# The mathematical models for penetration of a liquid jets into a pool 

IVAN V. KAZACHKOV<br>Department of Energy Technology, Division of Heat and Power<br>Royal Institute of Technology<br>Brinellvägen, 68, Stockholm, 10044<br>SWEDEN<br>Ivan.Kazachkov@energy.kth.se http://www.kth.se/itm/inst?l=en_UK


#### Abstract

The peculiarities of a jet penetrating the liquid pool of different density were examined by means of the non-linear and linear mathematical models derived including bending instability. Based on experimental observations reported in the literature for a number of situations, the penetration behaviour was assumed to govern the buoyancy-dominated regime. A new analytical solution of the one-dimensional non-linear model was obtained for the jet penetration in this condition, as function of Froude number, jet/ambient fluid density ratio and other parameters. The solution was analysed for a number of limit cases. Analytical solution of the non-linear second-order equation obtained can be of interest for other researchers as the mathematical result.


Key-Words: - Jet, Penetration, Pool of Liquid, Non-linear, Analytical Solution, Bifurcation, Bending

## 1 Introduction

The penetration dynamics of a liquid jet into the other liquid (or solid) medium has been investigated by a number of researchers [1-15]. Most of the earlier studies have been performed in the metal and nuclear industries, e.g. [1, 4-7, $9-11]$. But the problem still remains, especially in the case of the thick jets when they are penetrating a pool of other liquid without disintegration and in case of dominated inertia, drag and buoyancy forces.

For the thin jets it has been shown [16] that the jet instability might be caused by the bending perturbations of its axis. The objective of present paper is determining the penetration behaviours of a thick jet into a fluid pool and deriving a penetration depth as a function of the conditions and properties of a jet and a pool.

The jet penetration non-linear model is developed and some analysis is made for a number of limit cases, which maybe of interest for some practical applications.

General scheme of the penetration process is illustrated by experimental data borrowed from [17] shown in Fig. 1. It is clearly observed that the penetrating jet is going first with approximately stable radius and then changing its radius abruptly to another bigger one. This bifurcation point is explained from the analytical solution obtained below.


Fig. 1. Experimental illustration of a jet penetrating the pool of other liquid

General scheme of the penetration process is illustrated by experimental data borrowed from [17] shown in Fig. 1. It is clearly observed that the penetrating jet is going first with approximately stable radius and then changing its radius abruptly to a bigger one. This is some interesting bifurcation point, which has got explanation from the analytical solution obtained in this paper.

Some amount of air may also be entrained into a pool together with a jet. As shown in a number of papers [18-20], when liquid jet
impacts a liquid pool, air is entrained in a pool if jet's velocity exceeds the threshold value.

Phenomenologically based correlations for an air entrainment have been proposed in a few papers, for example in [20]. Then it has been considered [19] that instability responsible for the air entrainment was caused by the gas viscosity.

The analysis presented in [18] is based on inviscid flow theory assuming that the air entrainment was a result of a Helmholtz-Taylor instability. This is an interesting complex problem for a separate study, therefore an influence of the air entrainment on a jet penetration features is not considered here.
G. K. Batchelor [21] has also given the equation to compute the momentum looses by a shock of the jet on a liquid pool surface at the initial moment of a jet penetration when moving jet touches a pool having liquid with a zero velocity. Using those equations one can compute the abrupt change of a jet velocity at the entrance to a pool. This phenomenon is not taken into account here because it is easy to do and it does not influence the solution considered in this paper.

## 2 Problem Formulation

### 2.1 Physical model of a jet penetration

Consider a jet penetrating the pool of other liquid as a body of a variable mass assuming that the jet is moving under an inertia force acting against the drag and buoyancy forces (see Fig. 2). The surface forces are supposed to be negligible comparing to those ones.

Then a jet radius is assumed approximately constant during the jet penetration or at least during some part of the depth of penetration, which allows considering the jet being partly of a nearly constant radius. It allows calculating the jet penetration step by step in general case approximately taking the first constant jet radius, then next constant jet radius, and so on.

Strictly saying, such assumptions are always satisfied in case of a solid rod penetration into the liquid pool. But mainly it is also attainable assumption in case of a thick jet penetration into the pool because all the forces taken in a
consideration are of an order of a jet crosssection and a surface tension is of an order of a jet circular.


