Effects of Thermal Radiation and Free Convection Currents on the Unsteady Couette Flow Between Two Vertical Parallel Plates with Constant Heat Flux at one Boundary

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Abstract: - An exact analysis of the natural convection in unsteady Couette flow of a viscous incompressible fluid confined between two vertical parallel plates in the presence of thermal radiation is performed. The flow is induced by means of Couette motion and free convection currents occurring as a result of application of constant heat flux on the wall with a uniform vertical motion in its own plane while constant temperature on the stationary wall. The fluid considered here is a gray, absorbing-emitting but non-scattering medium, and the Rosseland approximation is used to describe the radiative heat flux in the analysis. The dimensionless governing partial differential equations are solved using Laplace transform technique. Numerical results for the velocity, the temperature, the skin-friction, the Nusselt number, the volume flow rate and the vertical heat flux are shown graphically. The effect of different parameters like thermal radiation parameter, Grashof number, Prandtl number and time are discussed. It is observed that the momentum and thermal boundary layer thickness decreases owing to an increase in the value of the radiation parameter. An increase in the Grashof number is found to increase the velocity of air and water and to decrease the skin-friction at the moving plate.

Key-Words: - Vertical channel, Natural convection, Couette flow, Constant heat flux, Radiation.

1 Introduction

The fluid flow between parallel plates by means of Couette motion is a classical fluid mechanics problem that has applications in magnetohydro-dynamic (MHD) power generators and pumps, accelerators, aerodynamics heating, electrostatic precipitation, polymer technology, petroleum industry, purification of crude oil, and also in many material processing applications such as extrusion, metal forming, continuous casting, wire and glass fiber drawing, etc. This problem has received considerable attention in the case of horizontal parallel plates [1]-[15] than vertical parallel plates. An analysis of flow formation in Couette motion between vertical parallel plates was presented by Schlichting and Gersten [16]. This problem is of fundamental importance as it provides the exact solution and reveals how the velocity profiles varies with time, approaching a linear distribution asymptotically, and how the boundary layer spreads throughout the flow field.

Free convection in vertical channels has been studied widely in the last few decades under different physical effects [17]-[27] due to its importance in many engineering applications such as cooling of electronic equipments, design of passive solar systems for energy conversion, cooling of nuclear reactors, design of heat exchangers, chemical devices and process equipment, geothermal systems, and others. However, very few papers deal with free convection in Couette motion between vertical parallel plates. Singh [28] studied the effect of free convection in Couette motion. He has considered the unsteady free-convective flow of a viscous incompressible fluid between two vertical parallel plates at constant but different temperatures and one of which is impulsively started in its own plane and the other is kept stationary. This problem was further extended for magnetohydrodynamic case by Jha [29]. Fully-developed laminar free convection Couette flow between two vertical parallel plates with transverse sinusoidal injection of the fluid at the stationary plate and its corresponding removal by constant suction through the plate in uniform motion has been analyzed by Jain and Gupta [30]. The physical effect of external shear in the form of Couette flow of a Bingham fluid in a vertical parallel plane channel with constant temperature differential across the walls was investigated analytically by Barletta and Magyari

[31]. Steady fully-developed combined forced and free convection Couette flow with viscous dissipation in a vertical channel has been investigated analytically by Barletta et al. [32]. In their study, the moving wall is thermally insulated and the wall at rest is kept at a uniform temperature.

The aim of the present paper is to provide an exact analysis of unsteady free convection in Couette motion between two vertical parallel plates in the presence of thermal radiation, where the moving plate is subject to constant heat flux and the plate at rest is isothermal. Exact solutions are derived for the velocity and temperature fields using Laplace transform technique. These solutions are useful to gain a deeper knowledge of the underlying physical processes and it provides the possibility to get a benchmark for numerical solvers with reference to basic flow configurations. The mathematical analysis and the solution of the velocity field for two different cases – one valid for fluids with Prandtl numbers different from unity and the other for which the Prandtl number is unity – have been presented in Section 2, the results are discussed in Section 3 and the conclusions are set out in Section 4.

