Blow-up solutions for a degenerate parabolic problem with a localized nonlinear term

P. SAWANGTONG[1], B. NOVAPRATEEP[1,2*] and W. JUMPEN[1,2]

[1]Dept of Mathematics, Faculty of science, Mahidol University,Thailand
[2]Center of Excellence in Mathematics, PERDO, CHE, Thailand
g4836373@student.mahidol.ac.th, scbnv@mahidol.ac.th and scwjp@mahidol.ac.th
*Corresponding author

Abstract: - In this article, we establish a blow-up solution and the blow-up set of such a solution of the degenerate parabolic problem with a localized nonlinear term: $k(x)u_t - (p(x)u_x)_x = k(x)f(u(x_0,t))$ where k, p and f are given functions and x_0 is a fixed point in the domain of x. In order to ensure the occurrence for blow-up in finite time, the sufficient condition to blow-up in finite time is shown. We furthermore study the particular problem of the previous problem: $x^q u_t + (x^\beta u_x)_x = x^q f(u(x_0,t))$ where q and β are specified constants. Under suitable assumptions on f, we obtain the same results as before.

Key-Words: - Blow-up in finite time, Blow-up set, Complete blow-up, Localized nonlinear terms, Semilinear parabolic problems

1 Introduction

Without loss of generality and for simplicity, we take the interval of x to [0,1]. Let $I = (0,1)$, $Q_T = I \times (0,T)$, $\bar{I}$ and $\bar{Q}_T$ be the closure of I and Q_T, respectively. We here study the following degenerate semilinear parabolic problem with a localized nonlinear term:

$$\left.\begin{aligned} k(x)u_t - \left(p(x)u_x\right)_x = k(x)f(u(x_0,t)) \text{ for } (x,t) \in Q_T, \\ u(0,t) = 0 = u(1,t) \text{ for } t \in (0,T), \\ u(x,0) = u_0(x) \text{ for } x \in \bar{I}, \end{aligned}\right\} \tag{1.1}$$

where u_t denotes partial differentiation of u with respect to t and k, p, f and u_0 are given functions. .In 2010 P. Sawangtong, B. Novaprateep and W. Jumpen [14] studied the degenerate parabolic problem (1.1). In this article we continuous to study the degenerate parabolic problem (1.1) and the purpose of this paper is to prove that before blow-up occurs, there exists a $T_1(>0)$ such that problem (1.1) has a unique nonnegative continuous solution u on the time interval $[0,T_1]$ for any $x \in \bar{I}$. In addition to prove the existence and uniqueness of solution, the sufficient condition to blow up in finite and the blow-up set of such a solution u are given. A solution u of problem (1.1) is said to blows up at $x = b$ in finite time t_b if there exists a sequence (x_n,t_n) with $t_n < t_b$ such that $(x_n,t_n) \to (b,t_b)$ as $n \to \infty$ and $\lim_{n\to\infty} u(x_n,t_n) = \infty$. The set of all blow-up points of solution u is called the blow-up set. In order to obtain our results, throughout this paper, we need following assumptions.

(A) $p \in C^1(\bar{I})$, $p(0) = 0$, p is positive on $(0,1]$ and p' is nonnegative on $\bar{I}$.

(B) $k \in C(\bar{I})$, $k(0) = 0$, k is positive on $(0,1]$.

(C) $f \in C^2[0,\infty)$ is convex with $f(0) = 0$ and $f(s) > 0$ for $s > 0$.

(D) $u_0 \in C^2(\bar{I})$, $u_0(0) = 0 = u_0(1)$, u_0 is nonnegative on I, $u_0(x_0) > 0$, and u_0 satisfies for any $x \in I$,

$$\frac{d}{dx}\left(p(x)\frac{du_0(x)}{dx}\right) + k(x)f(u_0(x_0)) \geq \varsigma k(x)u_0(x) \tag{1.2}$$

for some positive constant ς. By separation of variables, we obtain the corresponding singular eigenvalue problem to (1.1) defined by

$$\left.\begin{aligned} \frac{d}{dx}\left(p(x)\frac{d\varphi(x)}{dx}\right)+\lambda k(x)\varphi(x)=0 \text{ on } I,\\ \varphi(0)=0=\varphi(1). \end{aligned}\right\} \quad (1.3)$$

We note that conditions (A) and (B) implies that the point $x=0$ is a singular point of problem (1.3). By proposition 2.1 [12], condition (C) yields that f is increasing and locally Lipschitz on $[0,\infty)$. By equation (1.3), we have

$$\varphi''(x)+\frac{p'(x)}{p(x)}\varphi'(x)+\lambda\frac{k(x)}{p(x)}\varphi(x)=0 \text{ for } x\in I.$$

Multiplying both sides of above equation by x^2, we can rewrite equation (1.3) in a new form:

$$\left.\begin{aligned} x^2\varphi''(x)+x\left[x\frac{p'(x)}{p(x)}\right]\varphi'(x)+x^2\left[\lambda\frac{k(x)}{p(x)}\right]\varphi(x)=0 \text{ on } I,\\ \varphi(0)=0=\varphi(1). \end{aligned}\right\} \quad (1.4)$$

We have to add some conditions on functions p and k to make the point $x=0$ to be regular singular point, that is,

(E) The limit of $\frac{xp'(x)}{p(x)}$ and $\frac{x^2k(x)}{p(x)}$ are finite as $x\to 0$ and $\frac{xp'(x)}{p(x)}$ and $\frac{x^2k(x)}{p(x)}$ are analytic at $x=0$.

We note that theorem 5.7.1 [2] yields that eigenfunctions φ_n and eigenvalues λ_n of a corresponding singular eigenvalue problem (1.4) exist. Completeness of eigenfunctions φ_n of problem (1.4) follows from next assumption.

(E) $\int_0^1\int_0^1 H(x,\xi)^2 k(x)k(\xi)d\xi dx$ is finite where H is the corresponding Green's function to problem (1.4).

