Conservative Averaging and Finite Difference Methods for Transient Heat Conduction in 3D Fuse

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Abstract: – Three-dimensional mathematical model of the automotive fuse is considered in this paper. Initially, partial differential equations of the transient heat conduction are given to describe heat-up process in the fuse. Conservative averaging method is used to obtain analytical approximation of these equations by the system of three ordinary differential equations. Finite difference scheme is given if conservative averaging procedure is stopped one step before, i.e., after 1D problem of partial differential equations is obtained.

Key-Words: – Heat conduction, Quasi-linear, Transient process, Three-dimensional, Analytical reduction, Conservative averaging, Finite difference scheme.

1 Introduction

Usually, mathematical modeling of the fuse is implemented by making one dimensional assumptions [1]-[4]. In this paper, we use original method of conservative averaging to transform initial 3D statement of the problem to the statement of new type that consists of three ordinary differential equations. Approximate analytical 3D solution is obtainable from the solution of the transformed problem. Conservative averaging method is theoretically well founded for linear partial differential equations [6]-[12]. Here (as in [13], [14]) we investigate quasi-linear problem.

2 Geometry of the Model

We start with geometric assumptions of the typical car fuse (Fig.1 and Fig.2).

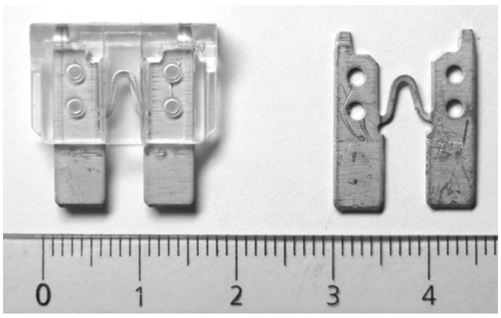

Fig.1. Example of automotive fuse with and without plastic shell

We seemingly straighten out the fuse and use geometry of the model as shown in Fig.3.

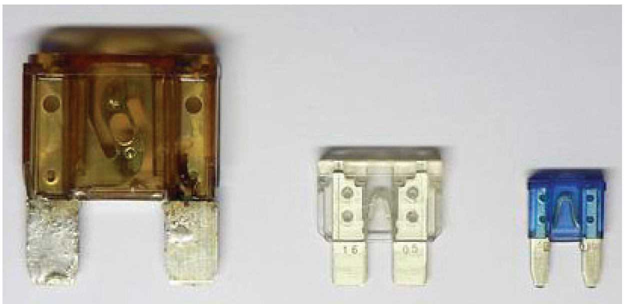

Fig.2

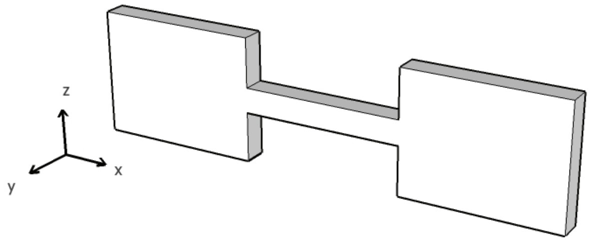

Fig.3

Because of the symmetry, it is enough to use only the shaded part of the model (Fig.4 and Fig.5)

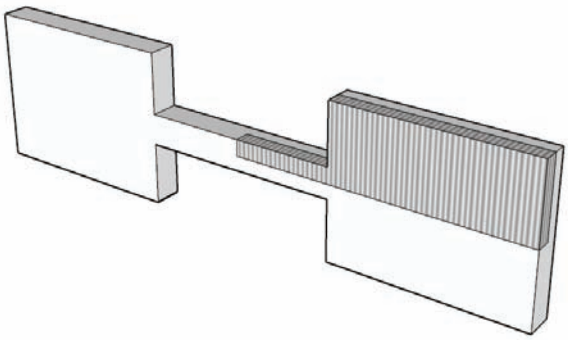

Fig.4

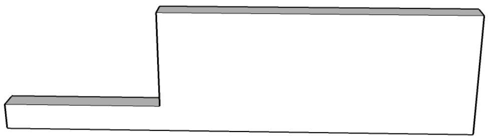

Fig.5

We give brief description of the method in the next chapter, further follow mathematical statement of this problem and usage of the conservative averaging.

3 Short Description of Conservative Averaging Method

Conservative averaging method was developed as approximate analytical and numerical method for solving partial differential equations with piecewise continuous coefficients. The usage of this method for separate relatively thin sub-domain or for sub-domain with large heat conduction coefficient leads to reduction of the domain in which the solution must be found. Method can be applied for several sub-domains simultaneously.

To apply this method for all sub-domains of the layered media, a special type of the spline is constructed: the integral averaged values interpolating parabolic spline. Usage of this spline allows diminishing the dimensions of initial problem per one. It is important that the original R^{n+1} problem with discontinuous coefficients transforms to problem with continuous coefficients in R^n in all cases. More detailed description of the method is given in papers [6]-[13]. Built on concrete steady-state heat conduction example, main idea of the method is given here.

Let us assume that we have domain D that consists of two sub-domains (rectangles) G_0 and G_1 (Fig.6).

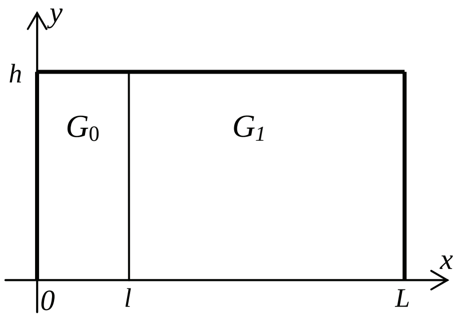

Fig.6

$G_0 = \{(x,y) \mid x \in (0,l), y \in (0,h)\}$,

$G_1 = \{(x,y) \mid x \in (l,L), y \in (0,h)\}$,

$D = G_0 \cup G_1 \cup \{(x,y) \mid x = l, y \in (0,h)\}$.

Objective is to find function $U_0(x,y)$ (continuous in domain $\bar{G}_0$) and function $U_1(x,y)$ (continuous in domain $\bar{G}_1$) that fulfills following equations:

a) differential equations:

$$\frac{\partial}{\partial x}\left(k_0 \frac{\partial U_0}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_0 \frac{\partial U_0}{\partial y}\right) + F_0(x,y) = 0 \qquad (1)$$

$$\frac{\partial}{\partial x}\left(k_1 \frac{\partial U_1}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_1 \frac{\partial U_1}{\partial y}\right) + F_1(x,y) = 0 \qquad (2)$$

b) conjugation conditions at $x = l$:

$$U_0\big|_{x=l} = U_1\big|_{x=l} \qquad (3)$$

$$k_0 \frac{\partial U_0}{\partial x}\bigg|_{x=l-0} = k_1 \frac{\partial U_1}{\partial x}\bigg|_{x=l+0} \qquad (4)$$

c) boundary conditions:

$$\frac{\partial U_0}{\partial x}\bigg|_{x=0} = 0, \qquad (5)$$

$$\frac{\partial U_1}{\partial x}\bigg|_{x=L} = 0, \qquad (6)$$

$$\frac{\partial U_i}{\partial x}\bigg|_{y=0} = 0, \quad i = \overline{0,1} \qquad (7)$$

$$\left[k_i \frac{\partial U_i}{\partial x} - h_y\left(U_i - \Theta\right)\right]\bigg|_{y=h} = 0 \qquad (8)$$

We require that all derivates of the equations (1), (2) are continuous in corresponding sub-domains. Solution of this mathematical problem can be treated as temperature in two layer media for heat transfer process. We assume that all coefficients are constant here. Temperature dependant coefficients are considered in the next chapter where conservative averaging is applied for the model of the fuse.

Let us assume that domain G_0 is thin in *x*-direction or it is made by material that has relatively better heat conductivity than the other one (or both conditions take place). We can obviously assume that temperature is almost constant in *x*-direction then. If this assumption is not the case, we can assume that distribution of the temperature differs from some other curve only slightly, i.e., polynomial or function of the exponential behavior. Therefore, the first thing is to understand in which domain and in which direction the behavior of the unknown function is predictable.

Before we choose specific representation, we introduce integral averaged value function over chosen interval. If we take domain G_0 and interval $x \in [0,l]$ then the definition of this function is

$$u_0(y) = \frac{1}{l}\int_0^l U_0(x,y)dx . \qquad (9)$$

In our case, function u_0 represents averaged temperature in interval $x \in [0,l]$ on given line *y*.

