\int_0^t d\tau\int_0^b \theta^*(\delta+l,\eta,\tau)\Gamma(\delta,y,\delta+l,\eta,t-\tau)d\eta$$
$$+a_\tau^2\int_0^t d\tau\int_\delta^{\delta+l} \theta^*(\xi,0,\tau)\Gamma(\delta,y,\xi,0,t-\tau)d\xi$$
$$+a_\tau^2\int_0^t d\tau\int_\delta^{\delta+l} \theta^*(\xi,b,\tau)\Gamma(\delta,y,\xi,b,t-\tau)d\xi$$
$$+\int_0^t d\tau\int_\delta^{\delta+l} d\xi\int_0^b \psi^*(\xi,\eta,\tau)\Gamma(\delta,y,\xi,\eta,t-\tau)d\eta\,. \quad (58)$$

Here $\Gamma^0=\left(\frac{\partial}{\partial x}-\beta\right)G^0$, $\Gamma=\left(\frac{\partial}{\partial x}+\beta\right)G$.

Now it remains to plug expression (57) into (58) to obtain non-homogeneous Volterra-Fredholm integral equation of the 2nd kind on the border between both parts of the L-shape sample

$$F^0(y,t)=$$
$$=\Phi^0(y,t)-\int_0^t d\iota\int_0^1 F^0(\upsilon,\iota)K^0(y,\upsilon,t,\iota)d\upsilon\,. \quad (59)$$

Where $\Phi^0(y,t)$ and $K^0(y,\upsilon,t,\iota)$, the kernel, are continuous real-valued functions.

$$K^0(y,\upsilon,t,\iota)=\left(a_\tau^2\right)^2\widetilde{\Gamma}^0(y,\delta,\upsilon,t,\iota),$$

where

$$\widetilde{\Gamma}^0(y,\zeta,\upsilon,t,\iota)=$$
$$=\int_0^t d\tau\int_0^b \Gamma^0(\delta,\eta,\zeta,\upsilon,\tau-\iota)\Gamma(\delta,y,\delta,\eta,t-\tau)d\eta\,.$$

But the non-homogeneous term is in the form

$$\Phi^0(y,t)=$$
$$=\int_\delta^{\delta+l} d\xi\int_0^b \overline{w}_0(\xi,\eta)\Gamma(\delta,y,\xi,\eta,t)d\eta$$
$$+\int_\delta^{\delta+l} d\xi\int_0^b v_0(\xi,\eta)\frac{\partial}{\partial t}\Gamma(\delta,y,\xi,\eta,t)d\eta$$
$$-a_\tau^2\int_0^\delta d\zeta\int_0^1 \overline{w}_0^0(\zeta,\upsilon)\widetilde{\Gamma}^0(y,\zeta,\upsilon,t,0)d\upsilon$$
$$-a_\tau^2\int_0^\delta d\zeta\int_0^1 v_0^0(\zeta,\upsilon)d\upsilon\times$$
$$\int_0^t d\tau\int_0^b \frac{\partial}{\partial \tau}\Gamma^0(\delta,\eta,\zeta,\upsilon,\tau)\Gamma(\delta,y,\delta,\eta,t-\tau)d\eta$$
$$-\left(a_\tau^2\right)^2\int_0^t d\iota\int_0^1 \theta^*(0,\upsilon,\iota)\widetilde{\Gamma}^0(y,0,\upsilon,t,\iota)d\upsilon$$
$$-\left(a_\tau^2\right)^2\int_0^t d\iota\int_0^\delta \theta^*(\zeta,0,\iota)\widetilde{\Gamma}^0(y,\zeta,0,t,\iota)d\zeta$$
$$-\left(a_\tau^2\right)^2\int_0^t d\iota\int_0^\delta \theta^*(\zeta,1,\iota)\widetilde{\Gamma}^0(y,\zeta,1,t,\iota)d\zeta$$
$$-a_\tau^2\int_0^t d\iota\int_0^1 d\upsilon\int_0^\delta \psi^*(\zeta,\upsilon,\iota)\widetilde{\Gamma}^0(y,\zeta,\upsilon,t,\iota)d\zeta$$
$$+a_\tau^2\int_0^t d\tau\int_0^b \theta^*(\delta+l,\eta,\tau)\Gamma(\delta,y,\delta+l,\eta,t-\tau)d\eta$$
$$+a_\tau^2\int_0^t d\tau\int_\delta^{\delta+l} \theta^*(\xi,0,\tau)\Gamma(\delta,y,\xi,0,t-\tau)d\xi$$
$$+a_\tau^2\int_0^t d\tau\int_\delta^{\delta+l} \theta^*(\xi,b,\tau)\Gamma(\delta,y,\xi,b,t-\tau)d\xi$$
$$+\int_0^t d\tau\int_\delta^{\delta+l} d\xi\int_0^b \psi^*(\xi,\eta,\tau)\Gamma(\delta,y,\xi,\eta,t-\tau)d\eta.$$

Finding the function $F^0(y,t)$ is crucial because of the following reasons. If the function $F^0(y,t)$ is obtained, we can fin a solution in the base, including the border $x=\delta$ and calculate the function $F(y,t)$. Wherewith, we find a solution in the second part of the sample too.