Previously there are mathematicians who studied blow-up problems of parabolic type with a localized nonlinear term. In 1992, J. M. Chadam, A. Peirce and H. M. Yin [3] investigated the blow-up behaviour of solutions to heat equation with a localized reaction term: let Ω be a bounded domain in $\mathbb{R}^n$ and x_0 a fixed point in Ω,

$$\left.\begin{aligned} u_t-\nabla^2 u = f(u(x_0,t)) \text{ for } (x,t)\in\Omega\times(0,T),\\ u(x,t)=0 \text{ for } (x,t)\in\partial\Omega\times(0,T),\\ u(x,0)=u_0(x) \text{ for } x\in\overline{\Omega}, \end{aligned}\right\} \quad (1.5)$$

where f and u_0 are given functions and $\partial\Omega$ and $\overline{\Omega}$ denote boundary and closure of Ω, respectively. They showed that under some conditions the solution u of problem (1.5) exhibits global blow-up and the blow-up set is $\overline{\Omega}$. In 2000, C.Y. Chan and J. Yang [6] studied the degenerate semilinear parabolic problem with a localized nonlinear term: let q be a nonnegative constant:

$$\left.\begin{aligned} x^q u_t-u_{xx} = f(u(x_0,t)) \text{ for } (x,t)\in Q_T,\\ u(0,t)=0=u(1,t) \text{ for } t\in(0,T),\\ u(x,0)=u_0(x) \text{ for } x\in\overline{I}, \end{aligned}\right\} \quad (1.6)$$

where f and u_0 are given functions. They proved that under certain hypotheses a nonnegative classical solution u of problem (1.6) blows up at all points $x\in\overline{I}$ in finite time. Moreover they gave a sufficient condition for solution a u of problem (1.6) to blow-up in finite time. In 2004, Y.P. Chen and C.H. Xie [8] discussed the degenerate parabolic equation with the nonlocal term:

$$\left.\begin{aligned} u_t-(x^\beta u_x)_x = \int_0^1 f(u(x,t))dx \text{ for } (x,t)\in Q_T,\\ u(0,t)=0=u(1,t) \text{ for } t\in(0,T),\\ u(x,0)=u_0(x) \text{ for } x\in\overline{I}. \end{aligned}\right\} \quad (1.7)$$

They consider the local existence and uniqueness of a classical solution. Under appropriate hypotheses, the obtained some sufficient conditions for the global existence and blow-up of a positive solution of problem (1.7). In 2004, Y.P. Chen, Q. Liu and C.H. Xie [7] studied the degenerate nonlinear reaction-diffusion equation with nonlocal source:

$$\left.\begin{aligned} x^q u_t-(x^\beta u_x)_x = \int_0^1 u^p(x,t)dx \text{ for } (x,t)\in Q_T,\\ u(0,t)=0=u(1,t) \text{ for } t\in(0,T),\\ u(x,0)=u_0(x) \text{ for } x\in\overline{I}. \end{aligned}\right\} \quad (1.8)$$

They established the local existence and uniqueness of a classical solution of problem (1.8). Under appropriate hypotheses, they also get some sufficient conditions for a global existence and blow-up of a positive solution. Furthermore, under certain conditions, it is proved that the blow-up set of the solution of problem (1.8) is the whole domain. In 2010, P. Sawangtong and W. Jumpen [13] showed, under certain condition, the existence of a blow-up solution of the following degenerate parabolic problem:

$$\left.\begin{aligned} x^q u_t - (x^\beta u_x)_x = x^q f(u(x,t)),\ (x,t) \in Q_T, \\ u(0,t) = 0 = u(1,t),\ t \in (0,T), \\ u(x,0) = u_0(x),\ x \in \bar{I}, \end{aligned}\right\} \quad (1.9)$$

where q and β are given constants and f and u_0 are suitable functions. Furthermore the sufficient condition to blow-up in finite time and the blow-up of such a solution of problem (1.9) are shown.

This paper is organized as follows. In section 2, we find corresponding Green's function to the degenerate parabolic problem (1.1). In order to obtain the existence and uniqueness of a solution of the degenerate parabolic problem (1.1), we transform problem (1.1) into the equivalent integral equation (2.2). Before blow-up occurs, we prove the existence and uniqueness of a solution of the equivalent integral problem (2.2) by using the Banach fixed point theorem. Furthermore we show that the solution of the equivalent integral problem (2.2) blows up at the point x_0 if blow-up occurs. To guarantee occurrence for blow-up in finite time, we give the sufficient condition to blow-up in finite time in section 3. In section 4, we establish the blow-up set of such a blow-up solution of the degenerate parabolic problem (1.1). Finally we study the particular problem of (1.1) in the last section.

2 Local existenec and uniqueness

This section deal with the local existence and uniqueness of a nonnegative continuous solution u of problem (1.1). Referred to [15], we have well-know properties of eigenvalues λ_n and eigenfunctions φ_n of problem (1.4) as the following lemma.

Lemma 2.1.

2.1.1. $\int_0^1 k(x)\varphi_n(x)\varphi_m(x)dx = \begin{cases} 1 & \text{for } m = m, \\ 0 & \text{for } m \neq n. \end{cases}$

2.1.2. All eigenvalues are real and positive.

2.1.3. Eigenfunctions are complete with the weight function k.

2.1.4. $\lambda_1 < \lambda_2 < \lambda_3 < \dots$ and $\lim_{n\to\infty} \lambda_n = \infty$.

2.1.5. $\int_0^1 p(x)\varphi_n'(x)\varphi_m'(x)dx = \begin{cases} \lambda_n & \text{for } n = m, \\ 0 & \text{for n} \neq \text{m}. \end{cases}$

2.1.6. For any $n \in \mathbb{N}$, $\varphi_n \in C^\infty(\bar{I})$.

Let us construct Green's function $G(x,t,\xi,\tau)$ corresponding to problem (1.1). It is determined by the following system: for $x,\xi \in I$ and $t,\tau \in (0,T)$,

$$\left.\begin{aligned} G_t - \frac{1}{k(x)}\left(p(x)G_x\right)_x = \delta(x-\xi)\delta(t-\tau), \\ G(0,t,\xi,\tau) = 0 = G(1,t,\xi,\tau), \\ G(x,t,\xi,\tau) = 0 \text{ for } t > \tau, \end{aligned}\right\} \quad (2.1)$$

where δ is the Dirac delta function. Let

$$G(x,t,\xi,\tau) = \sum_{n=1}^{\infty} a_n(t)\varphi_n(x) \quad (2.2)$$

Substituting equation (2.2) into equation (2.1), we obtain

$$\sum_{n=1}^{\infty} k(x)a_n'(t)\varphi_n(x) - \sum_{n=1}^{\infty} a_n(t)\frac{d}{dx}\left(p(x)\frac{d\varphi_n}{dx}\right) = \delta(x-\xi)\delta(t-\tau).$$