Next, we select function that will approximate our unknown function in chosen domain and segment. It should describe particular physical situation. It means, better view of the situation we have, more appropriate function we can choose. For example, let us use exponential approximation in *x*-direction. General form of the function U_0 then is

$$U_0(x,y) = a(y) + b(y)e^x + c(y)e^{-x} . \qquad (10)$$

Representation of the function U_0 contains unknown functions $a(y)$, $b(y)$, $c(y)$. They are obtained in such way that they fulfill conditions on the boundaries $x=0$, $x=l$ and integral equality (9).

Practically hyperbolic functions could be used instead of exponent. In our case, better form of the exponential approximation is

$$\begin{aligned} U_0(x,y) = a(y) + b(y)\left(\cosh\left(\frac{x}{l}\right) - \sinh(1)\right) + \\ + c(y)\left(\sinh\left(\frac{x}{l}\right) - \cosh(1) + 1\right) \end{aligned} \qquad (11)$$

Taking into account derivative of this function

$$\frac{\partial U_0}{\partial x} = \frac{1}{l}b(y)\sinh\left(\frac{x}{l}\right) + \frac{1}{l}c(y)\cosh\left(\frac{x}{l}\right) \qquad (12)$$

and boundary condition on the border $x=0$ (5), we obtain that $c(y) \equiv 0$. If we apply integral (9) to the formula (11), we obtain:

$a(y) = u_0(y)$.

Conjugation condition (4) on the second boundary $x = l$ of the domain G_0 gives unknown function b:

$$b(y) = \frac{lk_1}{\sinh(1)} \left.\frac{\partial U_1}{\partial x}\right|_{x=l}$$

After unknown functions are found, we can rewrite approximation of the function U_0 in following form:

$$\begin{aligned} U_0(x,y) = u_0(y) + \\ + \frac{lk_1}{\sinh(1)}\left(\cosh\left(\frac{x}{l}\right) - \sinh(1)\right)\left.\frac{\partial U_1}{\partial x}\right|_{x=l} \end{aligned} \qquad (13)$$

The next step of the conservative averaging method is integration of the main differential equation (1) over the interval $x \in [0,l]$:

$$\frac{1}{l}\int_0^l \left[\frac{\partial}{\partial x}\left(k_0 \frac{\partial U_0}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_0 \frac{\partial U_0}{\partial y}\right) + F_0(x,y)\right] dx = 0$$

Let us take a look at the first addend:

$$\frac{1}{l}\int_0^l \frac{\partial}{\partial x}\left(k_0 \frac{\partial U_0}{\partial x}\right) dx =$$

$$= \frac{k_0}{l}\left.\frac{\partial U_0}{\partial x}\right|_{x=l} - \frac{k_0}{l}\left.\frac{\partial U_0}{\partial x}\right|_{x=0} = \frac{k_1}{l}\left.\frac{\partial U_1}{\partial x}\right|_{x=l} - 0$$

We used conditions (4), (5) on the borders of the domain G_0 to make analytical transformations here. On the other hand we can use representation (13) and it also leads to the same result.

Order of the integration and derivation is swapped for the other derivatives in the integral. Integral formula (9) is used after that:

$$\frac{1}{l}\int_0^l \frac{\partial}{\partial y}\left(k_0 \frac{\partial U_0}{\partial y}\right) dx =$$

$$= \frac{\partial}{\partial y} k_0 \frac{\partial}{\partial y} \frac{1}{l}\int_0^l U_0 dx = \frac{d}{dy}\left(k_0 \frac{du_0}{dy}\right)$$

Consequently, differential equation for the unknown averaged value function $u_0(y)$ is

$$\frac{d}{dy}\left(k_0 \frac{du_0}{dy}\right) + \frac{k_1}{l}\left.\frac{\partial U_1}{\partial x}\right|_{x=l} + f(y) = 0 \qquad (14)$$

where $f(y)$ is averaged value of the source function:

$$f(y) := \frac{1}{l}\int_0^l F_0(x,y)dx$$

We have transformed initial problem to the new one. Differential equation for the function $U_1(x,y)$ in the domain G_1 remains the same (2). The second differential equation (14) is for the averaged value function $u_0(y)$ in interval $y \in [0,h]$.

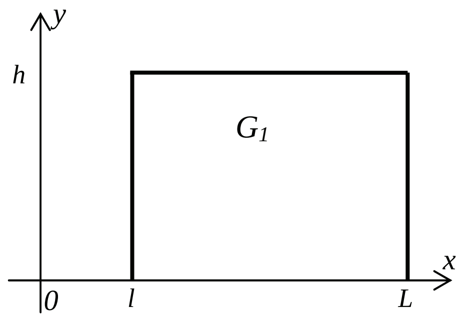

Fig.7

If we take into account representation of the averaged function (13), the conjugation condition (3) gives condition between functions u_0 and U_1:

$$u_0 = U_1\big|_{x=l} + lk_1\left(1-\tanh(1)\right)\frac{\partial U_1}{\partial x}\bigg|_{x=l} \quad (15)$$

This equation together with the equation (14) could be considered as non-classical boundary conditions for the border $x = l$ in domain G_1.

Boundary conditions when $y = 0$ and $y = h$ for the function u_0 are similar as for the function U_0:

$$\frac{du_0}{dy}\bigg|_{y=0} = 0 \quad (16)$$

$$\left(k_0\frac{du_0}{dy} + h_y\left(u_0 - \Theta\right)\right)\bigg|_{y=h} = 0 \quad (17)$$

To be accurate, these equalities are obtained after integral is applied to boundary conditions (7), (8). Transformations are similar to those that were done for the main differential equation (1).

Boundary conditions (6), (7), (8) remain the same for the function $U_1(x, y)$ of the domain G_1.

We have transformed original problem and reduced dimension of one domain. Usually, it is impossible to find analytical solution. The only choice is to solve it numerically. It takes less computer power to calculate such mathematical problem because of reduced dimension.

It is possible to reconstruct temperature distribution $U_0(x, y)$ in domain G_0 from the representation (13) after functions $u_0(y)$ and $U_1(x, y)$ are calculated. Note that it is possible to get value at any point of all interval - not only in some discrete points as it would be after applying finite difference method to initial problem.

Mathematical problem is reduced by one dimension for whole system if conservative averaging method is applied for the domain G_1 in *x*-direction over interval $x \in [l, L]$. Averaging procedure can be applied continuously in several directions, reducing dimensions of the problem one by one. Numerical calculations are also reduced by order.

Let us look back to the original problem (1)-(8). For both domains, conservative averaging could be applied in *y*-direction at first (Fig.8).

Fig.8

Boundary conditions at $y = 0$ and $y = h$ transfer to the new differential equations then. If constant approximation in *y*-direction is used

$U_0(x, y) = u_0(y)$, $U_1(x, y) = u_1(y)$

then differential equations for thin wire with convection is obtained:

$$\frac{d}{dx}\left(k_i\frac{du_i}{dx}\right) - \frac{h_y}{h}\left(u_i - \Theta\right) = -f_i \quad (18)$$

$f_i(x) = F_i(x, y)$, $i = \overline{0,1}$

Remaining boundary and conjugation conditions actually stay the same:

$$\frac{du_0}{dx}\bigg|_{x=0} = 0, \quad \frac{du_1}{dx}\bigg|_{x=L} = 0, \quad (19)$$

$$u_0\big|_{x=l} = u_1\big|_{x=l}, \quad k_0\frac{du_0}{dx}\bigg|_{x=l-0} = k_1\frac{du_1}{dx}\bigg|_{x=l+0} \quad (20)$$

We gave main steps of the conservative averaging method as a summary of this section. First, choose function of the approximation. Second, integrate main differential equation. Third, use boundary and conjugation conditions.

4 Original Problem and its Approximation by Conservative Averaging Method

4.1 Mathematical Statement of the Initial Problem

We continue with accurate formulation of the three-dimensional mathematical model of the transient heat conduction problem for fuse.

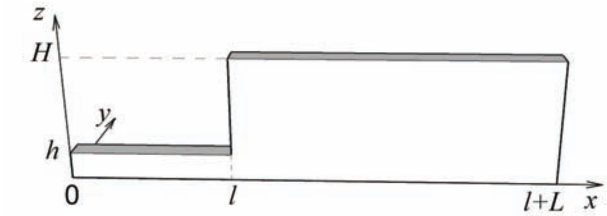

Fig.9

Let us treat main domain (Fig.9) as two connected sub-domains G_0 and G_1:

$G_0 = \left\{(x, y, z) \mid x \in [0, l], y \in [0, b], z \in [0, h]\right\}$,

$G_1 = \left\{(x, y, z) \mid x \in [l, l + L], y \in [0, b], z \in [0, H]\right\}$.