Fig. 2. Scheme of a jet penetration into the pool of other liquid: phases by penetration

### 2.2 Non-linear mathematical model of a jet penetration

Based on the above physical description of the problem, the equation of a jet momentum conservation (considering a jet as a body of a variable mass) is the following:

$$
\begin{equation*}
\rho_{1} \frac{d\left(h v_{1}\right)}{d t}=h\left(\rho_{1}-\rho_{2}\right) g-\frac{1}{2} \rho_{2} v_{1}^{2} \tag{1}
\end{equation*}
$$

where $h$ is a depth of a jet penetration into the pool, $\rho_{1}, \rho_{2}$ are densities of the jet and fluid in the pool, respectively, $v_{1}$ is the jet velocity. Obviously here is $v_{1}=d h / d t$.

For the thick jets one can neglect surface forces retaining the only drag force together with the buoyancy and inertia forces. To estimate this simplification, consider when the ratio of the surface force $\mu\left(\partial v_{1} / \partial z\right)$ taken by the entire jet surface to the drag force acting on a jet's head is negligibly
small. Here $\mu$ is the dynamic viscosity coefficient, $z$ is the coordinate perpendicular to a jet axis. Thus, it yields to the following condition:

$$
\mu_{1}\left(\partial v_{1} / \partial z\right)_{s} 2 \pi r_{0} h \ll \rho_{2} v_{1}^{2} \pi r_{0}^{2} / 2
$$

where from estimating the velocity gradient as $\left(\partial v_{1} / \partial z\right)_{s} \approx v_{1} / r_{0}$, one can finally get $\operatorname{Re} \gg 4\left(h / r_{0}\right) \rho_{1 / 2}$.

Here $\rho_{1 / 2}=\rho_{1} / \rho_{2}, \operatorname{Re}=v_{1} r_{0} \rho_{1} / \mu_{1}$ is the Reynolds number. For example, from the condition obtained follows that by $h / r_{0}=10, \rho_{1 / 2}=0.1$ surface force is negligible comparing to the drag force by $\operatorname{Re} \gg 4$.

### 2.3 Singularity of the initial conditions

The initial conditions for the jet's momentum equation (1) should be stated as follows:

$$
\begin{equation*}
t=0, \quad h=0, \quad d h / d t=u_{0} \tag{2}
\end{equation*}
$$

where $u_{0}$ is the initial jet velocity (before penetration into the pool).

In case of $\rho_{1}=\rho_{2}$ one can obtain from equation (1) the following simple equation:

$$
\begin{equation*}
h \frac{d^{2} h}{d t^{2}}+\frac{3}{2}\left(\frac{d h}{d t}\right)^{2}=0 \tag{3}
\end{equation*}
$$

which is integrated through the next transformation:

$$
\left(\frac{d h}{d t}\right)^{-1} \frac{d}{d t}\left(\frac{d h}{d t}\right)+\frac{3}{2 h} \frac{d h}{d t}=0
$$

where from yields

$$
\frac{d h}{d t} h^{3 / 2}=c_{1}
$$

so that

$$
\begin{equation*}
\frac{2}{5} h^{5 / 2}=c_{1} t+c_{2} \tag{4}
\end{equation*}
$$

As one can see we have here some singularity with the initial conditions (2) because at the initial moment of time $(t=0)$ the jet has velocity $u_{0}$, and the fluid pool at the jet/pool contact area ( $h=0$ ) changes
abruptly its velocity from 0 to $u_{0}$ (actually less than $u_{0}$, if energy dissipation is taken into account).

To avoid this singularity, let consider further the following initial conditions instead of the abovementioned conditions:

$$
\begin{equation*}
t=0, \quad h=h_{0}, \quad d h / d t=u_{p}, \tag{5}
\end{equation*}
$$

where $h_{0}$ and $u_{p}$ are the initial depth and velocity of a jet penetration (after a first contact of a jet with a pool), which should be calculated later on. For some limit cases they could be taken from the studies of a high-speed jet penetration [1, 5, 6, 13], e.g.

$$
\begin{equation*}
u_{p}=\frac{\lambda}{1+\lambda} u_{0} \tag{6}
\end{equation*}
$$

where $\lambda=\sqrt{\rho_{1 / 2}}$. Taking into account (5), (6), one can obtain for $\rho_{1 / 2}=1$ :

$$
\begin{align*}
& u_{p}=0.5, \quad h=h_{0}\left(\frac{5 t}{4 h_{0}}+1\right)^{2 / 5}, \\
& \frac{d h}{d t}=\frac{1}{2}\left(\frac{5 t}{4 h_{0}}+1\right)^{-3 / 5} . \tag{7}
\end{align*}
$$

Thus, in case of the same densities of a jet and a pool, the jet velocity tends to zero asymptotically. Then the jet velocity decreases twice at the depth $h=h_{0}$.