Multiplying both sides by φ_n and then integrating both sides with respect to x over its domain, we have

$$\int_0^1 k(x)\varphi_n(x)\sum_{n=1}^{\infty} a_n'(t)\varphi_n(x)dx - \int_0^1 \varphi_n(x)\sum_{n=1}^{\infty} a_n(t)\frac{d}{dx}\left(p(x)\frac{d\varphi_n}{dx}\right)dx$$
$$= \int_0^1 \varphi_n(x)\delta(x-\xi)\delta(t-\tau)dx.$$

By the orthonormal property of eigenfunctions φ_n and the property of Dirac delta function, we get

$$a_n'(t) + \lambda_n a_n(t) = \varphi_n(\xi)\delta(t-\tau)$$

or

$$\frac{d}{dt}\left(e^{\lambda_n t}a_n(t)\right) = \varphi_n(\xi)\delta(t-\tau)e^{\lambda_n t}.$$

Integrating both sides from t to t_1 with $t_1 < \tau$, we obtain

$$\int_{t_1}^{t} \frac{d}{ds}\left(e^{\lambda_n s}a_n(s)\right) = \int_{t_1}^{t} \varphi_n(\xi)\delta(t-\tau)e^{\lambda_n s}ds$$

or

$$e^{\lambda_n t}a_n(t) - e^{\lambda_n t_1}a_n(t_1) = \varphi_n(\xi)e^{\lambda_n \tau}.$$

Since $G(x,t,\xi,\tau) = 0$ for $t < \tau$, $a_n(t_1) = 0$ for all n. We therefore obtain that $a_n(t) = \varphi_n(\xi)e^{-\lambda_n(t-\tau)}$ for all n. By equation (2.2), the Green's function is defined by

$$G(x,t,\xi,\tau) = \sum_{n=1}^{\infty} \varphi_n(x)\varphi_n(\xi)e^{-\lambda_n(t-\tau)} \text{ for } t > \tau, \quad (2.3)$$

where φ_n and λ_n are eigenfunctions and eigenvalues of the singular eigenvalue problem (2.1), respectively.

By using Green's second identity, we get the integral equation equivalent to problem (1.1) given by

$$u(x,t) = \int_0^1 k(\xi)G(x,t,\xi,0)u_0(\xi)d\xi$$

$$+\int_0^t\int_0^1 k(\xi)G(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau. \qquad (2.4)$$

The following lemma is due to properties of G.

Lemma 2.2. Let $\lambda_n = O(n^s)$ for some $s>1$ as $n\to\infty$.
2.2.1. G is continuous for $x,\xi\in I$ and $0\le\tau<t<T$.
2.2.2. G is positive for $x,\xi\in I$ and $0\le\tau<t<T$.
2.2.3. $\lim_{t\to\tau^+} k(x)G(x,t,\xi,\tau)=\delta(x-\xi)$
2.2.4. For any $(x,t,\tau)\in I\times(0,T)\times(0,T)$,
$\int_0^1 k(\xi)G(x,t,\xi,\tau)d\xi\le C_0$ for some $C_0>0$.

Proof. By modifying proof of lemma 4.a and 4.c [5], we obtain the proof of 2.2.1 and 2.2.2, respectively. For proof of 2.2.3, let us consider the following problem:

$$k(x)w_t-\left(p(x)w_x\right)_x=0 \text{ for } x,\xi\in I \text{ and } 0<\tau<t<T,$$
$$w(0,t,\xi,\tau)=0=w(1,t,\xi,\tau) \text{ for } 0<\tau<t<T,$$
$$\lim_{t\to\tau^+} k(x)w(x,t,\xi,\tau)=\delta(x-\xi).$$

By equation (2.4), we have that for any $t>\tau$,

$$w(x,t,\xi,\tau)=\int_0^1 k(\zeta)G(x,t,\zeta,\tau)\frac{1}{k(\zeta)}\delta(\zeta-\xi)d\zeta=G(x,t,\xi,\zeta).$$

Hence, we obtain the proof of 2.2.3. We next prove 2.2.4.
Case 1. For any $t<\tau$.
Definition for G yields that $\int_0^1 k(\xi)G(x,t,\xi,\tau)d\xi=0.$
Case 2. $t=\tau$.
It follows lemma 2.2.3 and a property of Dirac delta function δ that

$$\int_0^1 k(\xi)G(x,t,\xi,\tau)d\xi=\int_0^1\delta(x-\xi)d\xi=1.$$

Case 3. For any $t>\tau$.
Let us consider the series

$$\sum_{n=1}^{\infty}\int_0^1 k(\xi)\varphi_n(\xi)\varphi_n(x)e^{-\lambda_n(t-\tau)}d\xi.$$

Since

$$\left|\int_0^1 k(\xi)\varphi_n(\xi)\varphi_n(x)e^{-\lambda_n(t-\tau)}d\xi\right|\le\left(\max_{x\in I}\varphi_n(x)\right)^2 e^{-\lambda_n(t-\tau)}$$

and the series $\sum_{n=1}^{\infty}e^{-\lambda_n(t-\tau)}$ converges, $\sum_{n=1}^{\infty}\int_0^1 k(\xi)\varphi_n(\xi)\varphi_n(x)e^{-\lambda_n(t-\tau)}d\xi$ converges uniformly for any $(x,t,\tau)\in I\times(0,T)\times(0,T)$. Hence we get the proof of 2.2.4. Therefore, the proof of lemma 2.2 is complete.

Next theorem says to local existence of a solution u of the equivalent integral equation (2.4).

Theorem 2.1. There exists a T_1 with $0<T_1<T$ such that the equivalent integral equation (2.4) has a unique continuous solution u for any $(x,t)\in\overline{Q}_{T_1}$.