If temperature in domain G_i is denoted as function $U_i(x, y, z, t)$, then differential equation for the heat transfer is

$$\frac{\partial}{\partial t}\left(\gamma U_i\right) = \frac{\partial}{\partial x}\left(k\frac{\partial U_i}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial U_i}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial U_i}{\partial z}\right) + F_i(x, y, z, t, U_i), \quad (21)$$

$(x, y, z) \in G_i$, $t > 0$, $i = \overline{0,1}$.

Source function F_i (heat produced by electrical current) can be approximated with linear function:

$$F_i(x,y,z,t,U_i)=B_i\left(1+\alpha\left(U_i-U_r\right)\right), \qquad (22)$$

where $B_0=\dfrac{\rho_{ref}I^2}{h^2b^2}$, $B_1=\dfrac{\rho_{ref}I^2}{H^2b^2}$.

Parameter ρ_{ref} is resistivity of the material at the reference temperature U_r, α is temperature coefficient at the same reference temperature; I – electrical current. Heat conductivity k and heat capacity (per volume) γ depend on temperature.

Besides main equations (21), we add symmetry conditions:

$$\left.\frac{\partial U_0}{\partial x}\right|_{x=0}=0,\quad \left.\frac{\partial U_0}{\partial y}\right|_{y=0}=0,\quad \left.\frac{\partial U_0}{\partial z}\right|_{z=0}=0, \qquad (23)$$

$$\left.\frac{\partial U_1}{\partial x}\right|_{x=l+L}=0,\quad \left.\frac{\partial U_1}{\partial y}\right|_{y=0}=0,\quad \left.\frac{\partial U_1}{\partial z}\right|_{z=0}=0 \qquad (24)$$

and heat exchange conditions on outer surfaces

$$\left.\left(k\frac{\partial U_0}{\partial y}+h_y\left(U_0-\Theta\right)\right)\right|_{y=b}=0,$$
$$\left.\left(k\frac{\partial U_1}{\partial y}+h_y\left(U_1-\Theta\right)\right)\right|_{y=b}=0, \qquad (25)$$

$$\left.\left(k\frac{\partial U_0}{\partial z}+h_z\left(U_0-\Theta\right)\right)\right|_{z=h}=0,$$
$$\left.\left(k\frac{\partial U_1}{\partial z}+h_z\left(U_1-\Theta\right)\right)\right|_{z=H}=0, \qquad (26)$$

$$\left.\left(-k\frac{\partial U_1}{\partial x}+h_z\left(U_1-\Theta\right)\right)\right|_{x=l+0,z\in[h,H]}=0, \qquad (27)$$

where $\Theta=\Theta(t)$ is temperature of the environment, but h_y,h_z are heat convection coefficients for surfaces in corresponding direction that also depend on temperature.

We also add conjugation conditions, i.e. continuity of the temperature and heat fluxes between both parts of the fuse:

$$U_0\big|_{x=l-0}=U_1\big|_{x=l+0},\quad \left.\frac{\partial U_0}{\partial x}\right|_{x=l-0}=\left.\frac{\partial U_1}{\partial x}\right|_{x=l+0}, \qquad (28)$$

$y\in[0,b]$, $z\in[0,h]$.

Finally we add initial conditions:

$$U_0\big|_{t=0}=U_1\big|_{t=0}=U^0=const. \qquad (29)$$

4.2 Conservative Averaging in *y*-direction

We introduce the integral average value of the functions $U_i(x,y,z,t)$ in the *y*-direction:

$$V_i(x,z,t)=\frac{1}{b}\int_0^b U_i(x,y,z,t)dy. \qquad (30)$$

In praxis, firstly, the thickness b is very small in comparison with the width of the fuse. Secondly, material of the fuse (metal) has high heat conductivity coefficient. These features allow us to use the simplest form of the conservative averaging method – approximation by the constant. Detailed procedure of the analytical transformations is given in previous section. Shortly, we integrate main equation (21) over the segment $y\in[0,b]$ and then we use boundary conditions (25) and linear representation of the source function (22). Finally, we take into account integral equality (30) and obtain:

$$\frac{\partial}{\partial t}\left(\gamma V_i\right)=\frac{\partial}{\partial x}\left(k\frac{\partial V_i}{\partial x}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial V_i}{\partial z}\right)-$$
$$-\frac{h_y}{b}\left(V_i-\Theta\right)+B_i\left(1+\alpha\left(V_i-U_r\right)\right),\qquad i=\overline{0,1}. \qquad (31)$$

Because of the linearity, the additional boundary conditions (BC) of new problem are the same as in the statement of the original problem (21)-(29):

$$\left.\frac{\partial V_0}{\partial x}\right|_{x=0}=\left.\frac{\partial V_1}{\partial x}\right|_{x=l+L}=0,\quad \left.\frac{\partial V_i}{\partial z}\right|_{z=0}=0,\quad i=\overline{0,1} \qquad (32)$$

$$\left.\left(k\frac{\partial V_0}{\partial z}+h_z\left(V_0-\Theta\right)\right)\right|_{z=h}=0,$$
$$\left.\left(k\frac{\partial V_1}{\partial z}+h_z\left(V_1-\Theta\right)\right)\right|_{z=H}=0, \qquad (33)$$

$$\left.\left(-k\frac{\partial V_1}{\partial x}+h_z\left(V_1-\Theta\right)\right)\right|_{x=l+0,z\in[h,H]}=0, \qquad (34)$$

We also add conjugation conditions at $z\in[0,h]$:

$$V_0\big|_{x=l-0}=V_1\big|_{x=l+0},\quad \left.\frac{\partial V_0}{\partial x}\right|_{x=l-0}=\left.\frac{\partial V_1}{\partial x}\right|_{x=l+0}, \qquad (35)$$

and initial conditions:

$$V_0\big|_{t=0}=V_1\big|_{t=0}=U^0=const. \qquad (36)$$

4.3 Conservative Averaging in *x*-direction

As the next step, we will make conservative averaging in the *x*-direction. We define one averaged value function over domain G_0 and two separate functions for the domain G_1 – the first for interval $z\in(0,h)$ and the second for interval

$z \in (h, H)$ because of different conditions on the line $x = l$:

$$W_0(z,t) = \frac{1}{l}\int_0^l V_0(x,z,t)dx, \ \ z \in (0,h),$$

$$W_1(z,t) = \frac{1}{L}\int_l^{l+L} V_1(x,z,t)dx, \ \ z \in (0,h), \tag{37}$$

$$W_2(z,t) = \frac{1}{L}\int_l^{l+L} V_1(x,z,t)dx, \ \ z \in (h,H).$$

In this case, we use exponential approximation in the following form:

$$V_0(x,z,t) = W_0(z,t) + p_0(z,t) \times \times \left[\cosh\left(\frac{x}{l}\right) - \sinh(1)\right], \tag{38}$$

$$V_1(x,z,t) = W_i(z,t) + p_i(z,t) \times \times \left[\cosh\left(\frac{x-l-L}{L}\right) - \sinh(1)\right], \quad i = \overline{1,2} \tag{39}$$

Equalities (38), (39) are chosen in such way that they fulfill integral equalities (37) (conservation of the heat energy) and BC (32) at $x = 0$ and $x = l + L$. We use conjugation conditions (35) to find unknown functions p_0, p_1 and afterwards obtain functions V_0, V_1:

$$V_0(x,z,t) = W_0(z,t) - C_0 l\left(W_0(z,t) - W_1(z,t)\right) \times \times \left[\cosh\left(\frac{x}{l}\right) - \sinh(1)\right], \quad C_0 = \frac{e}{l+L}, \tag{40}$$

$$V_1(x,z,t) = W_1(z,t) + C_0 L\left(W_0(z,t) - W_1(z,t)\right) \times \times \left[\cosh\left(\frac{x-l-L}{L}\right) - \sinh(1)\right], \quad z \in (0,h) \tag{41}$$

We find function p_2 and representation of function V_1 in interval $z \in (h, H)$ from expression (39) by means of BC (34):

$$V_1(x,z,t) = W_2(z,t) - C_1\left(W_2(z,t) - \Theta(t)\right) \times \times \left[\cosh\left(\frac{x-l-L}{L}\right) - \sinh(1)\right], \quad z \in (h,H) \tag{42}$$

$$C_1 = \frac{2eLh_z}{k\left(e^2 - 1\right) + 2Lh_z}.$$

Discontinuity for the temperature field could appear on the line $z = h$. This kind of discontinuities was considered in papers [13], [14].