Here and further the penetration depth is dimensionless value, and the scale is the jet's radius $r_{0}$.

It is also interesting to calculate the characteristic distance where the jet loses its velocity of a given value. This is easily determined from the equation (4):

$$
\begin{equation*}
v_{1}=u_{p}\left(h_{0} / h\right)^{3 / 2} \tag{8}
\end{equation*}
$$

where $u_{p}=0.5$. As one can see from the equation (8), the velocity of a jet penetration into a pool is decreasing by the jet penetration depth as $1 / h^{3 / 2}$. So far in a case of the same densities, the jet looses a half of its velocity at the depth $h_{0}$, then a half of that velocity at the depth $h_{1}=2^{2 / 3} h_{0} \approx 1,6 h_{0}$, and then ten times velocity decrease happens at the depth $h_{10}=10^{2 / 3} h_{0} \approx 4,5 h_{0}$.

In a general case of the different densities of a jet and a pool ( $\rho_{1 / 2} \neq 1$ ) one needs to solve the nonlinear equation (1), which has the following dimensionless form:

$$
\begin{equation*}
h \frac{d^{2} h}{d t^{2}}+\left(1+\frac{\rho_{2 / 1}}{2}\right)\left(\frac{d h}{d t}\right)^{2}+\frac{\rho_{2 / 1}-1}{F r} h=0 \tag{9}
\end{equation*}
$$

where $u_{0}$ is the velocity scale and $r_{0} / u_{0}$ is the time scale, $F r=u_{0}{ }^{2} /\left(g r_{0}\right)$ is the Froude number, which characterizes the inertia and buoyancy forces' ratio.

As one can see from the above, the Froude number and the density ratio totally predetermine the process of a thick jet penetration into a pool of other liquid.

### 2.4 The initial depth and corrected initial velocity of a jet penetration into a pool

The equation (9) is solved with the initial conditions (5), where $h_{0}$ and $u_{p}$ are determined using the equation of a jet momentum and the Bernoulli equation in the form:

$$
\begin{gather*}
\rho_{1} H u_{0}=\rho_{1} H u_{p}+\rho_{2} h_{0} u_{p}, \\
\frac{1}{2} \rho_{1} u_{0}^{2}=\frac{1}{2} \rho_{1} u_{p}^{2}+\left(\rho_{1}-\rho_{2}\right) g h_{0} / u_{0}^{2}-\frac{1}{2} \rho_{2} u_{p}^{2}, \tag{10}
\end{gather*}
$$

where $H$ is the initial length for the finite length jet falling into the pool. In case of a jet spreading out from a nozzle (not of a finite length), this value is determined by the pressure at the outlet.

Now an analytical solution to the equation array (10) is presented in the following dimensionless form:

$$
\begin{gather*}
h_{0}=\frac{H}{\rho_{2 / 1}}\left(\frac{\sqrt{1+\rho_{2 / 1}}}{\sqrt{1+2\left(1-\rho_{2 / 1}\right) h_{0} / F r}}-1\right),  \tag{11}\\
u_{p}=\sqrt{\frac{1+2 h_{0}\left(1-\rho_{2 / 1}\right) / F r}{1+\rho_{2 / 1}}} .
\end{gather*}
$$

Then, by a small density difference or by a small initial depth of a jet penetration (comparing to the Froude number) when $h_{0}\left(1-\rho_{2 / 1}\right) \ll F r$, the simpler approximations follow from (11):

$$
\begin{equation*}
h_{0}=\frac{H}{1+\sqrt{1+\rho_{2 / 1}}}, \quad u_{p}=\frac{1}{\sqrt{1+\rho_{2 / 1}}} . \tag{12}
\end{equation*}
$$

The last formula above corresponds to (6), which was taken from the literature for the highspeed jet/solid rod penetration into the liquid pools and solid plates $[1,5,6,13]$. Now (6) is rewritten as

$$
\begin{equation*}
u_{p}=\frac{u_{0}}{1+\sqrt{\rho_{2 / 1}}} . \tag{13}
\end{equation*}
$$

The correspondence of (12) and (13) by $u_{p}$ is clearly observed from the Table 1 below.