3.3 Solution of 3D Problem

Solution of 3D problem can be sought in the same way as for 2D. As a result one obtain an integral equation similar to (59) with another integral addend with respect to the third spatial coordinate

$$F^0(y,z,t)=$$
$$=\Phi^0(y,z,t)$$

$$-\int_0^t d\iota \int_0^{\varpi} d\varpi \int_0^b F^0(\upsilon,\varpi,\iota) K^0(y,\upsilon,z,\varpi,t,\iota) d\upsilon . \quad (60)$$

Green's function for the base in 3D case has a form

$$G^0(x,y,\zeta,\upsilon,z,\varpi,t) =$$

$$= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{h=1}^{p_{i,j}-1} \varphi_i(x)\varphi_i(\zeta)\phi_j(y)\phi_j(\upsilon)\gamma_h(z)\gamma_h(\varpi)\times$$

$$\frac{\sinh\left(t\sqrt{|f_{i,j,h}|}\right)}{\|\varphi_i\|^2\|\phi_j\|^2\|\gamma_h\|^2\sqrt{|f_{i,j,h}|}}$$

$$+ \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{h=p_{i,j}}^{\infty} \varphi_i(x)\varphi_i(\zeta)\phi_j(y)\phi_j(\upsilon)\gamma_h(z)\gamma_h(\varpi)\times$$

$$\frac{\sin\left(t\sqrt{f_{i,j,h}}\right)}{\|\varphi_i\|^2\|\phi_j\|^2\|\gamma_h\|^2\sqrt{f_{i,j,h}}}, \quad (61)$$

where

$$f_{i,j,h} = a_\tau^2(\lambda_i^2 + k_j^2 + \nu_h^2) + c .$$

The eigenfunctions have the following expressions

$$\varphi_i(x) = \sin(\lambda_i x + d_i), \quad d_i = \arctan\left(\frac{\lambda_i}{\delta}\right),$$

$$\phi_j(y) = \sin(k_j y + e_j), \quad e_j = \arctan(k_j),$$

$$\gamma_h(z) = \sin(\nu_h z + g_h), \quad g_h = \arctan\left(\frac{\nu_h}{\omega}\right),$$

$$\|\varphi_i\|^2 = \frac{\delta}{2} + \frac{\beta}{\lambda_i^2+\beta^2}, \quad \|\phi_j\|^2 = \frac{1}{2} + \frac{\beta}{k_j^2+\beta^2},$$

$$\|\gamma_h\|^2 = \frac{\omega}{2} + \frac{\beta}{\nu_h^2+\beta^2} .$$

But the eigenvalues can be obtained from these transcendental equations

$$\cot(\lambda_i\delta) = \frac{\lambda_i^2-\beta^2}{2\lambda_i\beta}, \quad \cot(k_j) = \frac{k_j^2-\beta^2}{2k_j\beta},$$

$$\cot(\nu_h\omega) = \frac{\nu_h^2-\beta^2}{2\nu_h\beta} .$$

4 Inverse Problems for 2D L-shape Sample

As it was mentioned before, in this case the initial temperatures and the initial heat fluxes are not known. But it is required to determine initial heat flux densities for their comparison with critical heat flux densities. Instead of initial conditions we have the temperature distribution and the heat flux densities at the final moment (at the end of the process)

$$u^0\Big|_{t=T} = \exp\left(\frac{T}{2\tau_r}\right) v_T^0(x,y), \frac{\partial u^0}{\partial t}\Big|_{t=T} = \overline{w}_T^0(x,y), \quad (62)$$

$$u\Big|_{t=T} = \exp\left(\frac{T}{2\tau_r}\right) v_T(x,y), \frac{\partial u}{\partial t}\Big|_{t=T} = \overline{w}_T(x,y), \quad (63)$$

where

$$\overline{w}_T^0(x,y) = \exp\left(\frac{T}{2\tau_r}\right)\left[w_T^0(x,y) + \frac{v_T^0(x,y)}{2\tau_r}\right],$$

$$\overline{w}_T(x,y) = \exp\left(\frac{T}{2\tau_r}\right)\left[w_T(x,y) + \frac{v_T(x,y)}{2\tau_r}\right].$$