Proof. We will use the fixed point theorem to prove existence of a solution u of the equivalent integral equation (2.4). Let $M=\max_{x\in I}|u_0(x)|+1.$ Locally Lipschitz property of f implies that there exists a positive constant $L(M)$ depending on M such that $|f(x)-f(y)|\le L(M)|x-y|$ for any $x,y\in\mathbb{R}$ with $|x|\le M$ and $|y|\le M$. We then choose

$$T_1<\min\left\{\frac{1}{C_0 f(M)},\frac{1}{L(M)C_0}\right\}.$$

Define a set E by

$$E=\left\{u\in C(\overline{Q}_{T_1}) \text{ such that } \max_{(x,t)\in\overline{Q}_{T_1}}|u(x,t)|\le M\right\}.$$

Then E is a Banach space equipped with the norm $|u|_E=\max_{(x,t)\in\overline{Q}_T}|u(x,t)|$. Let

$$\Lambda u(x,t)=\int_0^1 k(\xi)G(x,t,\xi,0)u_0(\xi)d\xi$$
$$+\int_0^t\int_0^1 k(\xi)G(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau. \qquad (2.5)$$

for any $u\in E$. We next show that the operator Λ defined by (2.5) maps E into itself and that Λ is contractive. Let $u,v\in E$. We then have that

$$|\Lambda u(x,t)|\le\left|\int_0^1 k(\xi)G(x,t,\xi,0)u_0(\xi)d\xi\right|$$
$$+\left|\int_0^t\int_0^1 k(\xi)G(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau\right|. \qquad (2.6)$$

Let us consider the following auxiliary problem:

$$\left.\begin{aligned} &k(x)u_t-\left(p(x)u_x\right)_x=0 \text{ for } (x,t)\in Q_{T_1},\\ &u(0,t)=0=u(1,t) \text{ for } t\in(0,T_1),\\ &u(x,0)=u_0(x) \text{ for } x\in\overline{I}.\end{aligned}\right\} \qquad (2.7)$$

It follows from (2.4) that a solution u of problem (2.7) is given by

$$u(x,t)=\int_0^1 k(\xi)G(x,t,\xi,0)u_0(\xi)d\xi \text{ for } (x,t)\in\overline{Q}_{T_1}.$$

Moreover, maximum principle for parabolic type implies that $0\le u(x,t)\le\max_{x\in I}|u_0(x)|$ for any $(x,t)\in\overline{Q}_{T_1}$. Thus, we obtain that

$\int_0^1 k(\xi)G(x,t,\xi,0)d\xi \le 1$. From (2.6) and lemma 2.2.4.,

$$|\Lambda u(x,t)| \le \max_{x\in\bar{I}} |u_0(x)| + f(M)\int_0^t\int_0^1 k(\xi)G(x,t,\xi,\tau)d\xi d\tau.$$
$$\le \max_{x\in\bar{I}} |u_0(x)| + f(M)C_0T_1.$$

By definition of T_1, $\Lambda u \in E$ for any $u \in E$. Since

$$|\Lambda u(x,t) - \Lambda v(x,t)|$$
$$\le \left|\int_0^t\int_0^1 k(\xi)G(x,t,\xi,\tau) f(u(x_0,\tau)) - f(v(x_0,\tau))d\xi d\tau.\right|$$
$$\le \int_0^t\int_0^1 k(\xi)G(x,t,\xi,\tau)\left| f(u(x_0,\tau)) - f(v(x_0,\tau))\right| d\xi d\tau$$
$$\le L(M)\int_0^t\int_0^1 k(\xi)G(x,t,\xi,\tau)d\xi d\tau |u-v|_E$$
$$\le C_0T_1L(M)|u-v|_E, \tag{2.8}$$

definition of T_1 and (2.8) yield that Λ is contractive. The fixed point theorem then implies that there exists a unique u in E satisfying the integral equation (2.4). Therefore, the proof is complete.

Lemma 2.3. Let v be a classical solution of the following problem:

$$\left.\begin{aligned} k(x)v_t - \left(p(x)v_x\right)_x &\ge B(x,t)v(x_0,t) \text{ for } (x,t)\in Q_T,\\ v(0,t) = 0 &= v(1,t) \text{ for } t\in(0,T),\\ v(x,0) = u_0(x) &\ge 0 \text{ for } x\in \bar{I}, \end{aligned}\right\} \tag{2.9}$$

where $B(x,t)$ is a nonnegative and bounded function on $\bar{Q}_T$. Then $v(x,t)\ge 0$ for any $(x,t)\in\bar{Q}_T$.

Proof. In order to prove this lemma we have to add a nonnegative continuous function $z(x,t)$ on $\bar{Q}_T$ to right-hand side of equation (2.9) and then we have that

$$\left.\begin{aligned} k(x)v_t - \left(p(x)v_x\right)_x &= B(x,t)v(x_0,t) + z(x,t) \text{ on } Q_T,\\ v(0,t) = 0 &= v(1,t) \text{ for } t\in(0,T),\\ v(x,0) = u_0(x) &\ge 0 \text{ for } x\in \bar{I}. \end{aligned}\right\} \tag{2.10}$$

From equation (2.4), we obtain that for $(x,t)\in\bar{Q}_T$,

$$\begin{aligned} v(x,t) &= \int_0^1 k(\xi)G(x,t,\xi,0)u_0(\xi)d\xi \\ &+\int_0^t\int_0^1 k(\xi)G(x,t,\xi,\tau)B(\xi,\tau)v(x_0,\tau)d\xi d\tau \\ &+\int_0^t\int_0^1 k(\xi)G(x,t,\xi,\tau)z(\xi,\tau)d\xi d\tau. \end{aligned} \tag{2.11}$$

From (2.11), we have

$$\begin{aligned} v(x_0,t) &= \int_0^1 k(\xi)G(x_0,t,\xi,0)u_0(\xi)d\xi \\ &+\int_0^t\int_0^1 k(\xi)G(x_0,t,\xi,\tau)B(\xi,\tau)v(x_0,\tau)d\xi d\tau \\ &+\int_0^t\int_0^1 k(\xi)G(x_0,t,\xi,\tau)z(\xi,\tau)d\xi d\tau. \end{aligned}$$

Let $h_0(t) = \int_0^1 k(\xi)G(x_0,t,\xi,0)u_0(\xi)d\xi$

$$+\int_0^t\int_0^1 k(\xi)G(x_0,t,\xi,\tau)z(\xi,\tau)d\xi d\tau.$$

Since functions k, z, G and u_0 are nonnegative, h_0 is nonnegative. Let $u(x_0,t) = h(t)$ for $t\in[0,T]$. Define an operator Φ mapping from $C[0,T]$ to $C[0,T]$ by

$$\Phi h(t) = \int_0^t\int_0^1 k(\xi)G(x_0,t,\xi,\tau)B(\xi,\tau)h(\tau)d\xi d\tau.$$

By corollary 5.2.1. [10], there exists a $T_2(<T)$ such that

$$v(x_0,t) = h(t) = \sum_{m=0}^{\infty}\Phi^{(m)}h_0(t) \tag{2.12}$$

where $\Phi^{(0)}h_0(t) = h_0(t)$ and $\Phi^{(m+1)}h_0(t) = \Phi\left[\Phi^{(m)}h_0(t)\right]$ for $m\in\mathbb{N}$. Mathematical induction yields that $\Phi^{(m)}h_0(t)\ge 0$ for $m\in\mathbb{N}$. Thus, from equation (2.12), we obtain that $v(x_0,t)\ge 0$ for any $t\in[0,T_2]$. It follows from equation (2.11) that $v(x,t)\ge 0$ on $\bar{Q}_{T_2}$. Finally, we can repeat the previous procedure to obtain the desired result for $(x,t)\in\bar{Q}_T$.