Table 1. The initial velocity and the depth of a jet penetration

|  | $u_{p}$ by <br> $(12)$ | $u_{p}$ by <br> $(13)$ | $h_{0}$ by (12) |
| :---: | :---: | :---: | :---: |
| $\rho_{2 / 1}=0$ | 1 | 1 | 0 |
| $\rho_{2 / 1}=1$ | $1 / \sqrt{2}$ | $1 / 2$ | $(\sqrt{2}-1) H$ |
| $\rho_{2 / 1} \gg$ | $1 / \sqrt{\rho_{2 / 1}}$ | $1 / \sqrt{\rho_{2 / 1}}$ | $\sqrt{\rho_{2 / 1}} H\left(1-\sqrt{\rho_{2 / 1}}\right.$ |

### 2.5 The analytical solution of the secondorder non-linear differential equation

The equation (9) can be solved using the following special coupled transformations for the both dependent and independent variables, which were found by the method described in [22]:

$$
\begin{align*}
& h=\left(\frac{2 A+1}{2}\right)^{\frac{2}{2 A+1}} X^{\frac{2}{2 A+1}}, \\
& d t=\left(\frac{1}{2 A+1}\right)^{\frac{1}{2 A+1}} X^{\frac{1}{2 A+1}} d \tau, \tag{14}
\end{align*}
$$

where are: $\rho_{2 / 1}=\rho_{2} / \rho_{1}, A=1+\rho_{2 / 1} / 2$.

Implementation of (14) and a few further simple transformations lead to the following linear secondorder equation in the new variables:

$$
\begin{equation*}
\frac{d^{2} y}{d \tau^{2}}+\frac{\rho_{2 / 1}^{2}-1}{2 F r^{2}}=0 \tag{15}
\end{equation*}
$$

Here $y$ is the new variable given by $X=e^{y}$. The solution of (15) is $y=c_{1} e^{k \tau}+c_{2} e^{-k \tau}$, where $c_{1}, c_{2}$ are the constants computed using the initial conditions (5). The eigen value $k$ is

$$
\begin{equation*}
k=\sqrt{\left(1-\rho_{2 / 1}\right)\left[1+0.5\left(1+\rho_{2 / 1}\right)\right] / F r} . \tag{16}
\end{equation*}
$$

In case of $\rho_{2 / 1}>1$ (a pool is denser than a jet), the eigen values are imaginary, and the solution is

$$
\begin{equation*}
c_{1}^{\prime} \cos k \tau+c_{2}^{\prime} \sin k \tau \tag{17}
\end{equation*}
$$

### 2.5.1 Dimensionless time

The dimensionless time $t$ is determined through the variable $\tau$ by (14), which gives

$$
\begin{align*}
& \rho_{2 / 1}<1  \tag{18}\\
& t=\left(\frac{1}{3+\rho_{2 / 1}}\right)^{\frac{1}{3+\rho_{2 / 1}}} \int e^{\frac{c_{1} e^{k \tau}+c_{2} e^{-k \tau}}{\rho_{2 / 1}+3}} d \tau+c_{3} \\
& \rho_{2 / 1}>1  \tag{19}\\
& t=\left(\frac{1}{3+\rho_{2 / 1}}\right)^{\frac{1}{3+\rho_{2 / 1}}} \int e^{\frac{c_{1}^{\prime} \cos k \tau+c_{2}^{\prime} \sin k \tau}{\rho_{2 / 1}+3}} d \tau+c_{3}
\end{align*}
$$

where the constants $c_{3}$ are calculated later on. For $\tau \ll 1$, the following linear approximations by $k \tau$ are satisfied: $e^{ \pm k \tau} \approx 1 \pm k \tau, \cos k \tau \approx 1, \sin k \tau \approx k \tau$.

Thus, the equations (18), (19) yield:

$$
\begin{aligned}
& \rho_{2 / 1}<1 \\
t= & \left(\frac{1}{3+\rho_{2 / 1}}\right)^{\frac{1}{3+\rho_{2 / 1}}} \frac{3+\rho_{2 / 1}}{k\left(c_{1}-c_{2}\right)} e^{\frac{1}{3+\rho_{2 / 1}}\left[\left(c_{1}+c_{2}\right)+k\left(c_{1}-c_{2}\right) \tau\right]}+c_{3} . \\
& \rho_{2 / 1}>1
\end{aligned}
$$

$$
t=\left(\frac{1}{3+\rho_{2 / 1}}\right)^{\frac{1}{3+\rho_{2 / 1}}} \frac{3+\rho_{2 / 1}}{c_{2}^{\prime} k} e^{\frac{c_{1}+c_{2}^{\prime} k \tau}{3+\rho_{2} / 1}}+c_{3}
$$

From these equations requiring $t=0$, which leads to $\tau=0$, the constants $c_{3}$ are got.