We integrate differential equations (31) on the first step of averaging in order to obtain equations for the second step. We use representations (40), (41), (42) of the functions V_0, V_1 and integral equalities (37) to make approximate analytical reduction of 2D system to 1D system of partial differential equations.

$$\frac{\partial}{\partial t}(\bar{\gamma} W_0) = \frac{\partial}{\partial z}\left(k\frac{\partial W_0}{\partial z}\right) + \frac{D_0}{l}\left(W_1 - W_0\right) - -\frac{h_y}{b}\left(W_0 - \Theta\right) + B_0\left(1 + \alpha\left(W_0 - U_r\right)\right), \tag{43}$$

$$\frac{\partial}{\partial t}(\bar{\bar{\gamma}} W_1) = \frac{\partial}{\partial z}\left(k\frac{\partial W_1}{\partial z}\right) + \frac{D_0}{L}\left(W_0 - W_1\right) - -\frac{h_y}{b}\left(W_1 - \Theta\right) + B_1\left(1 + \alpha\left(W_1 - U_r\right)\right), \tag{44}$$

$$\frac{\partial}{\partial t}\left(\bar{\bar{\bar{\gamma}}} W_2\right) = \frac{\partial}{\partial z}\left(k\frac{\partial W_2}{\partial z}\right) - D_1\left(W_2 - \Theta\right) - -\frac{h_y}{b}\left(W_2 - \Theta\right) + B_1\left(1 + \alpha\left(W_2 - U_r\right)\right), \tag{45}$$

$$D_0 = C_0 k \sinh(1), \ \ D_1 = \frac{C_1 k \sinh(1)}{L^2}.$$

As we mentioned in the introduction, we consider the quasi-linear problem here. This problem significantly differs from the problem considered in our papers [10], [11]. We have made the averaging procedure over the sub-domain with linear differential equation in the earlier statement of the problem. Therefore we will explain deeper the averaging procedure for the left hand side of the equation (43) (procedure for the equations (44), (45) can be realized in the same way). In this paper, we use *enthalpy* form of the heat equation (see, e.g. [5], chapter 7). This form is substantially more suitable for the use of the mean value theorem:

$$\frac{1}{h}\int_0^h \frac{\partial}{\partial t}\left[\gamma(V_0)V_0\right]dz = \frac{\partial}{\partial t}\left[\gamma(\bar{V}_0)\frac{1}{h}\int_0^h V_o dz\right] = = \frac{\partial}{\partial t}\left[\bar{\gamma} W_0\right], \qquad \bar{\gamma} = \gamma(\bar{V}_0), \ \ \bar{V}_0 = V_0\left(\bar{x}, z, t\right).$$

It is possible to choose the mean value more or less freely. We propose to use the corresponding middle point, i.e

. $\bar{x} = l/2$, $\bar{\bar{x}} = \bar{\bar{\bar{x}}} = L/2$, or averaged temperature: $\bar{\gamma} = \gamma(W_0)$, $\bar{\bar{\gamma}} = \gamma(W_1)$, $\bar{\bar{\bar{\gamma}}} = \gamma(W_2)$.

Again, boundary and initial conditions are the same as in the original problem because of the linearity:

$$\left.\frac{\partial W_0}{\partial z}\right|_{z=0} = 0, \ \ \left.\frac{\partial W_1}{\partial z}\right|_{z=0} = 0, \tag{46}$$

$$\left.\left(k\frac{\partial W_0}{\partial z}+h_z\left(W_0-\Theta\right)\right)\right|_{z=h}=0,$$
$$\left.\left(k\frac{\partial W_2}{\partial z}+h_z\left(W_2-\Theta\right)\right)\right|_{z=H}=0, \tag{47}$$

$$W_0\big|_{t=0}=W_1\big|_{t=0}=W_2\big|_{t=0}=U^0=const. \tag{48}$$

We also ask for continuity of the averaged temperature and fluxes on the line $z=h$. That gives additional conjugation conditions:

$$W_1\big|_{z=h-0}=W_2\big|_{z=h+0},\quad \left.\frac{\partial W_1}{\partial z}\right|_{z=h-0}=\left.\frac{\partial W_2}{\partial z}\right|_{z=h+0} \tag{49}$$

4.4 Conservative Averaging in *z*-direction

Finally, we will make conservative averaging procedure in the *z*-direction. We introduce three new functions for this purpose:

$$u_0(t)=\frac{1}{h}\int_0^h W_0(z,t)dz,$$
$$u_1(t)=\frac{1}{h}\int_0^h W_1(z,t)dz, \tag{50}$$
$$u_2(t)=\frac{1}{H-h}\int_h^H W_2(z,t)dz.$$

We use exponential approximation in the form

$$W_0(z,t)=u_0(t)+q_0(t)\left[\cosh\left(\frac{z}{h}\right)-\sinh(1)\right],$$
$$W_1(z,t)=u_1(t)+q_1(t)\left[\cosh\left(\frac{z}{h}\right)-\sinh(1)\right], \tag{51}$$
$$W_2(z,t)=u_2(t)+q_2(t)\left[\cosh\left(\frac{z-h}{H-h}\right)-\sinh(1)\right]+$$
$$+q_3(t)\left[\sinh\left(\frac{z-h}{H-h}\right)-\cosh(1)+1\right].$$

We fulfill the integral equalities (conservation of the heat energy (50)) and the symmetry conditions (46) at $z=0$ by this representation. Using BC (47) and conjugation conditions (49) we can find four unknown parameters in the representation (51). This gives:

$$W_0(z,t)=u_0+e_0\left(u_0-\Theta\right)\left[\cosh\left(\frac{z}{h}\right)-\sinh(1)\right],$$
$$W_1(z,t)=u_1+$$
$$+\left[e_1\left(u_1-u_2\right)+e_2\left(u_2-\Theta\right)\right]\left[\cosh\left(\frac{z}{h}\right)-\sinh(1)\right], \tag{52}$$
$$W_2(z,t)=u_2+$$
$$+\left[e_3\left(u_1-u_2\right)+e_4\left(u_2-\Theta\right)\right]\left[\cosh\left(\frac{z-h}{H-h}\right)-\sinh(1)\right]+$$
$$+\left[e_5\left(u_1-u_2\right)+e_6\left(u_2-\Theta\right)\right]\left[\sinh\left(\frac{z-h}{H-h}\right)-\cosh(1)+1\right],$$
$$u_0=u_0(t),\quad u_1=u_1(t),\quad u_2=u_2(t),$$

where constants e_i are:

$$e_0=\frac{-2ehh_z}{k(e^2-1)+2hh_z},$$
$$e_1=-eh\left(k(e^2-1)+2h_z(H-h)\right)/e_7,$$
$$e_2=2ehh_z(H-h)(e^2-2e-1)/e_7,$$
$$e_3=(H-h)(e^2-1)\left(k(e^2+1)+2h_z(e-1)(H-h)\right)/2e_7$$
$$e_4=-h_z(H-h)\left(2eh+(e^2-1)(e-1)^2(H-h)\right)/e_7$$
$$e_5=-(H-h)(e^2-1)\left(k(e^2+1)+2h_z(e-1)\right)/2e_7$$
$$e_6=h_z(H-h)^2(e^2-1)(e^2-2e-1)/e_7$$
$$e_7=k(e^2-1)(2H-h)+$$
$$+h_z(H-h)\left(2h-(e^2-1)(e-3)(e-1)(H-h)\right)$$

After integration of equations (43)-(45), we finally obtain system of ordinary differential equations:

$$\frac{d}{dt}(\tilde{\gamma}u_0)=\frac{D_0}{l}\left(u_1-u_0\right)-E_0\left(u_0-\Theta\right)+$$
$$+B_0\left(1+\alpha\left(u_0-U_r\right)\right), \tag{53}$$

$$\frac{d}{dt}(\tilde{\tilde{\gamma}}u_1)=\frac{D_0}{L}\left(u_0-u_1\right)-\frac{h_y}{b}\left(u_1-\Theta\right)+$$
$$+E_1\left[e_1\left(u_1-u_2\right)+e_2\left(u_2-\Theta\right)\right]+ \tag{54}$$
$$+B_1\left(1+\alpha\left(u_1-U_r\right)\right),$$

$$\frac{d}{dt}(\hat{\gamma}u_2)=-E_2\left(u_1-u_2\right)-$$
$$-\left[E_3+D_1+\frac{h_y}{b}\right]\left(u_2-\Theta\right)+ \tag{55}$$
$$+B_1\left(1+\alpha\left(u_2-U_r\right)\right),$$