Next lemma gives additional properties of a solution u of problem (1.1).

Lemma 2.4. Let u be a continuous solution of problem (1.1). Then $u(x,t)\ge u_0(x)$ and $u_t(x,t)\ge 0$ for any $(x,t)\in\bar{Q}_{T_1}$.

Proof. Let $z(x,t) = u(x,t) - u_0(x)$ on $\bar{Q}_{T_1}$. Let us consider that for any $(x,t)\in Q_{T_1}$,

$$k(x)z_t - \left(p(x)z_x\right)_x = k(x)f(u(x_0,t)) + \frac{d}{dx}\left(p(x)\frac{du_0(x)}{dx}\right).$$

Equation (1.2) yields

$\frac{d}{dx}\left(p(x)\frac{du_0(x)}{dx}\right) \ge -k(x)f(u_0(x_0))$ on I and then we obtain that for any $(x,t)\in Q_{T_1}$,

$$\begin{aligned} k(x)z_t - \left(p(x)z_x\right)_x &= k(x)f(u(x_0,t)) - k(x)f(u_0(x_0)) \\ &\ge k(x)f'(\eta_1)z(x_0,t) \end{aligned}$$

where η_1 is between $u(x_0,t)$ and $u_0(x_0)$. Moreover, for any $(x,t)\in\{0,1\}\times(0,T)\cup\bar{I}\times\{0\}$, $z(x,t)=0$. Lemma

2.3 implies that $z \geq 0$ on $\overline{Q}_{T_1}$ or $u \geq u_0$ on $\overline{Q}_{T_1}$. Let h be any positive constant less that T and $w(x,t) = u(x,t+h) - u(x,t)$ on $\overline{Q}_{T_1}$. Then we have that on Q_{T_1},

$$k(x)w_t - \left(p(x)w_x\right)_x = k(x)f(u(x_0,t+h)) - k(x)f(u(x_0,t))$$

$$= k(x)f'(\eta_2)w(x_0,t),$$

for η_2 between $u(x_0,t+h)$ and $u(x_0,t)$. Furthermore, $w=0$ on $\{0,1\}\times(0,T_1)$ and $w \geq 0$ on $\overline{I}\times\{0\}$. It then follows from lemma 2.3 that $w \geq 0$ on $\overline{Q}_{T_1}$. This shows that $u_t \geq 0$ on $\overline{Q}_{T_1}$.

We note that before blow-up occurs, there exists a positive constant M such that $|u(x,t)| \leq M$ for all $(x,t) \in \overline{Q}_{T_1}$. Locally Lipschitz continuity of f yields that there exists a positive constant $L(M)$ depending on M such that $|f(u(x_0,t))| \leq L(M)|u(x_0,t)|$ for any $t \in [0,T_1]$.

Lemma 2.5. If $f'(u_0(x_0)) \geq L(M)$, then $u_t(x,t) \geq L(M)u(x,t)$ on $\overline{Q}_{T_1}$.

Proof. Let $z(x,t) = u_t(x,t) - L(M)u(x,t)$ on $\overline{Q}_{T_1}$. We then have that for $(x,t) \in Q_{T_1}$,

$$k(x)z_t - \left(p(x)z_x\right)_x = k(x)f'(u(x_0,t))u_t(x_0,t) - k(x)L(M)f(u(x_0,t))$$

Locally Lipschitz continuity of f implies that for $(x,t) \in Q_{T_1}$,

$$\begin{aligned} k(x)z_t - \left(p(x)z_x\right)_x &\geq k(x)f'(u(x_0,t))u_t(x_0,t) - k(x)L^2(M)u(x_0,t) \\ &\geq k(x)\left[f'(u_0(x_0))u_t(x_0,t) - L^2(M)u(x_0,t)\right] \\ &\geq k(x)L(M)z(x_0,t). \end{aligned}$$

From lemma 2.4, $z(0,t) = u_t(0,t) \geq 0$ and $z(1,t) = u_t(1,t) \geq 0$ for $t \in (0,T_1)$. If we set $\varsigma = L(M)$, then equation (1.2) implies that for any $x \in I$,

$$\begin{aligned} z(x,0) &= \lim_{t\to 0} u_t(x,t) - L(M)u_0(x) \\ &= \frac{1}{k(x)}\left(p(x)\frac{du_0(x)}{dx}\right) + f(u(x_0)) - L(M)u_0(x) \\ &\geq 0. \end{aligned}$$

Therefore, by lemma 2.3, the proof is complete.

Lemma 2.6. If $u_0(x_0) \geq u_0(x)$ for any $x \in \overline{I}$, then $u(x_0,t) \geq u(x,t)$ on $\overline{Q}_{T_1}$.

Proof. Let $z(x,t) = u(x_0,t) - u(x,t)$ on $\overline{Q}_{T_1}$. We then have that on Q_{T_1}, lemma 2.5 yields that

$$\begin{aligned} k(x)z_t - \left(p(x)z_x\right)_x &= k(x)\left[u_t(x_0,t) - f(u(x_0,t))\right] \\ &= k(x)\left[u_t(x_0,t) - L(M)u(x_0,t)\right] \\ &\geq 0. \end{aligned}$$

Since
$z(0,t) = u(x_0,t) \geq u_0(x) \geq 0$, $z(1,t) = u(x_0,t) \geq u_0(x) \geq 0$ for $t \in (0,T_1)$, and $z(x,0) = u_0(x_0) - u_0(x) \geq 0$ for any $x \in \overline{I}$, by lemma 2.3, the proof of this lemma is complete.

Theorem 2.2. Let $T_{\max}$ be the supremum of all T_1 such that the continuous solution u of an equivalent integral equation (2.4) exists. If $T_{\max}$ is finite, then $u(x_0,t)$ is unbounded as t tends to $T_{\max}$.