Consequently, the real dimensionless time $t$ is expressed through the artificial variable $\tau$ :

$$
\begin{align*}
& \rho_{2 / 1}<1, \\
& t=\left(\frac{1}{3+\rho_{2 / 1}}\right)^{\frac{1}{3+\rho_{2 / 1}} \frac{3+\rho_{2 / 1}}{k\left(c_{1}-c_{2}\right)}} \begin{array}{l}
\cdot\left\{e^{\frac{1}{3+\rho_{2 / 1}}\left[\left(c_{1}+c_{2}\right)+k\left(c_{1}-c_{2}\right) \tau\right]}-e^{\frac{c_{1}+c_{2}}{3+\rho_{2 / 1}}}\right\} \\
\rho_{2 / 1}>1, \\
t=\left(\frac{1}{3+\rho_{2 / 1}}\right)^{\frac{1}{3+\rho_{2 / 1}} \frac{3+\rho_{2 / 1}}{c_{2}^{\prime} k}} \\
\quad\left(e^{\frac{c_{1}^{\prime}+c_{2} k \tau}{3+\rho_{2 / 1}}}-e^{\frac{c_{1}^{\prime}}{3+\rho_{2 / 1}}}\right)
\end{array},
\end{align*}
$$

Strictly saying, these equations are satisfied in a small $\varepsilon$-surrounding of $\tau=0$. In general case one needs to compute integrals in (18), (19) numerically. But for $\rho_{2 / 1} \sim 1$ and $F r \gg 1$, the multiplayer of $\tau$ has to be small value, which is possible using approximations (20), (21) in a wider region of $\tau$, and even if $\tau$ is not small but the condition $k \tau \ll 1$ is satisfied.

And further the expression (20) is presented in the form:
$t=\left(\frac{1}{3+\rho_{2 / 1}}\right)^{\frac{1}{3+\rho_{2 / 1}}} \frac{3+\rho_{2 / 1}}{k\left(c_{1}-c_{2}\right)} e^{\frac{c_{1} e^{k \tau}+c_{2} e^{-k \tau}}{3+\rho_{2 / 1}}}-t_{0}$,
$t_{0}=\left(\frac{1}{3+\rho_{2 / 1}}\right)^{\frac{1}{3+\rho_{2 / 1}}} \frac{3+\rho_{2 / 1}}{k\left(c_{1}-c_{2}\right)} e^{\frac{c_{1}+c_{2}}{3+\rho_{2 / 1}}}$.

For $\rho_{2 / 1}>1$ the corresponding expressions are obtained from (21) similarly.

### 2.5.2 Caclulation of the constants

Now using the initial condition (5) and correlations (11), one can substitute (14) into (5) and calculate constants. Thus, for $\rho_{2 / 1}<1$ (a jet is denser than a pool) the equations for constants are:

$$
\begin{aligned}
c_{1}+c_{2}= & \ln \left[\left(\frac{2}{3+\rho_{2 / 1}}\right) h_{0}^{\frac{3+\rho_{2 / 1}}{2}}\right] \\
c_{1}-c_{2}= & \frac{u_{p}}{\sqrt{h_{0}}} \sqrt{\frac{F r}{1-\rho_{2 / 1}}}\left(\frac{2}{3+\rho_{2 / 1}}\right)^{\frac{1}{3+\rho_{2 / 1}}-\frac{1}{2}} . \\
& \cdot\left(3+\rho_{2 / 1}\right)^{-\frac{1}{3+\rho_{2 / 1}}}
\end{aligned}
$$

where from yields:

$$
\begin{align*}
& c_{1}=\frac{1}{2}\left[\ln \left(\frac{2}{3+\rho_{2 / 1}} h_{0}^{\frac{3+\rho_{2 / 1}}{2}}\right)+\frac{u_{p}}{\sqrt{h_{0}}} \sqrt{\frac{F r}{1-\rho_{2 / 1}}} .\right. \\
& \left.\cdot\left(\frac{2}{3+\rho_{2 / 1}}\right)^{\frac{1}{3+\rho_{2 / 1}}-\frac{1}{2}}\left(3+\rho_{2 / 1}\right)^{-\frac{1}{3+\rho_{2 / 1}}}\right], \\
& c_{2}=\frac{1}{2}\left[\ln \left(\frac{2}{3+\rho_{2 / 1}} h_{0}^{\frac{3+\rho_{2 / 1}}{2}}\right)-\frac{u_{p}}{\sqrt{h_{0}}} \sqrt{\frac{F r}{1-\rho_{2 / 1}}} .\right.  \tag{24}\\
& \left.\cdot\left(\frac{2}{3+\rho_{2 / 1}}\right)^{\frac{1}{3+\rho_{2 / 1}-\frac{1}{2}}}\left(3+\rho_{2 / 1}\right)^{-\frac{1}{3+\rho_{2 / 1}}}\right] .
\end{align*}
$$

For $\rho_{2 / 1}>1$ (a pool is denser than a jet), from (14), (5), accounting (17), yield the constants $c_{1,2}^{\prime}$ :

$$
\begin{equation*}
c_{1}^{\prime}=\ln \left[\left(\frac{2}{3+\rho_{2 / 1}}\right) h_{0}^{\frac{3+\rho_{2 / 1}}{2}}\right] \tag{25}
\end{equation*}
$$

$c_{2}^{\prime}=\frac{u_{p}}{\sqrt{h_{0}}} \sqrt{\frac{F r}{1-\rho_{2 / 1}}}\left(\frac{2}{3+\rho_{2 / 1}}\right)^{\frac{1}{3+\rho_{2 / 1}-\frac{1}{2}}}\left(3+\rho_{2 / 1}\right)^{\frac{1}{3+\rho_{2 / 1}}}$,
so that $c_{1}=\left(c_{1}^{\prime}+c_{2}^{\prime}\right) / 2, c_{2}=\left(c_{1}^{\prime}-c_{2}^{\prime}\right) / 2$.

Then from the equations (23), (24) follows $t_{0} \approx 1.02\left(1+\sqrt{\rho_{2 / 1}}\right) h_{0}, t_{0}=2 h_{0} / u_{p}$. By $\sqrt{\rho_{2 / 1}} \ll 1$, there is $t_{0} \approx h_{0}$, and by $\sqrt{\rho_{2 / 1}} \gg 1$ there is $t_{0} \approx \sqrt{\rho_{2 / 1}} h_{0}$.

### 2.5.3 Explicit form of the solution obtained

The solution (14) can be transformed to an explicit form as the function of $t$ (exclude the artificial time $\tau$ ). For this purpose, from (23), (24) yields

$$
t+t_{0}=\left(\frac{1}{3+\rho_{2 / 1}}\right)^{\frac{1}{3+\rho_{2 / 1}}} \frac{3+\rho_{2 / 1}}{k\left(c_{1}-c_{2}\right)}
$$

$$
\cdot e^{\frac{1}{3+\rho_{2 / 1}}\left[\left(c_{1}+c_{2}\right)+k\left(c_{1}-c_{2}\right) \tau\right]}
$$

and further it goes to

$$
\begin{align*}
& e^{k \tau}=\left[\left(t+t_{0}\right) k\left(c_{1}-c_{2}\right)\right. \\
& \left.\cdot\left(\frac{1}{3+\rho_{2 / 1}}\right)^{\frac{1}{3+\rho_{2 / 1}}-1} e^{-\frac{c_{1}+c_{2}}{3+\rho_{2 / 1}}}\right]^{\frac{3+\rho_{2 / 1}}{c_{1}-c_{2}}} \tag{26}
\end{align*}
$$

or

$$
\begin{equation*}
e^{k \tau}=\left[\left(t+t_{0}\right) \frac{u_{p}}{2 h_{0}}\right]^{\alpha} \tag{27}
\end{equation*}
$$

With account of (16) yields:

$$
\begin{gather*}
\alpha=\frac{\sqrt{1-\rho_{2 / 1}}}{u_{p}} \sqrt{\frac{h_{0}}{F r}}\left(\frac{2}{3+\rho_{2 / 1}}\right)^{\frac{\rho_{2 / 1}+1}{2\left(3+\rho_{2 / 1}\right)}} .  \tag{28}\\
\cdot\left(3+\rho_{2 / 1}\right)^{\frac{1}{3+\rho_{2 / 1}}+1}
\end{gather*}
$$