Here, constants E_0, E_1, E_2, E_3 and coefficients γ:

$$E_0=\frac{h_y}{b}+\frac{h_z}{h}\left(1+\frac{e_0}{e}\right),$$
$$E_1=\frac{k\sinh(1)}{h^2},$$
$$E_2=\frac{k\left(e_3\sinh(1)+e_5(\cosh(1)-1)\right)}{\left(H-h\right)^2},$$

$$E_3 = \frac{k\left(e_4 \sinh(1) + e_6(\cosh(1)-1)\right)}{\left(H-h\right)^2}$$

$\tilde{\gamma} = \overline{\gamma}\left(W_0(\tilde{z},t)\right)$, $\tilde{\tilde{\gamma}} = \overline{\overline{\gamma}}\left(W_1(\tilde{z},t)\right)$, $\hat{\gamma} = \overline{\overline{\overline{\gamma}}}\left(W_1(\hat{z},t)\right)$,

$\tilde{z} = h/2$, $\hat{z} = (H+h)/2$

or $\tilde{\gamma} = \overline{\gamma}\left(u_0(t)\right)$, $\tilde{\tilde{\gamma}} = \overline{\overline{\gamma}}\left(u_1(t)\right)$, $\hat{\gamma} = \overline{\overline{\overline{\gamma}}}\left(u_2(t)\right)$,

This system of three ordinary differential equations must be supplemented with initial conditions:

$$u_0\big|_{t=0} = u_1\big|_{t=0} = U^0. \tag{56}$$

4.5 Simplified Averaged System of Ordinary Differential Equations

The main goal of this mathematical model is to predict time before melting of the material in the thinnest sub-domain G_0 caused by inadmissible strong current. According to expression (22), density of the electrical current is H^2 / h^2 times bigger in this sub-domain. This reason allows us to propose another model besides the first one. As the second step of the averaging, we use the simplest approximation in the z-direction – approximation by constant.

We introduce averaged values:

$$\begin{aligned} w_0(x,t) &= \frac{1}{h}\int_0^h V_0(x,z,t)dz, \\ w_1(x,t) &= \frac{1}{H}\int_0^H V_1(x,z,t)dz. \end{aligned} \tag{57}$$

We assume that temperature is constant in z-direction because it changes only slightly in comparison with x-direction:

$$\begin{aligned} w_0(x,t) &= V_0(x,z,t), \\ w_1(x,t) &= V_1(x,z,t). \end{aligned} \tag{58}$$

Integration of the differential equations (11) immediately gives system of two 1D partial differential equations:

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\gamma(w_0)w_0\right) = \frac{\partial}{\partial x}\left(k\frac{\partial w_0}{\partial x}\right) - \\ &-\left(\frac{h_y}{b}+\frac{h_z}{h}\right)\left(w_0-\Theta\right) + B_0\left(1+\alpha\left(w_0-U_r\right)\right), \\ &\frac{\partial}{\partial t}\left(\gamma(w_1)w_1\right) = \frac{\partial}{\partial x}\left(k\frac{\partial w_1}{\partial x}\right) - \\ &-\left(\frac{h_y}{b}+\frac{h_z}{H}\right)\left(w_1-\Theta\right) + B_1\left(1+\alpha\left(w_1-U_r\right)\right). \end{aligned} \tag{59}$$

Boundary conditions remain the same:

$$\left.\frac{\partial w_0}{\partial x}\right|_{x=0} = \left.\frac{\partial w_1}{\partial x}\right|_{x=l+L} = 0. \tag{60}$$

The second conjugation condition changes substantially because of convective heat losses over the surface $\{x = l, z \in [h, h+H]\}$:

$$\begin{aligned} &w_0\big|_{x=l-0} = w_1\big|_{x=l+0}, \\ &hk\left.\frac{\partial w_0}{\partial x}\right|_{x=l-0} = \left[Hk\frac{\partial w_1}{\partial x} - h_z\left(H-h\right)\left(w_1-\Theta\right)\right]\bigg|_{x=l+0}. \end{aligned} \tag{61}$$

Integration of the boundary condition and conjugation conditions was made to obtain previous equation. By the way, such type of the second conjugation condition was used in paper [4]. The initial conditions remain the same:

$$w_0\big|_{t=0} = w_1\big|_{t=0} = U^0. \tag{62}$$

As the last step, we will apply the conservative averaging method in x-direction. We will use exponential approximation as the form used earlier:

$$\begin{aligned} w_0(x,t) &= u_0(t) + p_0(t)\left[\cosh\left(\frac{x}{l}\right) - \sinh(1)\right], \\ w_1(x,t) &= u_1(t) + p_1(t)\left[\cosh\left(\frac{x-l-L}{L}\right) - \sinh(1)\right] \end{aligned} \tag{63}$$

We have introduced the average integral values again:

$$\begin{aligned} u_0(t) &= \frac{1}{l}\int_0^l w_0(x,t)dx, \\ u_1(t) &= \frac{1}{L}\int_l^{l+L} w_1(x,t)dx. \end{aligned} \tag{64}$$

We obtain parameters $p_0(t)$, $p_1(t)$ of the representations (63) from the conjugation conditions (61):

$$p_1(t) = e\left[u_0(t) - u_1(t)\right] + p_0(t),$$

$$p_0(t) = e\frac{g_1\left(u_1-u_0\right) - g_2\left(u_0-\Theta\right)}{g_3},$$

i.e.,

$$p_1(t) = e\frac{g_0\left(u_0-u_1\right) - g_2\left(u_1-\Theta\right)}{g_3}.$$

Here

$$g_0 = k\frac{h}{l}\left(e^2-1\right), \quad g_1 = k\frac{H}{L}\left(e^2-1\right),$$

$$g_2 = 2h_z\left(H-h\right), \quad g_3 = g_0 + g_1 + g_2.$$

Finally, we integrate partial differential equations (59) and obtain system of ordinary differential equations:

$$\frac{d}{dt}(\bar{\gamma}u_0) = \frac{G}{l^2}\left[(g_1+g_2)(u_1-u_0) - g_2(u_1-\Theta)\right] - \\ -\left(\frac{h_y}{b}+\frac{h_z}{h}\right)(u_0-\Theta) + B_0\left(1+\alpha\left(u_0-U_r\right)\right), \quad (65)$$

$$\frac{d}{dt}(\bar{\bar{\gamma}}u_1) = \frac{G}{L^2}\left[g_0(u_0-u_1) - g_2(u_1-\Theta)\right] - \\ -\left(\frac{h_y}{b}+\frac{h_z}{H}\right)(u_1-\Theta) + B_1\left(1+\alpha\left(u_1-U_r\right)\right). \quad (66)$$

Here

$\bar{\gamma} = \gamma\left(w_0(\bar{x},t)\right)$ or $\bar{\gamma} = \gamma\left(u_0(t)\right)$, $\bar{x} = l/2$,

$\bar{\bar{\gamma}} = \gamma\left(w_1(\bar{\bar{x}},t)\right)$ or $\bar{\bar{\gamma}} = \gamma\left(u_1(t)\right)$, $\bar{x} = l/2$,

$$G = \frac{k\left(e^2-1\right)}{2g_3}.$$

It remains to add the initial conditions for the completeness of the full statement of the 0-D problem:

$$u_0\big|_{t=0} = u_1\big|_{t=0} = U^0. \quad (67)$$

5 Finite Difference Method for 1D Problem

To improve the accuracy of the conservative averaging method, we can use the finite difference method for the numerical approximation of the system of two 1D heat equations (59).

