

# Blow-up solutions for a degenerate parabolic problem with a localized nonlinear term

P. SAWANGTONG<sup>1</sup>, B. NOVAPRATEEP<sup>1,2\*</sup> and W. JUMPEN<sup>1,2</sup>

<sup>1</sup>Dept of Mathematics, Faculty of science, Mahidol University, Thailand

<sup>2</sup>Center of Excellence in Mathematics, PERDO, CHE, Thailand

g4836373@student.mahidol.ac.th, scbnv@mahidol.ac.th and scwjp@mahidol.ac.th

\*Corresponding author

*Abstract:* - In this article, we establish a blow-up solution and the blow-up set of such a solution of the degenerate parabolic problem with a localized nonlinear term:  $k(x)u_t - (p(x)u_x)_x = k(x)f(u(x_0, t))$  where  $k, p$  and  $f$  are given functions and  $x_0$  is a fixed point in the domain of  $x$ . In order to ensure the occurrence for blow-up in finite time, the sufficient condition to blow-up in finite time is shown. We furthermore study the particular problem of the previous problem:  $x^q u_t + (x^\beta u_x)_x = x^q f(u(x_0, t))$  where  $q$  and  $\beta$  are specified constants. Under suitable assumptions on  $f$ , we obtain the same results as before.

*Key-Words:* - Blow-up in finite time, Blow-up set, Complete blow-up, Localized nonlinear terms, Semilinear parabolic problems

## 1 Introduction

Without loss of generality and for simplicity, we take the interval of  $x$  to  $[0, 1]$ . Let  $I = (0, 1)$ ,  $Q_T = I \times (0, T)$ ,  $\bar{I}$  and  $\bar{Q}_T$  be the closure of  $I$  and  $Q_T$ , respectively. We here study the following degenerate semilinear parabolic problem with a localized nonlinear term:

$$\left. \begin{aligned} k(x)u_t - (p(x)u_x)_x &= k(x)f(u(x_0, t)) \text{ for } (x, t) \in Q_T, \\ u(0, t) &= 0 = u(1, t) \text{ for } t \in (0, T), \\ u(x, 0) &= u_0(x) \text{ for } x \in \bar{I}, \end{aligned} \right\} \quad (1.1)$$

where  $u_t$  denotes partial differentiation of  $u$  with respect to  $t$  and  $k, p, f$  and  $u_0$  are given functions. In 2010 P. Sawangtong, B. Novaprateep and W. Jumpen [14] studied the degenerate parabolic problem (1.1). In this article we continuous to study the degenerate parabolic problem (1.1) and the purpose of this paper is to prove that before blow-up occurs, there exists a  $T_1(> 0)$  such that problem (1.1) has a unique nonnegative continuous solution  $u$  on

the time interval  $[0, T_1]$  for any  $x \in \bar{I}$ . In addition to prove the existence and uniqueness of solution, the sufficient condition to blow up in finite and the blow-up set of such a solution  $u$  are given. A solution  $u$  of problem (1.1) is said to blows up at  $x = b$  in finite time  $t_b$  if there exists a sequence  $(x_n, t_n)$  with  $t_n < t_b$  such that  $(x_n, t_n) \rightarrow (b, t_b)$  as  $n \rightarrow \infty$  and  $\lim_{n \rightarrow \infty} u(x_n, t_n) = \infty$ . The set of all blow-up points of solution  $u$  is called the blow-up set. In order to obtain our results, throughout this paper, we need following assumptions.

(A)  $p \in C^1(\bar{I})$ ,  $p(0) = 0$ ,  $p$  is positive on  $(0, 1]$  and  $p'$  is nonnegative on  $\bar{I}$ .

(B)  $k \in C(\bar{I})$ ,  $k(0) = 0$ ,  $k$  is positive on  $(0, 1]$ .

(C)  $f \in C^2[0, \infty)$  is convex with  $f(0) = 0$  and  $f(s) > 0$  for  $s > 0$ .

(D)  $u_0 \in C^2(\bar{I})$ ,  $u_0(0) = 0 = u_0(1)$ ,  $u_0$  is nonnegative on  $I$ ,  $u_0(x_0) > 0$ , and  $u_0$  satisfies for any  $x \in I$ ,

$$\frac{d}{dx} \left( p(x) \frac{du_0(x)}{dx} \right) + k(x)f(u_0(x_0)) \geq \zeta k(x)u_0(x) \quad (1.2)$$









2.3 implies that  $z \geq 0$  on  $\bar{Q}_{T_1}$  or  $u \geq u_0$  on  $\bar{Q}_{T_1}$ . Let  $h$  be any positive constant less than  $T$  and  $w(x,t) = u(x,t+h) - u(x,t)$  on  $\bar{Q}_{T_1}$ . Then we have that on  $Q_{T_1}$ ,

$$k(x)w_t - (p(x)w_x)_x = k(x)f(u(x_0,t+h)) - k(x)f(u(x_0,t)) \\ = k(x)f'(\eta_2)w(x_0,t),$$

for  $\eta_2$  between  $u(x_0,t+h)$  and  $u(x_0,t)$ . Furthermore,  $w = 0$  on  $\{0,1\} \times (0,T_1)$  and  $w \geq 0$  on  $\bar{I} \times \{0\}$ . It then follows from lemma 2.3 that  $w \geq 0$  on  $\bar{Q}_{T_1}$ . This shows that  $u_t \geq 0$  on  $\bar{Q}_{T_1}$ .