Proof. Suppose that $u(x_0,T_{\max})$ is finite. Let $N = u(x_0,T_{\max}) + 1$. By theorem 2.1 and a fact that u is non-decreasing in t, there exists a finite time $\tilde{T}(> T_{\max})$ depending on N such that the equivalent integral equation (2.4) has a unique continuous solution on the time interval $[0,\tilde{T}]$ for any $x \in \overline{I}$. By the definition of $T_{\max}$, we get a contradiction.

A proof similar to that of theorem 3 of Chan and Tian [4] gives the following result.

Theorem 2.3 Such a continuous solution u of the equivalent integral equation (2.4) is a classical solution.

3 A sufficient condition to blow-up in finite time

Let φ_1 be the first eigenfunction of a singular eigenvalue problem (1.3) and let λ_1 be its corresponding eigenvalue. Without loss of generality we assume

$$\int_0^1 k(x)\varphi_1(x)dx = 1. \tag{3.1}$$

We then define a function H by

$$H(t) = \int_0^1 k(x)\varphi_1(x)u(x,t)dx.$$

Theorem 3.1. Assume that
3.1.1. u_0 attains its maximum at point x_0.
3.1.2. $f(\xi) \geq b\xi^p$ with $b > 0$ and $p > 1$.

3.1.3. $H(0) > \left(\frac{\lambda_1}{b}\right)^{\frac{1}{p-1}}$.

Then a solution u of problem (1.1) blows up in finite time.

Proof. Multiplying equation (1.1) by φ_1 and integrating equation (1.1) with respect to x over its domain yield

$$\frac{dH(t)}{dt} = -\lambda_1 H(t) + \int_0^1 k(x) f(u(x_0,t))\varphi_1(x)dx.$$

By lemma 2.6 and assumption 3.1.2, we have

$$\frac{dH(t)}{dt} \geq -\lambda_1 H(t) + \int_0^1 k(x) f(u(x,t))\varphi_1(x)dx$$
$$\geq -\lambda_1 H(t) + b\int_0^1 k(x) u^p(x,t)\varphi_1(x)dx. \quad (3.2)$$

Holder inequality implies that

$$\int_0^1 k(x)\varphi_1(x)u(x,t)dx$$
$$\leq \left(\int_0^1 k(x)\varphi_1(x)dx\right)^{\frac{p-1}{p}} \left(\int_0^1 k(x)\varphi_1(x)u^p(x,t)dx\right)^{\frac{1}{p}}.$$

From (3.1), we get

$$\int_0^1 k(x)\varphi_1(x)u^p(x,t)dx \geq \left(\int_0^1 k(x)\varphi_1(x)u(x,t)dx\right)^p = H^p(t). \quad (3.3)$$

Form equation (3.2) and (3.3), we obtain

$$H'(t) \geq -\lambda_1 H(t) + bH^p(t)$$

or

$$H^{p-1}(t) \geq \frac{1}{\frac{b}{\lambda_1} + \left[H^{1-p}(0) - \frac{b}{\lambda_1}\right]e^{-\lambda_1(1-p)t}}.$$

It then follows from assumption 3.1.3 that there exists a $\hat{T}(>0)$ such that H tends to infinity as t converges to $\hat{T}$. By the definition of H, we find that

$$H(t) \leq \left(\int_0^1 k(x)\varphi_1(x)dx\right)u(x_0,t) = u(x_0,t).$$

Therefore, a solution u of problem (1.1) blows up at point x_0 as t tends to $\hat{T}$.

3 The blow-up set

In this section, we establish the blow-up set of the degenerate parabolic problem (1.1).

Theorem 4.1. The blow-up set of a solution u of problem (1.1) is $\bar{I}$.

Proof. From equation (2.4), we have that for $t \in (0, T_{\max})$,

$$u(x_0,t) = \int_0^1 k(\xi)G(x_0,t,\xi,0)u_0(\xi)d\xi$$
$$+\int_0^t\int_0^1 k(\xi)G(x_0,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau$$
$$\leq \max_{x\in\bar{I}}|u_0(x)| + C_0\int_0^t f(u(x_0,\tau))d\tau. \quad (4.1)$$

By theorem 2.2, we obtain that as t tends to $T_{\max}$,

$$\int_0^{T_{\max}} f(u(x_0,\tau))d\tau = \infty. \quad (4.2)$$

On the other hand, by positivity of k, G, and u_0, we get that for any $(x,t) \in Q_{T_{\max}}$,

$$u(x,t) \geq \int_0^t\int_0^1 k(\xi)G(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau.$$

Since there exists a positive constant C_1 such that

$$\int_0^1 k(\xi)G(x,t,\xi,\tau)d\xi \geq C_1,$$

we obtain that

$$u(x,t) \geq C_1\int_0^t f(u(x_0,\tau))d\tau \quad \text{for all } (x,t) \in Q_{T_{\max}}.$$

Hence, the solution u tends to infinity for all $x \in I$ as t approaches to $T_{\max}$. Furthermore, for $x \in \{0,1\}$, we can find a sequence $\{(x_n,t_n)\}$ such that $\lim_{n\to\infty} u(x_n,t_n) \to \infty$. Hence, the blow-up set of a solution of a degenerate parabolic problem (1.1) is $\bar{I}$. Therefore the proof of this theorem is complete.

4 The particular problem

In this section, we consider the particular problem of the degenerate parabolic problem (1.1): let q and β be some constants with $q \geq 0$, $0 \leq \beta < 1$ and $q + \beta \neq 0$,

$$\left.\begin{aligned} x^q u_t - (x^\beta u_x)_x &= x^q f(u(x_0,t)) \text{ for } (x,t) \in Q_T, \\ u(0,t) = 0 &= u(1,t) \text{ for } t \in (0,T), \\ u(x,0) &= u_0(x) \text{ for } x \in \bar{I}. \end{aligned}\right\} \quad (5.1)$$

By using separation of variables as [9] and [11] on the homogenous problem corresponding to problem (5.1), we obtain the singular eigenvalue problem,

$$\left.\begin{aligned} \frac{d}{dx}\left(x^\beta \frac{d\tilde{\varphi}}{dx}\right) + \tilde{\lambda}x^q\tilde{\varphi}(x) = 0 \text{ for } x \in I, \\ \tilde{\varphi}(0) = 0 = \tilde{\varphi}(1). \end{aligned}\right\} \quad (5.2)$$