2 Mathematical Analysis

Consider the unsteady free-convective Couette flow of an incompressible viscous radiating fluid between two infinite vertical parallel plates separated by a distance h. The x'- axis is taken along one of the plates in the vertically upward direction and the y'- axis is taken normal to the plate. Initially, at time $t' \le 0$, the two plates and the fluid are assumed to be at the same temperature T_h' and stationary. At time $t' > 0$, the plate at $y' = 0$ starts moving in its own plane with an impulsive velocity U and is heated by supplying heat at constant rate whereas the plate at $y' = h$ is stationary and maintained at a constant temperature T_h'. It is also assumed that the radiative heat flux in the x'- direction is negligible as compared to that in the y'- direction. As the plates are infinite in length, the velocity and temperature fields are functions of y' and t' only. Then under the usual Boussinesq's approximation, the flow of a radiating fluid is shown to be governed by the following system of equations:

$$\frac{\partial u'}{\partial t'} = g\beta(T' - T_h') + \nu \frac{\partial^2 u'}{\partial y'^2} \tag{1}$$

and

$$\rho C_p \frac{\partial T'}{\partial t'} = k \frac{\partial^2 T'}{\partial y'^2} - \frac{\partial q_r}{\partial y'} \tag{2}$$

The initial and boundary conditions are as follows:

$$\left.\begin{array}{lll} t' \le 0 : u' = 0, T' = T_h' & \text{for} & 0 \le y' \le h, \\ t' > 0 : u' = U, \dfrac{\partial T'}{\partial y'} = -\dfrac{q}{k} & \text{at} & y' = 0, \\ \quad u' = 0, T' = T_h' & \text{at} & y' = h. \end{array}\right\} \tag{3}$$

where g is the acceleration due to gravity, β the volumetric co-efficient of thermal expansion, ν the kinematic viscosity, ρ the density, k the thermal conductivity, C_p the specific heat at constant pressure, q the constant heat flux, q_r the radiative heat flux in y'- direction, T' the fluid temperature, and u' is the fluid velocity.

The radiative heat flux term is simplified by making use of the Rosseland approximation [33] as

$$q_r = -\frac{4\sigma}{3k^*} \frac{\partial T'^4}{\partial y'} \tag{4}$$

where σ is the Stefan-Boltzmann constant and k^* is the mean absorption coefficient. It should be noted that by using the Rosseland approximation we limit our analysis to optically thick fluids. If temperature differences within the flow are sufficiently small such that T'^4 may be expressed as a linear function of the temperature, Then the Taylor series for T'^4 about T_h', after neglecting higher order terms, is given by

$$T'^4 \cong 4T_h'^3 T' - 3T_h'^4 \tag{5}$$

It is emphasized here that equation (5) is widely used in computational fluid dynamics involving radiation absorption problems [34] in expressing the term T'^4 as a linear function.
In view of Eqs. (4) and (5), Eq. (2) reduces to

$$\rho C_p \frac{\partial T'}{\partial t'} = k \frac{\partial^2 T'}{\partial y'^2} + \frac{16\sigma T_h'^3}{3k^*} \frac{\partial^2 T'}{\partial y'^2} \tag{6}$$

In order to solve the governing equations in dimensionless form, we introduce the following non-dimensional quantities:

$$\left.\begin{aligned} & y=\frac{y'}{h},\, t=\frac{t'\nu}{h^2},\, u=\frac{u'}{U},\, \theta=\frac{T'-T_h'}{(hq/k)}, \\ & \Pr=\frac{\mu C_p}{k},\, R=\frac{kk^*}{4\sigma T_h'^3},\, Gr=\frac{g\beta h^3 q}{Uk\nu} \end{aligned}\right\} \tag{7}$$

where Gr is the thermal Grashof number, $\Pr$ the Prandtl number, R the radiation parameter, t the dimensionless time, u the dimensionless velocity, y the dimensionless coordinate axis normal to the plate, μ the coefficient of viscosity and θ is the dimensionless temperature.

Then in view of Eqs. (7), Eqs. (1), (6) and (3) reduces to the following non-dimensional form of equations:

$$\frac{\partial u}{\partial t}=Gr\,\theta+\frac{\partial^2 u}{\partial y^2} \tag{8}$$

$$3R\Pr\frac{\partial \theta}{\partial t}=(3R+4)\frac{\partial^2 \theta}{\partial y^2} \tag{9}$$

The initial and boundary conditions are

$$\left.\begin{aligned} & t\le 0:\; u=0, \theta=0 \quad \text{for } 0\le y\le 1, \\ & t>0:\; u=1, \frac{\partial\theta}{\partial y}=-1 \quad \text{at} \quad y=0, \\ & \qquad u=0, \theta=0 \quad \text{at} \quad y=1. \end{aligned}\right\} \tag{10}$$