5.1 The Statement of the 1D Problem

The system of two quasi-linear 1D heat equations has the form:

$$\frac{\partial}{\partial t}\left(\gamma(w_0)w_0\right) = \frac{\partial}{\partial x}\left(k\frac{\partial w_0}{\partial x}\right) + f_0(x,t,w_0), \\ \frac{\partial}{\partial t}\left(\gamma(w_1)w_1\right) = \frac{\partial}{\partial x}\left(k\frac{\partial w_1}{\partial x}\right) + f_1(x,t,w_1). \quad (68)$$

Here, source functions $f_i(x,t,w_i)$, $i=\overline{0,1}$ can play the role of the heat sources or heat sinks depending on values of the first or the second term in expressions (69):

$$f_0(x,t,w_0) = B_0\left(1+\alpha\left(w_0-U_r\right)\right) - \\ -\left(\frac{h_y}{b}+\frac{h_z}{h}\right)\left(w_0-\Theta\right), \\ f_1(x,t,w_1) = B_1\left(1+\alpha\left(w_1-U_r\right)\right) - \\ -\left(\frac{h_y}{b}+\frac{h_z}{H}\right)\left(w_1-\Theta\right) \quad (69)$$

Boundary conditions are taken as the generalization of the homogeneous boundary conditions (40):

$$\left.\frac{\partial w_0}{\partial x}\right|_{x=0} = q_0(t), \quad \left.\frac{\partial w_1}{\partial x}\right|_{x=l+L} = q_1(t). \quad (70)$$

Conjugation conditions differ from the ideal thermal contact conditions as well as from non-ideal thermal contact conditions:

$$w_0\big|_{x=l-0} = w_1\big|_{x=l+0}, \\ hk\left.\frac{\partial w_0}{\partial x}\right|_{x=l-0} = \\ \left.\left[Hk\frac{\partial w_1}{\partial x} - h_z\left(H-h\right)\left(w_1-\Theta\right)\right]\right|_{x=l+0}. \quad (71)$$

Initial conditions are non-homogeneous:

$$w_0\big|_{t=0} = w_1\big|_{t=0} = U^0(x). \quad (72)$$

From mathematical point of view, important is a fact that the functions $f_i(x,t,w_i)$ (as well as function $\gamma(w_i)$) fulfill following estimations:

$$\left|\frac{\partial f_j(x,t,w_j)}{\partial w_j}\right| \le M, \quad j=\overline{0,1}, \quad \left|\frac{\partial \gamma}{\partial w_j}\right| \le M.$$

These constraints guarantee the uniqueness of the solution of the problem (68)-(72).

5.2 The Construction of the Finite Difference Scheme

The finite difference method for heat transfer problems are well explained in literature, e.g. [18]. The finite difference solution of the 1D problem will be denoted as $v_{j,i}^n \approx w_j(x_i,t_n)$. We will use uniform time step: $t_n = n\tau$, $n=\overline{0,N}$; the space step will be piece-wise constant:

$$x_i = i\Delta x_0, \quad i=\overline{0,i_0}, \quad i_0 = \frac{l}{\Delta x_0};$$

$$x_i = l + (I-i_0)\Delta x_1, \quad i=\overline{i_0,I}, \quad I = i_0 + \frac{L-l}{\Delta x_1}.$$

We approximate heat conductivity term in the following way (temporarily we will omit the notation of the time-level):

$$\Lambda v_i = [a(x,v)v_{\bar{x}}]_{x,i}, \quad i \ne 0, i_0, I. \quad (73)$$

Here we have used traditional notations, e.g. [18]:

$$v_{\bar{x},i} = \frac{v_i - v_{i-1}}{\Delta x}, \quad v_{x,i} = \frac{v_{i+1}-v_i}{\Delta x},$$

where

$\Delta x = \Delta x_0$, $0 < i < i_0$; $\Delta x = \Delta x_1$, $i_0 < i < I$.

For coefficient $a(x,v)$ in (73), several equivalent expressions in the sense of the order of approximation $O(h^2)$ can be applied, e.g.

$$a(x,v)_i = k\left(x_i - \frac{\Delta x}{2}, \frac{v_i + v_{i-1}}{2}\right) \tag{74}$$

or

$$a(x,v)_i = \frac{k\left(x_i - \Delta x, v_{i-1}\right) + k(x_i, v_i)}{2}. \tag{75}$$

Now, we can propose two-step *predictor-corrector-type* finite difference scheme for the differential equations (68) (it is important to show the time level here, but notation j of the sub-segment may be omitted):

$$\begin{gathered}\gamma(v_i^n)\frac{\tilde{v}_i^{n+1} - v_i^n}{\tau} = [a(x,v^n)v_{\bar{x}}^n]_{x,i} + f(x,t,v)_i^n,\\ \gamma(\tilde{v}_i^n)\frac{v_i^{n+1} - v_i^n}{\tau} = \\ [a(x,\tilde{v}^{n+1})v_{\bar{x}}^{n+1}]_{x,i} + f(x,t,\tilde{v})_i^{n+1};\end{gathered} \tag{76}$$

$$0 < i < i_0,\ i_0 < i < I\ \left(i \neq 0, i \neq i_0, i \neq I\right).$$

Now we will pay special attention to obtain approximation of the boundary conditions (70) and conjugations conditions (71), (72) with the same order of approximation $O(\Delta x^2)$ as the finite difference equations (76). To guaranty the second order of the approximation, we employ the idea of use of main differential equation on the border [19]. We start with Taylor series expansions for the functions $v_{j,i}^n$ as differentiable functions of arguments x,t:

$$k_i v_{i\pm 1} = k_i v_i \pm \Delta x\, k_i \left.\frac{\partial v}{\partial x}\right|_i + \frac{\Delta x^2}{2} k_i \left.\frac{\partial^2 v}{\partial x^2}\right|_i + O(\Delta x^3).$$

We draw the reader's attention to following nuance: the heat conductivity coefficient k is taken in the fixed point $x = x_i$. This assumption allows us to rewrite the last formula in the form:

$$k_i v_{i\pm 1} = k_i v_i \pm \Delta x\, k_i \left.\frac{\partial v}{\partial x}\right|_i + \frac{\Delta x^2}{2}\frac{\partial}{\partial x}\left.\left(k_i \frac{\partial v}{\partial x}\right)\right|_i + O(\Delta x^3). \tag{77}$$

Next two equalities follow from (77):

$$k_i \left.\frac{\partial v}{\partial x}\right|_i = k_i \frac{v_i - v_{i-1}}{\Delta x} + \frac{\Delta x}{2}\frac{\partial}{\partial x}\left.\left(k\frac{\partial v}{\partial x}\right)\right|_i + O(\Delta x^2),$$

$$k_i \left.\frac{\partial v}{\partial x}\right|_i = k_i \frac{v_{i+1} - v_i}{\Delta x} - \frac{\Delta x}{2}\frac{\partial}{\partial x}\left.\left(k\frac{\partial v}{\partial x}\right)\right|_i + O(\Delta x^2).$$

The assumption that functions $v(x,t)$ fulfill differential equations (68) gives following expressions for the first derivatives:

$$k\left.\frac{\partial v}{\partial x}\right|_i = k_i \frac{v_i - v_{i-1}}{\Delta x} + \frac{\Delta x}{2}\left.\left[\frac{\partial}{\partial t}\left(\gamma v\right) - f(x,t,v)\right]\right|_i + O(\Delta x^2),$$

$$k\left.\frac{\partial v}{\partial x}\right|_i = k_i \frac{v_{i+1} - v_i}{\Delta x} - \frac{\Delta x}{2}\left.\left[\frac{\partial}{\partial t}\left(\gamma v\right) - f(x,t,v)\right]\right|_i + O(\Delta x^2).$$

It remains to use boundary conditions (69), and, in accordance with difference scheme (76), we obtain second order finite difference approximation of both boundary conditions. We have following difference equations for the predictor step:

$$\begin{gathered}\frac{\Delta x_0}{2}\gamma(v_{0,0}^n)\frac{\tilde{v}_{0,0}^{n+1} - v_{0,0}^n}{\tau} - a(x,v_{0,1}^n)(\tilde{v}_{0,0}^{n+1})_x = \\ = \frac{\Delta x_0}{2} f_0(x,t,v_0)_0^n - q_0(t_n),\\ \frac{\Delta x_1}{2}\gamma(v_{1,I}^n)\frac{\tilde{v}_{1,I}^{n+1} - v_{1,I}^n}{\tau} + a(x,v_{1,I}^n)(\tilde{v}_{1,I}^{n+1})_{\bar{x}} = \\ = \frac{\Delta x_1}{2} f_1(x,t,v_1)_I^n + q_1(t_n).\end{gathered} \tag{77}$$

The difference equations for the corrector step are:

$$\begin{gathered}\frac{\Delta x_0}{2}\gamma(\tilde{v}_{0,0}^{n+1})\frac{v_{0,0}^{n+1} - v_{0,0}^n}{\tau} - a(x,\tilde{v}_{0,1}^{n+1})(v_{0,0}^{n+1})_x = \\ = \frac{\Delta x_0}{2} f_0(x_0,t_{n+1},\tilde{v}_{0,0}^{n+1}) - q_0(t_{n+1}),\\ \frac{\Delta x_1}{2}\gamma(\tilde{v}_{1,I}^{n+1})\frac{v_{1,I}^{n+1} - v_{1,I}^n}{\tau} + a(x,\tilde{v}_{1,I}^{n+1})(v_{1,I}^{n+1})_{\bar{x}} = \\ = \frac{\Delta x_1}{2} f_1(x_I,t_{n+1},\tilde{v}_{1,I}^{n+1}) + q_1(t_{n+1}).\end{gathered} \tag{78}$$

We make similar construction of the second order approximation on the border between both parts (in the point $i = i_0$). Here we need to be carefully with notation and use different indexes for differences to the left (and right) from the border point.