Let $\tilde{\varphi}(x) = x^{\frac{1-\beta}{2}} y(x)$. Then

$$\tilde{\varphi}'(x) = x^{\frac{1-\beta}{2}} y'(x) + \left(\frac{1-\beta}{2}\right) x^{\frac{-\beta-1}{2}} y(x) \tag{5.3}$$

and

$$\tilde{\varphi}''(x) = x^{\frac{1-\beta}{2}} y''(x) + (1-\beta) x^{\frac{-\beta-1}{2}} y'(x) + \left(\frac{1-\beta}{2}\right)\left(\frac{-\beta-1}{2}\right) x^{\frac{-\beta-3}{2}} y(x). \tag{5.4}$$

Substituting equations (5.3) and (5.4) into equation (5.2), we obtain

$$x^{\beta}\left[x^{\frac{1-\beta}{2}} y''(x) + (1-\beta) x^{\frac{-\beta-1}{2}} y'(x) + \left(\frac{1-\beta}{2}\right)\left(\frac{-\beta-1}{2}\right) x^{\frac{-\beta-3}{2}} y(x)\right]$$

$$+\beta x^{\beta-1}\left[x^{\frac{1-\beta}{2}} y'(x) + \left(\frac{1-\beta}{2}\right) x^{\frac{-\beta-1}{2}} y(x)\right] + \tilde{\lambda} x^{q} x^{\frac{1-\beta}{2}} y(x) = 0$$

or

$$x^{\frac{1+\beta}{2}} y''(x) + x^{\frac{\beta-1}{2}} y'(x) + \left[\tilde{\lambda} x^{q} x^{\frac{1-\beta}{2}} - \left(\frac{1-\beta}{2}\right)^2 x^{\frac{\beta-3}{2}}\right] y(x) = 0. \tag{5.5}$$

Dividing both sides of equation (5.5) by $x^{\frac{1+\beta}{2}}$, we get

$$y''(x) + \frac{1}{x} y'(x) + \left[\tilde{\lambda} x^{q-\beta} - \left(\frac{1-\beta}{2}\right)^2 \frac{1}{x^2}\right] y(x) = 0. \tag{5.6}$$

Multiplying both side of equation (5.6) by x^2, the singular eigenvalue problem (5.2) becomes

$$\left.\begin{array}{c} x^2 y''(x) + x y'(x) + \left[\tilde{\lambda} x^{q-\beta+2} - \left(\frac{1-\beta}{2}\right)^2\right] y(x) = 0, \\ y(0) \text{ is bounded and } y(1) = 0. \end{array}\right\} \tag{5.7}$$

Again, we set $x = z^{\frac{2}{q-\beta+2}}$. Then

$$y'(x) = \left(\frac{q-\beta+2}{2}\right) z^{\frac{q-\beta}{q-\beta+2}} y'(z)$$

and

$$y''(x) = \left(\frac{q-\beta+2}{2}\right)^2 z^{\frac{2(q-\beta)}{q-\beta+2}} y''(z) + \left(\frac{q-\beta}{2}\right)\left(\frac{q-\beta+2}{2}\right) z^{\frac{q-\beta-2}{q-\beta+2}} y'(z).$$

It follows from equation (5.7) that we have

$$z^{\frac{4}{q-\beta+2}}\left[\left(\frac{q-\beta+2}{2}\right)^2 z^{\frac{2(q-\beta)}{q-\beta+2}} y''(z) + \left(\frac{q-\beta}{2}\right)\left(\frac{q-\beta+2}{2}\right) z^{\frac{q-\beta-2}{q-\beta+2}} y'(z)\right]$$

$$+\left(\frac{q-\beta+2}{2}\right) z y'(z) + \left[\tilde{\lambda} z^2 - \left(\frac{1-\beta}{2}\right)^2\right] y(z) = 0$$

or

$$\left.\begin{array}{c} z^2 y''(z) + z y'(z) + \left[\frac{4\tilde{\lambda} z^2}{(q-\beta+2)^2} - \frac{(1-\beta)^2}{(q-\beta+2)^2}\right] y(z) = 0, \\ z(0) \text{ is bounded and } z(1) = 0. \end{array}\right\} \tag{5.8}$$

Thus, we see that equation (5.8) is a Bessel equation. Its general solution of a Bessel equation (5.8) is given by

$$y(z) = A J_{\mu}(\omega z) + B J_{-\mu}(\omega z)$$

where $\mu = \frac{1-\beta}{q-\beta+2}$, $\omega = \frac{2\tilde{\lambda}^{\frac{1}{2}}}{q-\beta+2}$, A and B are arbitrary constants and J_{μ} denotes the Bessel function of the first kind of order $\mu(>0)$. Turning to the boundary condition, at $z=0$ leads to $B=0$ and then we obtain

$$y(z) = A J_{\mu}(\omega z). \tag{5.9}$$

The boundary condition at $z=1$ gives the following equation,

$$J_{\mu}(\omega) = 0. \tag{5.10}$$

Then, by equation (5.9), the appropriate eigenfunctions φ_n of the singular eigenvalue problem (5.2) are

$$\tilde{\varphi}_n(x) = Ax^{\frac{1-\beta}{2}} J_\mu(\omega_n x^{\frac{q-\beta+2}{2}}) \tag{5.11}$$

where ω_n is the n^{th} root of equation (5.10). In order to obtain the orthonormal property of eigenfunctions $\tilde{\varphi}_n$ with the weight function x^q,

$\int_0^1 x^q \tilde{\varphi}_n(x)\tilde{\varphi}_m(x)dx = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$, we use the orthogonality of Bessel functions, that is,

$$\int_0^1 xJ_\mu(\omega_n x)J_\mu(\omega_m x)dx = \begin{cases} \frac{1}{2}J_{\mu+1}^2(\omega_n) & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases},$$

to determine the value of constant A. To do so, we consider

$$\int_0^1 x^q \tilde{\varphi}_n^2(x)dx = A^2\int_0^1 x^{q-\beta+1}J_{\mu+1}^2(\omega_n x^{\frac{q-\beta+2}{2}})dx. \tag{5.12}$$