The solutions of Eqs. (8) and (9) under the initial and boundary conditions (10) by Laplace transform technique is given by

$$\begin{aligned} u(y,t) &= \sum_{n=0}^{\infty}\left[f_1(a,t)-f_1(b,t)\right] \\ &+\frac{Gr}{(N-1)\sqrt{N}}\sum_{n=0}^{\infty}\Big[f_3(b,t)-f_3(a,t) \\ &+2\sum_{m=0}^{n}(-1)^m\left\{f_3(c,t)-f_3(d,t)\right\} \\ &+(-1)^n\left\{f_3\left(b\sqrt{N},t\right)-f_3\left(a\sqrt{N},t\right)\right\}\Big] \end{aligned} \tag{11}$$

$$\theta(y,t)=\frac{1}{\sqrt{N}}\sum_{n=0}^{\infty}(-1)^n\left[f_2\left(a\sqrt{N},t\right)-f_2\left(b\sqrt{N},t\right)\right] \tag{12}$$

where

$$a=2n+y,\, b=2n+2-y,\, N=\frac{3R\Pr}{3R+4},$$

$$c=2m\sqrt{N}-2m+a,\, d=2m\sqrt{N}-2m+b,$$

$$f_1(z,t)=\operatorname{erfc}\left(\frac{z}{2\sqrt{t}}\right),$$

$$f_2(z,t)=2\sqrt{\frac{t}{\pi}}\exp\left(-\frac{z^2}{4t}\right)-z\operatorname{erfc}\left(\frac{z}{2\sqrt{t}}\right),$$

$$\begin{aligned} f_3(z,t) &= \frac{1}{3}\left(4t+z^2\right)\sqrt{\frac{t}{\pi}}\exp\left(-\frac{z^2}{4t}\right) \\ &\quad -\left(zt+\frac{z^3}{6}\right)\operatorname{erfc}\left(\frac{z}{2\sqrt{t}}\right), \end{aligned}$$

z is a dummy variable and f_1, f_2, f_3 are functions of dummy variable.

Using the expression (11), the skin-friction at the moving hot plate $y=0$ in non-dimensional form is given by

$$\tau_0=\frac{\tau_0' \, h}{\mu U}=-\left.\frac{\partial u}{\partial y}\right|_{y=0}$$

$$\begin{aligned} \tau_0 &= \sum_{n=0}^{\infty}\left[f_4(n,t)+f_4(n+1,t)\right] \\ &-\frac{Gr}{(N-1)\sqrt{N}}\sum_{n=0}^{\infty}\Big[\left\{f_5(n,t)+f_5(n+1,t)\right\} \\ &-2\sum_{m=0}^{n}(-1)^m\left\{f_5(w,t)+f_5(w+1,t)\right\} \\ &+(-1)^n\left\{f_6\left(n\sqrt{N},t\right)+f_6\left((n+1)\sqrt{N},t\right)\right\}\Big] \end{aligned} \tag{13}$$

and the skin-friction at the stationary plate $y=1$ is given by

$$\begin{aligned} \tau_1 &= -\left.\frac{\partial u}{\partial y}\right|_{y=1} \\ &= 2\sum_{n=0}^{\infty} f_4\left(\frac{2n+1}{2},t\right) \\ &-\frac{2Gr}{(N-1)\sqrt{N}}\sum_{n=0}^{\infty}\Bigg[f_5\left(\frac{2n+1}{2},t\right) \\ &-2\sum_{m=0}^{n}(-1)^m f_5\left(\frac{2w+1}{2},t\right) \\ &+(-1)^n f_6\left(\frac{(2n+1)\sqrt{N}}{2},t\right)\Bigg] \end{aligned} \tag{14}$$