The second conjugation condition (71) can be rewritten in following equivalent form:

$$\chi k\frac{\partial w_0}{\partial x}\Big|_{x=l-0} = \left[k\frac{\partial w_1}{\partial x} - h_z\lambda\left(w_1 - \Theta\right)\right]\Big|_{x=l+0}, \qquad (79)$$

$$\chi = \frac{h}{H}, \quad \lambda = \frac{(H-h)}{H}.$$

We approximate the left hand side flux of equation (79) for the predictor stage as follow:

$$\chi\frac{\Delta x_0}{2}\left[\gamma(v_{0,i}^n)\frac{\tilde{v}_{0,i}^{n+1} - v_{0,i}^n}{\tau} - f_0(x,t,v_0)_i^n\right] +$$

$$\chi a(x, v_{0,i}^n)(\tilde{v}_{0,i}^{n+1})_{\bar{x}} = J_{0,i}.$$

We obtain similar expression for the right hand side:

$$a(x, v_{1,i}^n)(\tilde{v}_{1,i}^{n+1})_x - h_z\lambda(\tilde{v}_{1,i}^{n+1} - \Theta) -$$

$$\frac{\Delta x_1}{2}\left[\gamma(v_{1,i}^n)\frac{\tilde{v}_{1,i}^{n+1} - v_{1,i}^n}{\tau} - f_1(x,t,v_1)_i^n\right] = J_{1,i}.$$

Taking into account the first conjugation condition (71) (continuity: $v_{1,i_0} = v_{0,i_0}$), finally we have at the border:

$$J_{0,i_0} = J_{1,i_0}$$

or such equation for the predictor stage:

$$\begin{aligned}&\chi\frac{\Delta x_0}{2}\left[\gamma(v_{0,i}^n)\frac{\tilde{v}_{0,i}^{n+1} - v_{0,i}^n}{\tau} - f_0(x,t,v_0)_i^n\right]\\&+\chi a(x, v_{0,i}^n)(\tilde{v}_{0,i}^{n+1})_{\bar{x}} = a(x, v_{1,i}^n)(\tilde{v}_{1,i}^{n+1})_x\\&-h_z\lambda(\tilde{v}_{1,i}^{n+1} - \Theta) -\\&\frac{\Delta x_1}{2}\left[\gamma(v_{1,i}^n)\frac{\tilde{v}_{1,i}^{n+1} - v_{1,i}^n}{\tau} - f_1(x,t,v_1)_i^n\right].\end{aligned} \qquad (80)$$

We have following equation for the corrector stage:

$$\begin{aligned}&\chi\frac{\Delta x_0}{2}\left[\gamma(\tilde{v}_{0,i}^{n+1})\frac{v_{0,i}^{n+1} - v_{0,i}^n}{\tau} - f_0(x,t,\tilde{v}_0)_i^{n+1}\right]\\&+\chi a(x, \tilde{v}_{0,i}^{n+1})(v_{0,i}^{n+1})_{\bar{x}} = a(x, \tilde{v}_{1,i}^{n+1})(v_{1,i}^{n+1})_x\\&-h_z\lambda(v_{1,i}^{n+1} - \Theta) -\\&-\frac{\Delta x_1}{2}\left[\gamma(\tilde{v}_{1,i}^{n+1})\frac{v_{1,i}^{n+1} - v_{1,i}^n}{\tau} - f_1(x,t,\tilde{v}_1)_i^{n+1}\right].\end{aligned} \qquad (81)$$

The difference equations (77)-(81) together with self evident initial conditions

$$\begin{aligned}&v_{0,i}^0 = U^0(x_i), 0 \le i \le i_0,\\&v_{1,i}^0 = U^0(x_i), i_0 \le i \le I\end{aligned} \qquad (82)$$

are complete difference scheme of the second order of approximation.

6 Numerical Examples – 50A Fuse

Automotive fuse of the nominal current 50A (Fig.10) is taken as a sample.

Fig.10. 50A Fuse without shell

Geometry is transferred to our mathematical model (Fig.9). One eighth of the fuse is considered because of the symmetry. Notations of the dimensions are as in paragraphs 4 and 5: $l = 13mm$, $L = 27mm$, $b = 0.2mm$, $h = 1.9mm$, $H = 8mm$. Dimensions are obtained by measuring the fuse.

Fuse is made of zinc. Properties of the material depend on temperature. Values are known at some reference temperatures. Spline is constructed from them. It is satisfactory to use linear spline (Fig.11, Fig.12).

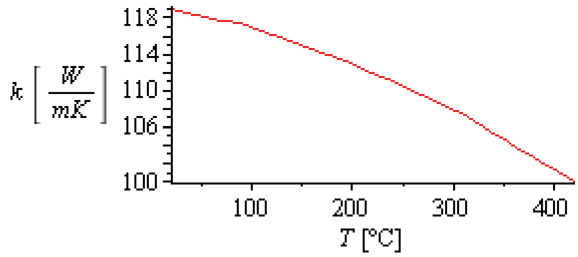

Fig.11. Heat conductivity of zinc

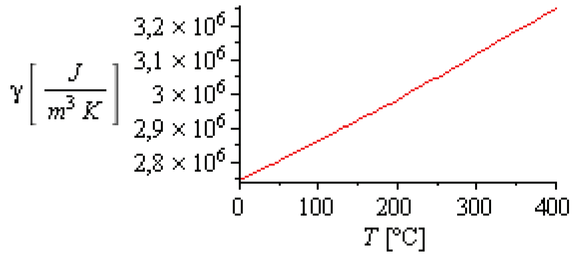

Fig.12. Heat capacity of zinc (per volume)

Next figures show other parameters. Fig.13 shows heat convection coefficient h_y and h_z. Solid line is for the thinnest part of the fuse $x \in (0,l)$; dash line is for the interval $x \in (l, L)$.

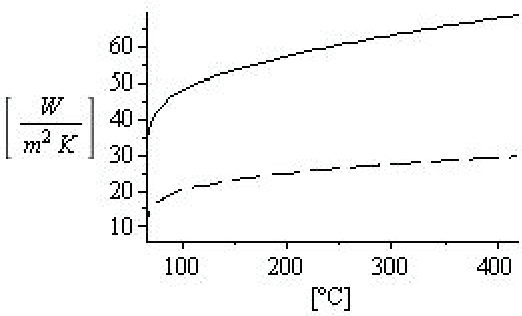

Fig.13. Heat convection to the air

Next figures show heat produced by electrical current in the thinner part of the fuse F_0 and in the blades F_1 at different current values.