We note that before blow-up occurs, there exists a positive constant  $M$  such that  $|u(x,t)| \leq M$  for all  $(x,t) \in \bar{Q}_{T_1}$ . Locally Lipschitz continuity of  $f$  yields that there exists a positive constant  $L(M)$  depending on  $M$  such that  $|f(u(x_0,t))| \leq L(M)|u(x_0,t)|$  for any  $t \in [0,T_1]$ .

**Lemma 2.5.** If  $f'(u_0(x_0)) \geq L(M)$ , then  $u_t(x,t) \geq L(M)u(x,t)$  on  $\bar{Q}_{T_1}$ .

**Proof.** Let  $z(x,t) = u_t(x,t) - L(M)u(x,t)$  on  $\bar{Q}_{T_1}$ . We then have that for  $(x,t) \in Q_{T_1}$ ,

$$k(x)z_t - (p(x)z_x)_x = k(x)f'(u(x_0,t))u_t(x_0,t) \\ - k(x)L(M)f(u(x_0,t))$$

Locally Lipschitz continuity of  $f$  implies that for  $(x,t) \in Q_{T_1}$ ,

$$k(x)z_t - (p(x)z_x)_x \\ \geq k(x)f'(u(x_0,t))u_t(x_0,t) - k(x)L^2(M)u(x_0,t) \\ \geq k(x)[f'(u_0(x_0))u_t(x_0,t) - L^2(M)u(x_0,t)] \\ \geq k(x)L(M)z(x_0,t).$$

From lemma 2.4,  $z(0,t) = u_t(0,t) \geq 0$  and  $z(1,t) = u_t(1,t) \geq 0$  for  $t \in (0,T_1)$ . If we set  $\zeta = L(M)$ , then equation (1.2) implies that for any  $x \in I$ ,

$$z(x,0) = \lim_{t \rightarrow 0} u_t(x,t) - L(M)u_0(x) \\ = \frac{1}{k(x)} \left( p(x) \frac{du_0(x)}{dx} \right) + f(u(x_0)) - L(M)u_0(x) \\ \geq 0.$$

Therefore, by lemma 2.3, the proof is complete.

**Lemma 2.6.** If  $u_0(x_0) \geq u_0(x)$  for any  $x \in \bar{I}$ , then  $u(x_0,t) \geq u(x,t)$  on  $\bar{Q}_{T_1}$ .

**Proof.** Let  $z(x,t) = u(x_0,t) - u(x,t)$  on  $\bar{Q}_{T_1}$ . We then have that on  $Q_{T_1}$ , lemma 2.5 yields that

$$k(x)z_t - (p(x)z_x)_x = k(x)[u_t(x_0,t) - f(u(x_0,t))] \\ = k(x)[u_t(x_0,t) - L(M)u(x_0,t)] \\ \geq 0.$$

Since

$z(0,t) = u(x_0,t) - u_0(x) \geq 0$ ,  $z(1,t) = u(x_0,t) - u_0(x) \geq 0$  for  $t \in (0,T_1)$ , and  $z(x,0) = u_0(x_0) - u_0(x) \geq 0$  for any  $x \in \bar{I}$ , by lemma 2.3, the proof of this lemma is complete.

**Theorem 2.2.** Let  $T_{\max}$  be the supremum of all  $T_1$  such that the continuous solution  $u$  of an equivalent integral equation (2.4) exists. If  $T_{\max}$  is finite, then  $u(x_0,t)$  is unbounded as  $t$  tends to  $T_{\max}$ .

**Proof.** Suppose that  $u(x_0,T_{\max})$  is finite. Let  $N = u(x_0,T_{\max}) + 1$ . By theorem 2.1 and a fact that  $u$  is non-decreasing in  $t$ , there exists a finite time  $\tilde{T} (> T_{\max})$  depending on  $N$  such that the equivalent integral equation (2.4) has a unique continuous solution on the time interval  $[0,\tilde{T}]$  for any  $x \in \bar{I}$ . By the definition of  $T_{\max}$ , we get a contradiction.

A proof similar to that of theorem 3 of Chan and Tian [4] gives the following result.

**Theorem 2.3** Such a continuous solution  $u$  of the equivalent integral equation (2.4) is a classical solution.

### 3 A sufficient condition to blow-up in finite time

Let  $\varphi_1$  be the first eigenfunction of a singular eigenvalue problem (1.3) and let  $\lambda_1$  be its corresponding eigenvalue. Without loss of generality we assume

$$\int_0^1 k(x)\varphi_1(x)dx = 1. \tag{3.1}$$

We then define a function  $H$  by

$$H(t) = \int_0^1 k(x)\varphi_1(x)u(x,t)dx.$$

**Theorem 3.1.** Assume that

3.1.1.  $u_0$  attains its maximum at point  $x_0$ .

3.1.2.  $f(\xi) \geq b\xi^p$  with  $b > 0$  and  $p > 1$ .