As it is important to calculate initial heat fluxes, let's introduce a new time variable

$$\tilde{t} = T - t . \quad (64)$$

By means of that we can formulate new problems for the functions

$$\tilde{u}^0(x,y,\tilde{t}) \overset{def}{\equiv} u^0(x,y,T-\tilde{t}),$$

$$\tilde{u}(x,y,\tilde{t}) \overset{def}{\equiv} u(x,y,T-\tilde{t}).$$

The equations are in the same form as (38), (39)

$$\frac{\partial^2\tilde{u}^0}{\partial\tilde{t}^2} =$$

$$= a_\tau^2\left(\frac{\partial^2\tilde{u}^0}{\partial x^2} + \frac{\partial^2\tilde{u}^0}{\partial y^2}\right) - c\tilde{u}^0 + \tilde{\psi}^*(x,y,\tilde{t}), \quad (65)$$

$$x\in(0,\delta), y\in(0,1), \tilde{t}\in(0,T];$$

$$\frac{\partial^2\tilde{u}}{\partial\tilde{t}^2} =$$

$$= a_\tau^2\left(\frac{\partial^2\tilde{u}}{\partial x^2} + \frac{\partial^2\tilde{u}}{\partial y^2}\right) - c\tilde{u} + \tilde{\psi}^*(x,y,\tilde{t}), \quad (66)$$

$$x\in(\delta,\delta+l), y\in(0,b), \tilde{t}\in(0,T].$$

Establishing appropriate conditions, one can find solutions to these equations. The boundary and conjugation conditions are written as in the direct problem but the initial conditions take the form

$$\tilde{u}^0\Big|_{\tilde{t}=0} = u^0\Big|_{t=T} \equiv \tilde{w}_T^0(x,y),$$

$$\frac{\partial\tilde{u}^0}{\partial\tilde{t}}\Big|_{\tilde{t}=0} = -\frac{\partial u^0}{\partial t}\Big|_{t=T} = -\overline{w}_T^0(x,y) \equiv \widetilde{W}_T^0, \quad (67)$$

$$\tilde{u}\Big|_{\tilde{t}=0} = u\Big|_{t=T} \equiv \tilde{w}_T(x,y),$$

$$\frac{\partial\tilde{u}}{\partial\tilde{t}}\Big|_{\tilde{t}=0} = -\frac{\partial u}{\partial t}\Big|_{t=T} = -\overline{w}_T(x,y) \equiv \widetilde{W}_T, \quad (68)$$

by means of which we obtain a direct problem instead of inverse. We look for the solution of this boundary-value problem as before in the preceding section. Differentiating calculated solutions with respect to time and calculating the derivatives at the moment $\tilde{t} = T$

$$\left.\frac{\partial \tilde{u}^0(x,y,\tilde{t})}{\partial \tilde{t}}\right|_{\tilde{t}=T} =$$

$$= \int_0^\delta d\zeta \int_0^1 \tilde{W}_T^0(\zeta,\upsilon) \left.\frac{\partial}{\partial \tilde{t}} G^0(x,y,\zeta,\upsilon,\tilde{t})\right|_{\tilde{t}=T} d\upsilon$$
$$+ \int_0^\delta d\zeta \int_0^1 \tilde{w}_T^0(\zeta,\upsilon) \left.\frac{\partial^2}{\partial \tilde{t}^2} G^0(x,y,\zeta,\upsilon,\tilde{t})\right|_{\tilde{t}=T} d\upsilon$$
$$+ a_\tau^2 \int_0^1 \tilde{\theta}^*(0,\upsilon,T) G^0(x,y,0,\upsilon,0) d\upsilon$$
$$+ a_\tau^2 \int_0^T d\iota \int_0^1 \tilde{\theta}^*(0,\upsilon,\iota) \left.\frac{\partial}{\partial \tilde{t}} G^0(x,y,0,\upsilon,\tilde{t}-\iota)\right|_{\tilde{t}=T} d\upsilon$$
$$+ a_\tau^2 \int_0^1 \tilde{F}^0(\upsilon,T) G^0(x,y,\delta,\upsilon,0) d\upsilon$$
$$+ a_\tau^2 \int_0^T d\iota \int_0^1 \tilde{F}^0(\upsilon,\iota) \left.\frac{\partial}{\partial \tilde{t}} G^0(x,y,\delta,\upsilon,\tilde{t}-\iota)\right|_{\tilde{t}=T} d\upsilon$$
$$+ a_\tau^2 \int_0^\delta \tilde{\theta}^*(\zeta,0,T) G^0(x,y,\zeta,0,0) d\zeta$$
$$+ a_\tau^2 \int_0^T d\iota \int_0^\delta \tilde{\theta}^*(\zeta,0,\iota) \left.\frac{\partial}{\partial \tilde{t}} G^0(x,y,\zeta,0,\tilde{t}-\iota)\right|_{\tilde{t}=T} d\zeta$$
$$+ a_\tau^2 \int_0^\delta \tilde{\theta}^*(\zeta,1,T) G^0(x,y,\zeta,1,0) d\zeta$$
$$+ a_\tau^2 \int_0^T d\iota \int_0^\delta \tilde{\theta}^*(\zeta,1,\iota) \left.\frac{\partial}{\partial \tilde{t}} G^0(x,y,\zeta,1,\tilde{t}-\iota)\right|_{\tilde{t}=T} d\zeta$$
$$+ \int_0^\delta d\zeta \int_0^1 \tilde{\psi}^*(\zeta,\upsilon,T) G^0(x,y,\zeta,\upsilon,0) d\upsilon$$
$$+ \int_0^T d\iota \int_0^\delta d\zeta \int_0^1 \tilde{\psi}^*(\zeta,\upsilon,\iota) \times \left.\frac{\partial}{\partial \tilde{t}} G^0(x,y,\zeta,\upsilon,\tilde{t}-\iota)\right|_{\tilde{t}=T} d\upsilon, \quad (69)$$