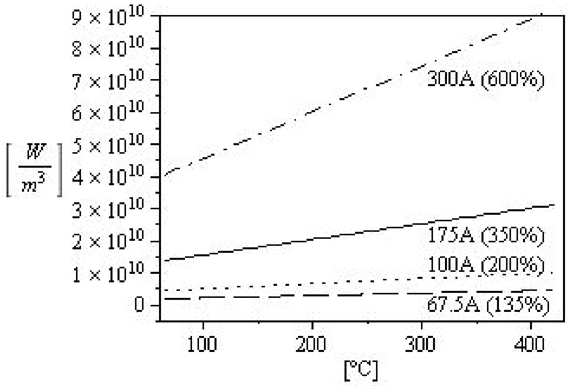

Fig.14. Heat produced by electrical current F_0

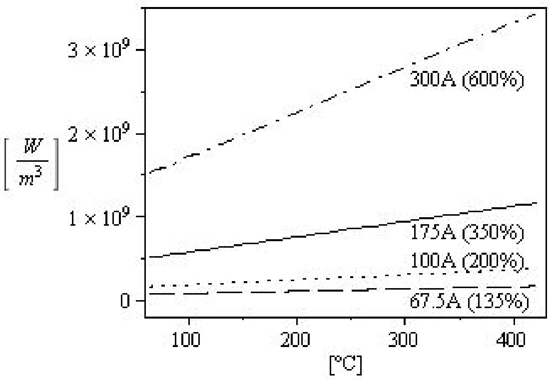

Fig.15. Heat produced by electrical current F_1

It is visible from the figures that heat production in the thinner part is more than 10 times larger than in the other part of the fuse.
Numerical calculations are done in 3 different ways. First, solution is obtained from the system of 3 ODE-s (53)-(55). Second, calculations are done from the system of 2 ODE-s (65)-(66). Third, 1D mathematical problem that consists of PDE-s (68) and additional conditions (70)-(72) is solved by applying difference scheme from the section 5.
Results are compared altogether and with standard DIN 72581-3 that define time interval of the burn-out of the fuses (Table 1).

% of Rated Current	Min.	Max.
600 %	0.04 s	1 s
350 %	0.2 s	7 s
200 %	2 s	60 s
135 %	60 s	1800 s

Table 1

Maximal temperature is reached in the middle of the fuse. Time to reach melting temperature is calculated and compared among all three mathematical models (Table 2, Fig.00).

% of Rating	1D PDE-s [s]	3 ODE-s [s]	2 ODE-s [s]
600 %	0.25	0.24	0.24
350 %	0.78	0.74	0.76
200 %	3.4	2.9	3.7
135 %	19	11.6	23

Table 2

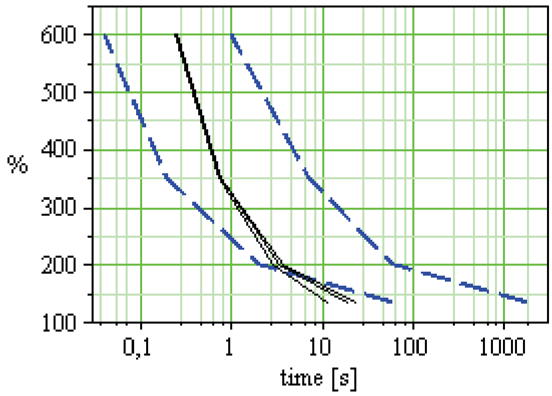

Fig.16. 50A Fuse time-current curve

Dash lines are time limits from the DIN standard.
Numerical results are appropriate if current is greater then 200% of the rating. Heat given away by radiation and conduction over blades plays greater role, if the current is close to nominal value, and real fuse breaking time is larger. This is not considered in this particular model, but could be added to original 3D mathematical model. Conservative averaging could be applied in the same manner.
For example, if we take into account radiation, additional term should be added to the boundary conditions (25)-(27):

$$\left.\left(k\frac{\partial U_i}{\partial y}+h_y\left(U_i-\Theta\right)+\varepsilon\sigma\left(U_i^4-\Theta^4\right)\right)\right|_{y=b}=0, \quad (83)$$

$$\left.\left(k\frac{\partial U_0}{\partial z}+h_z\left(U_0-\Theta\right)+\varepsilon\sigma\left(U_0^4-\Theta^4\right)\right)\right|_{z=h}=0,$$
$$\left.\left(k\frac{\partial U_1}{\partial z}+h_z\left(U_1-\Theta\right)+\varepsilon\sigma\left(U_1^4-\Theta^4\right)\right)\right|_{z=H}=0, \quad (84)$$

$$\left.\left(-k\frac{\partial U_1}{\partial x}+h_z\left(U_1-\Theta\right)\right)\right|_{x=l+0,z\in[h,H]} + \left.\varepsilon\sigma\left(U_i^4-\Theta^4\right)\right|_{x=l+0,z\in[h,H]}=0. \quad (85)$$

3D temperature distribution could be reconstructed whomever averaging is choused. It is enough to show temperature only in (x,z) plane because we have assumption about constant temperature distribution over y-dimension. Fig.17 shows temperature reconstruction after system of 2 ODE-s is solved in case of 100A (200%) current is applied to the fuse.

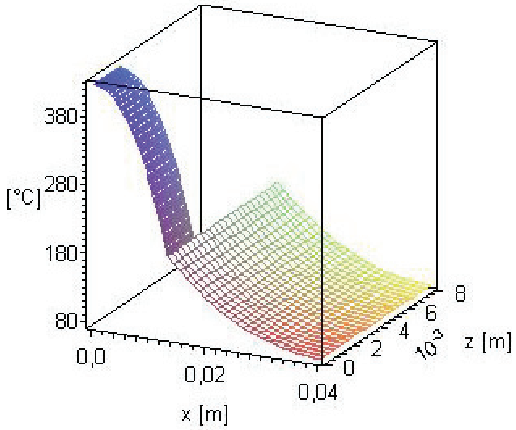

Fig.17

Similar graphic (Fig.18) could be obtained for the solution of 3 ODE-s by formulas (40)-(42), (51).

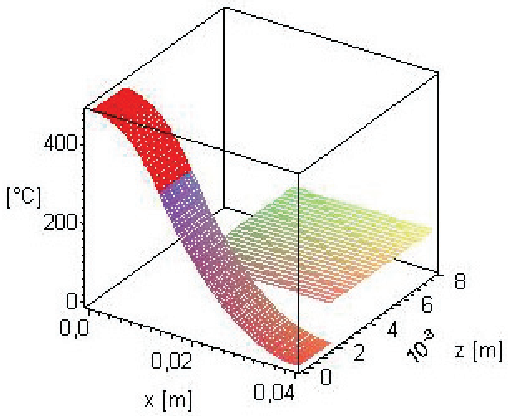

Fig.18

Solution is discontinuous because discontinuous approximation function $V_1(x,z,t)$ were used ((41)-(42)) although averaged values $w_i(x,t)$ are more precise than in previous case.

Solution of 1D PDE-s should be used if temperature distribution in the fuse is also important and not only fuse breaking time. Fig.19 shows temperature on the line $x\in[0,L]$.

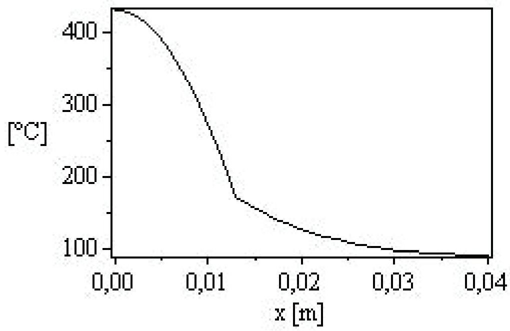

Fig.19

Next figure (Fig.20) contains temperature distribution on the line $x\in[0,L]$ for all approximations used.

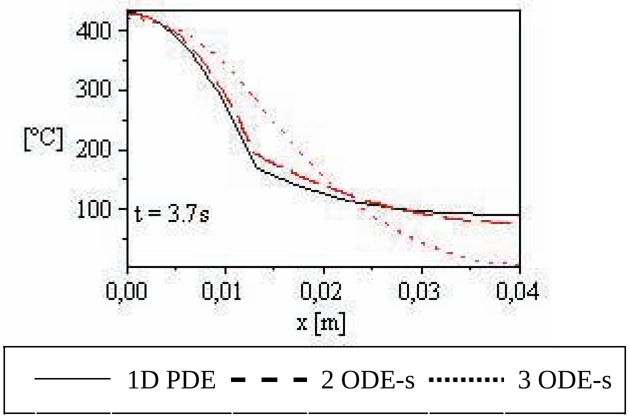

Fig.20

It takes about one minute on modern desktop computer to calculate particular example at given current. Calculation of ODE-s is even quicker. It is more efficient to calculate averaged mathematical problems rather then full 3D problems.

7 Conclusions

We have approximated 3D problem and reduced its solution to the solution of the time-dependent non-linear system of two or three ordinary differential equations. Reduction was realized in two different ways by different assumptions. Both systems have similar structure, but different coefficients. The systems of ordinary differential equations are solvable with standard techniques. Approximate analytical 3D solution could be easily obtained from the solution of the transformed problem afterwards.

Acknowledgements:

Research was supported by European Social Fund and Council of Sciences of Latvia (grant 05.1525). Special thanks to Prof. H.-D. Liess from University of the Bundeswehr Munich for useful consultations about physical background of the automotive fuses.

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