Using the expression (12), it is also interesting to study the rate of heat transfer at the moving hot plate $y=0$ which is expressed as Nusselt number by

$$Nu_0 = \frac{qh}{k(T'-T'_h)} = -\frac{1}{\theta(0,t)}\left.\frac{\partial\theta}{\partial y}\right|_{y=0} = \frac{1}{\theta(0,t)} \quad (15)$$

and the Nusselt number at the stationary plate $y=1$ is given by

$$Nu_1 = -\left.\frac{\partial\theta}{\partial y}\right|_{y=1} = 2\sum_{n=0}^{\infty}(-1)^n f_1\left((2n+1)\sqrt{N},t\right) \quad (16)$$

where

$w = m\left(\sqrt{N}-1\right)+n$,

$$f_4(z,t) = \frac{1}{\sqrt{\pi t}}\exp\left(-\frac{z^2}{t}\right),$$

$$f_5(z,t) = \left(2z^2+t\right)\operatorname{erfc}\left(\frac{z}{\sqrt{t}}\right) - 2z\sqrt{\frac{t}{\pi}}\exp\left(-\frac{z^2}{t}\right),$$

$$f_6(z,t) = \left(2z^2+t\sqrt{N}\right)\operatorname{erfc}\left(\frac{z}{\sqrt{t}}\right) - 2z\sqrt{\frac{Nt}{\pi}}\exp\left(-\frac{z^2}{t}\right),$$

$$\theta(0,t) = \frac{1}{\sqrt{N}}\sum_{n=0}^{\infty}(-1)^n\left[f_2\left(2n\sqrt{N},t\right) - f_2\left(2(n+1)\sqrt{N},t\right)\right],$$

τ'_0 is the dimensional skin-friction at the plate $y=0$ and f_4, f_5, f_6 are functions of dummy variable z.

Another two important quantities for this problem are the non-dimensional volume flow rate between the plates and the non-dimensional vertical heat flux defined, respectively, by the following equations [35]-[37]:

$$M = \int_0^1 u\, dy \quad (17)$$

and

$$Q = \int_0^1 u\theta\, dy \quad (18)$$

2.1 Solution in the absence of radiation

In the absence of thermal radiation, i.e. in the pure convection case which numerically corresponds to $R\to\infty$, the energy equation in non-dimensional form becomes

$$\Pr\frac{\partial\theta}{\partial y} = \frac{\partial^2\theta}{\partial y^2} \quad (19)$$

Since $N=\Pr$ as $R\to\infty$, therefore the solution of the problem in the absence of radiation can be obtained from the equations of (11) and (12) simply by replacing N by $\Pr$. Thus the velocity and temperature expressions in the absence of thermal radiation are given by

$$\begin{aligned} u(y,t) = &\sum_{n=0}^{\infty}\left[f_1(a,t)-f_1(b,t)\right] \\ &+ \frac{Gr}{(\Pr-1)\sqrt{\Pr}}\sum_{n=0}^{\infty}\Bigg[\left[f_3(b,t)-f_3(a,t)\right] \\ &+ 2\sum_{m=0}^{n}(-1)^m\left[f_3(c_1,t)-f_3(d_1,t)\right] \\ &+ (-1)^n\left[f_3\left(b\sqrt{\Pr},t\right)-f_3\left(a\sqrt{\Pr},t\right)\right]\Bigg] \end{aligned} \quad (20)$$

$$\theta(y,t) = \frac{1}{\sqrt{\Pr}}\sum_{n=0}^{\infty}(-1)^n\left[f_2\left(a\sqrt{\Pr},t\right) - f_2\left(b\sqrt{\Pr},t\right)\right] \quad (21)$$

where

$c_1 = 2m\sqrt{\Pr}-2m+a,\ d_1 = 2m\sqrt{\Pr}-2m+b$.

It is clear that the solution for temperature field given by Eq. (21) is valid for all values of the Pandtl number whereas the solution for velocity field given by Eq. (20) is not valid for fluids of Prandtl number unity. As the Prandtl number is a measure of the relative importance of the viscosity and thermal conductivity of the fluid, the case $\Pr=1$ corresponds to those fluids whose momentum and thermal boundary layer thicknesses are of the same order of magnitude. Thus the solution for the velocity field has to be re-derived from Eqs. (19) and (8) when $\Pr=1$. It can be shown that

$$u(y,t) = f_7(y,t) + \sum_{n=0}^{\infty}\left[f_1(a,t) - f_1(b-2,t)\right] + Gr\sum_{n=0}^{\infty}\left[f_8(c_2,t) - f_8(d_2,t)\right] + \frac{yGr}{2}\sum_{n=0}^{\infty}(-1)^n\left[f_8(a,t) + f_8(b,t)\right] \tag{22}$$

where

$$c_2 = 2(n+1) + a,\ d_2 = 2n + b$$

$$f_7(z,t) = 1 + \text{erf}\left(\frac{z}{2\sqrt{t}}\right)$$

$$f_8(z,t) = \left(\frac{z^2}{2} + t\right)\text{erfc}\left(\frac{z}{2\sqrt{t}}\right) - z\sqrt{(t/\pi)}\exp\left(-\frac{z^2}{4t}\right)$$