and

$$\left.\frac{\partial \tilde{u}(x,y,\tilde{t})}{\partial \tilde{t}}\right|_{\tilde{t}=T} =$$

$$= \int_\delta^{\delta+l} d\xi \int_0^b \tilde{W}_T(\xi,\eta) \left.\frac{\partial}{\partial \tilde{t}} G(x,y,\xi,\eta,\tilde{t})\right|_{\tilde{t}=T} d\eta$$
$$+ \int_\delta^{\delta+l} d\xi \int_0^b \tilde{w}_T(\xi,\eta) \left.\frac{\partial^2}{\partial \tilde{t}^2} G(x,y,\xi,\eta,\tilde{t})\right|_{\tilde{t}=T} d\eta$$
$$- a_\tau^2 \int_0^b \tilde{F}(\eta,T) G(x,y,\delta,\eta,0) d\eta$$
$$- a_\tau^2 \int_0^T d\tau \int_0^b \tilde{F}(\eta,\tau) \left.\frac{\partial}{\partial \tilde{t}} G(x,y,\delta,\eta,\tilde{t}-\tau)\right|_{\tilde{t}=T} d\eta$$
$$+ a_\tau^2 \int_0^b \tilde{\theta}^*(\delta+l,\eta,T) G(x,y,\delta+l,\eta,0) d\eta$$
$$+ a_\tau^2 \int_0^T d\tau \int_0^b \tilde{\theta}^*(\delta+l,\eta,\tau) \times \left.\frac{\partial}{\partial \tilde{t}} G(x,y,\delta+l,\eta,\tilde{t}-\tau)\right|_{\tilde{t}=T} d\eta$$
$$+ a_\tau^2 \int_\delta^{\delta+l} \tilde{\theta}^*(\xi,0,T) G(x,y,\xi,0,0) d\xi$$
$$+ a_\tau^2 \int_0^T d\tau \int_\delta^{\delta+l} \tilde{\theta}^*(\xi,0,\tau) \frac{\partial}{\partial \tilde{t}} G(x,y,\xi,0,\tilde{t}-\tau) d\xi$$
$$+ a_\tau^2 \int_\delta^{\delta+l} \tilde{\theta}^*(\xi,b,T) G(x,y,\xi,b,\tilde{t}-\tau) d\xi$$
$$+ a_\tau^2 \int_0^T d\tau \int_\delta^{\delta+l} \tilde{\theta}^*(\xi,b,\tau) \times \left.\frac{\partial}{\partial \tilde{t}} G(x,y,\xi,b,\tilde{t}-\tau)\right|_{\tilde{t}=T} d\xi$$
$$+ \int_\delta^{\delta+l} d\xi \int_0^b \tilde{\psi}^*(\xi,\eta,T) G(x,y,\xi,\eta,0) d\eta$$
$$+ \int_0^T d\tau \int_\delta^{\delta+l} d\xi \int_0^b \tilde{\psi}^*(\xi,\eta,\tau) \times \left.\frac{\partial}{\partial \tilde{t}} G(x,y,\xi,\eta,\tilde{t}-\tau)\right|_{\tilde{t}=T} d\eta, \quad (70)$$

we can get heat flux densities $\frac{\partial v^0}{\partial t}$, $\frac{\partial v}{\partial t}$ (or $\frac{\partial V^0}{\partial t}$, $\frac{\partial V}{\partial t}$ in 3D case) at the very beginning of the process.

Solution of the 3D problem can be sought in the same way as in 2D case.

5 Conclusion

We have constructed exact solution of hyperbolic heat equation for 2D L-shape sample in closed form of iterative integral equation of 2nd kind. It is Fredholm integral equation with respect to the space arguments and Volterra integral equation with respect to time.

The solutions of 2D time reverse problems are reduced to direct problem with a view to determine the initial heat flux densities.

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