Let $y = x^{\frac{q-\beta+2}{2}}$. Then $dy = \left(\frac{q-\beta+2}{2}\right)x^{\frac{q-\beta}{2}}dx.$ Let us consider the right-hand side of equation (5.12)

$$A^2\int_0^1 x^{q-\beta+1}J_{\mu+1}^2(\omega_n x^{\frac{q-\beta+2}{2}})dx = \frac{2A^2}{q-\beta+2}\int_0^1 yJ_\mu^2(\omega_n y)dy$$

$$= \frac{A^2}{q-\beta+2}J_{\mu+1}^2(\omega_n). \tag{5.13}$$

It follows from (5.12) and (5.13) that

$$\int_0^1 x^q\tilde{\varphi}_n^2(x)dx = \frac{A^2}{q-\beta+2}J_{\mu+1}^2(\omega_n).$$

Since the right-hand side of equation (5.12) must equal to 1, the value of constant A is determined by $A = \frac{(q-\beta+2)^{\frac{1}{2}}}{|J_{\mu+1}(\omega_n)|}$. Hence, the appropriate eigenfunctions $\tilde{\varphi}_n$ of the singular eigenvalue problem (5.2) are defined by

$$\tilde{\varphi}_n(x) = \frac{(q-\beta+2)^{\frac{1}{2}} x^{\frac{1-\beta}{2}} J_\mu(\omega_n x^{\frac{q-\beta+2}{2}})}{|J_{\mu+1}(\omega_n)|}. \tag{5.14}$$

The properties of eigenfunctions $\tilde{\varphi}_n$ and eigenvalues $\tilde{\lambda}_n$ of the singular eigenvalue problem (5.2) associating to the degenerate parabolic problem (5.1) is given in the following lemma.

Lemma 5.1.

5.1.1 $\int_0^1 x^q\tilde{\varphi}_n(x)\tilde{\varphi}_m(x)dx = \begin{cases} 1 & \text{for } n = m, \\ 0 & \text{for } n \neq m. \end{cases}$

5.1.2 All eigenvalues $\tilde{\lambda}_n$ are real and positive.

5.1.3 Eigenfunctions $\tilde{\varphi}_n$ are complete with respect to the weight function x^q.

5.1.4 $\tilde{\lambda}_n = O(n^2)$ as $n \to \infty$.

5.1.5 $\int_0^1 p(x)\tilde{\varphi}'_n(x)\tilde{\varphi}'_m(x)dx = \begin{cases} \tilde{\lambda}_n & \text{for } n = m, \\ 0 & \text{for } n \neq m. \end{cases}$

5.1.6 For any $x \in \bar{I}$, $|\tilde{\varphi}_n(x)| \leq C_2 x^{\frac{1-\beta}{2}}\tilde{\lambda}_n^{\frac{1}{4}}$ for some positive constant C_2.

As previous discussion, we establish the Green's function corresponding to the degenerate parabolic problem (5.1) by the following system: for $x, \xi \in I$ and $t, \tau \in (0,T)$,

$$x^q\tilde{G}_t(x,t,\xi,\tau) + (x^\beta\tilde{G}_x(x,t,\xi,\tau))_x = \delta(x-\xi)\delta(t-\tau),$$
$$\tilde{G}(0,t,\xi,\tau) = 0 = \tilde{G}(1,t,\xi,\tau),$$
$$\tilde{G}(x,t,\xi,\tau) = 0 \text{ for } t < \tau,$$

where δ is the Dirac delta function and then by the eigenfunction expansion the Green's function $\tilde{G}$ is given by

$$\tilde{G}(x,t,\xi,\tau) = \sum_{n=1}^{\infty}\tilde{\varphi}_n(x)\tilde{\varphi}_n(\xi)e^{-\tilde{\lambda}_n(t-\tau)} \text{ for } t > \tau.$$

The proofs of next lemma is similar to that of lemma 2.2 and then we obtain the following.

Lemma 5.2.

5.2.1 $\tilde{G}$ is continuous for $x, \xi \in I$ and $0 \leq \tau < t < T$.

5.2.2 $\tilde{G}$ is positive for $x, \xi \in I$ and $0 \leq \tau < t < T$.

5.2.3 $\lim_{t\to\tau^+} x^q G(x,t,\xi,\tau) = \delta(x-\xi)$

5.2.4. $\int_0^1 \xi^q G(x,t,\xi,\tau)d\xi \le C_3$ for some $C_3 > 0$ for any $(x,t,\tau) \in I \times (0,T) \times (0,T)$,

It follows from Green's second identity that the equivalent integral equation to the degenerate parabolic problem (5.1) is defined by

$$u(x,t) = \int_0^1 x^q \widetilde{G}(x,t,\xi,0)u_0(\xi)d\xi$$

$$+\int_0^t\int_0^1 x^q \widetilde{G}(x,t,\xi,\tau) f(u(x_0,\tau))d\xi d\tau. \qquad (5.15)$$

It is not necessary to go through much detail since the proofs are almost identical. For the proofs of local existence and uniqueness of a blow-up solution of the equivalent integral equation (5.15) and the blow-up set of such a solution, we can demonstrate in exactly the same way as before.

Theorem 5.3. There exists a T_2 with $0 < T_2 < T$ such that the equivalent integral equation (5.15) has a unique continuous solution u for any $(x,t) \in \overline{Q}_{T_2}$. Let $\widetilde{T}_{\max}$ be the supremum of all T_2 such that the continuous solution u of the equivalent integral equation (5.15) exists. If $\widetilde{T}_{\max}$ is finite, then $u(x_0,t)$ is unbounded as t tends to $\widetilde{T}_{\max}$.

Theorem 5.4. If the solution u of the degenerate parabolic problem (5.1) blows up, then the blow-up set of u is $\overline{I}$.

In order to ensure occurrence of blow-up in finite time, we give the sufficient condition for the degenerate parabolic problem (5.1) to blow up in finite time.

Theorem 5.5. Let $\widetilde{H}(t) = \int_0^1 x^q u(x,t)\widetilde{\varphi}_1(x)dx$ where $\widetilde{\varphi}_1$ is the first eigenfunction of the singular eigenvalue problem (5.2), $\widetilde{\lambda}_1$ is its corresponding eigenvalue and $\int_0^1 x^q \widetilde{\varphi}_1(x)dx = 1$. Suppose that

5.5.1. u_0 attains its maximum at point x_0.

5.5.2. $f(\xi) \ge b\xi^p$ with $b > 0$ and $p > 1$.

5.5.3. $\widetilde{H}(0) > \left(\frac{\lambda_1}{b}\right)^{\frac{1}{p-1}}$.

Then a solution u of the degenerate parabolic problem (5.1) blows up in finite time.

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