3 Results and Discussion

An exact solution to the problem of natural convection in unsteady Couette flow between two long vertical parallel plates in the presence of constant heat flux and thermal radiation have been presented in the preceding section. In order to get the physical insight into the problem, the numerical values of the temperature field, the velocity field, the skin-friction, the Nusselt number, the volume flow rate and the vertical heat flux are computed for different values of the system parameters such as Radiation parameter (*R*), Grashof number (*Gr*), Prandtl number (Pr) and time (*t*). Figure 1 presents the temperature profiles of air (Pr = 0.71) for different values of *t* and *R*. It is seen that the temperature increases with increasing time in the presence of radiation and in the case of pure convection (which numerically corresponds to $R \to \infty$) i.e. in the absence of radiation. Moreover, the temperature is found to decrease due to an increase in the radiation parameter. When radiation is present, the thermal boundary layer was always found to thicken, which may be explained by the fact that radiation provides an additional means to diffuse energy. For $R = 10$ the temperature profile is found to increase 4.26 % of the pure convection case at the moving plate when $t = 0.4$. The thickening of the thermal boundary layer is more significant for small values of *t* and *R*. Furthermore, the temperature profiles attain their maximum value near the moving hot wall $(y = 0)$ and decreases smoothly to zero at the stationary wall $(y = 1)$ of the vertical channel.

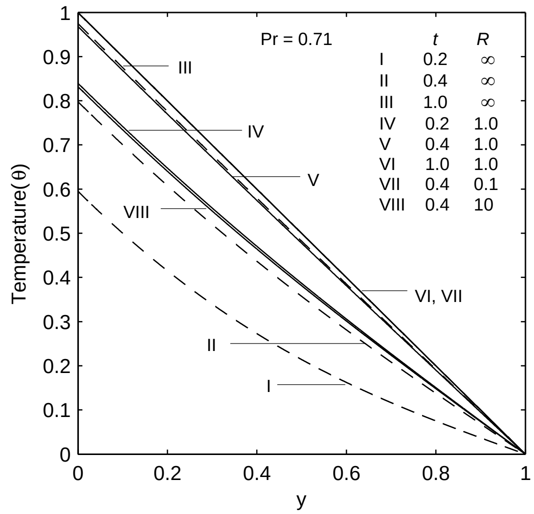

Fig. 1 Temperature profiles

Figure 2 presents the velocity profiles for both air and water (Pr = 7.0) in the case of pure convection $(R \to \infty)$ for different values of *Gr* and *t*. It is seen that the velocity of air and water increases with increasing *Gr* and *t*. At a smaller *t*, the velocity distribution is monotonic, but at a higher time it passes through a maximum near the moving plate when the buoyancy effect partly suppresses the inertial effects of the plate velocity. Moreover, the velocity of air is greater than the velocity of water. Physically this is possible because fluids with high Prandtl number have greater viscosity, which makes the fluid thick and hence move slowly.

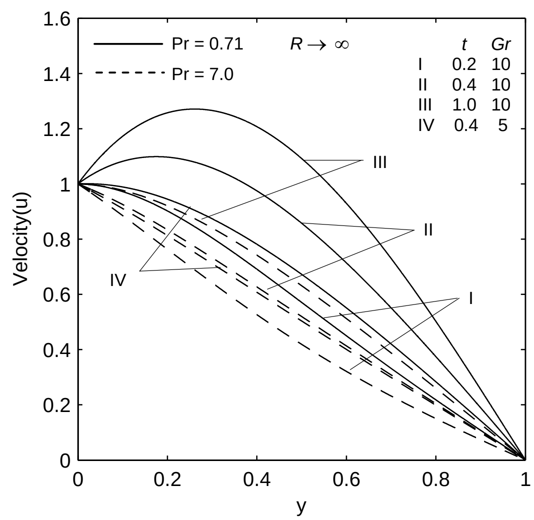

Fig. 2 Velocity profiles for different *t*, Pr and *Gr* (Pure convection case)