Accounting that $e^{c_{1} e^{k \tau}+c_{2} e^{-k \tau}}=e^{c_{1} e^{k \tau}} e^{c_{2} e^{-k \tau}}$, and using the equations (27), (28), (22)-(23), one can come to a solution (14) for the penetration depth $h$ as a function of the real temporal variable $t$ (for $k \tau \ll 1$ ):
$h=\left(\frac{3+\rho_{2 / 1}}{2}\right)^{\frac{2}{3+\rho_{2 / 1}}(1-c h k \tau)} h_{0}^{\operatorname{chk} \tau} e^{\frac{2}{\alpha} s h k \tau}$,
where $c h$, sh denote the hyperbolic cosine and sine, respectively, $e^{k \tau}$ is expressed through $t$ by (27). The velocity of a jet penetration into a pool is determined from (14) or (29) using the equations $v_{1}=d h / d t=(d h / d \tau) d \tau / d t$. In the cases of $\rho_{2 / 1}<1$ and $\rho_{2 / 1}>1$, it results in

$$
\begin{align*}
& \frac{d h}{d t}=\left(\frac{3+\rho_{2 / 1}}{2}\right)^{\frac{2}{3+\rho_{2 / 1}}-1}\left(3+\rho_{2 / 1}\right)^{\frac{1}{3+\rho_{2 / 1}}} .  \tag{30}\\
& \cdot k\left(c_{1} e^{k \tau}-c_{2} e^{-k \tau}\right) e^{\frac{c_{1} e^{k \tau}+c_{2} e^{-k \tau}}{3+\rho_{2 / 1}}} \\
& \frac{d h}{d t}=\left(\frac{3+\rho_{2 / 1}}{2}\right)^{\frac{2}{3+\rho_{2 / 1}}-1}\left(3+\rho_{2 / 1}\right)^{\frac{1}{3+\rho_{2 / 1}}} .  \tag{31}\\
& \cdot k\left(c_{2}^{\prime} \cos k \tau-c_{1}^{\prime} \sin k \tau\right) e^{\frac{c_{1} \cos k \tau+c^{\operatorname{cin}} k \tau}{3+\rho_{2 / 1}}}
\end{align*}
$$

correspondingly.

### 2.5.4 Parameters of a jet penetrating a pool

 The equations obtained, e.g. (29)-(31), allow computing the parameters of the jet penetrating the pool. For example, the penetration depth $h_{*}$ is determined by condition $d h / d t=0$, therefore $h_{*}$ and the correspondent penetration time $t *$, for $\rho_{2 / 1}<1$ and $\rho_{2 / 1}>1$ are computed as$$
\begin{align*}
& \rho_{2 / 1}<1, \quad h_{*}=\left(\frac{3+\rho_{2 / 1}}{2}\right)^{\frac{2}{3+\rho_{2 / 1}}} e^{\frac{4}{3+\rho_{2 / 1}} \sqrt{c_{1} c_{2}}},  \tag{32}\\
& t_{*}=\frac{\left(3+\rho_{2 / 1}\right)^{1-\frac{1}{3+\rho_{2 / 1}}}}{k\left(c_{1}-c_{2}\right)} e^{\frac{2 \sqrt{c_{c} c_{2}}}{3+\rho_{2 / 1}}}-\mathrm{t}_{0} ; \\
& \rho_{2 / 1}>1, \quad h_{*}=\left(\frac{3+\rho_{2 / 1}}{2}\right)^{\frac{2}{3+\rho_{2 / 1}}} e^{2 \frac{c_{1}^{\prime}+\frac{\left.c^{\prime}\right)^{2} / c_{1}^{\prime}}{3+\rho_{2 / 1}}}{},}  \tag{33}\\
& t_{*}=\frac{\left(3+\rho_{2 / 1}\right)^{1-\frac{1}{3+\rho_{2 / 1}}}}{k c_{2}^{\prime}} e^{\frac{c_{1}+\left(c_{2}^{\prime}\right)^{2} / c_{1}}{3+\rho_{2 / 1}}}-t_{0} .
\end{align*}
$$

Thus, here we have two different cases correspondingly for the pool, which is denser than a jet and for the inverse situation. Peculiarities of a jet penetration to the pool are different for these two cases.

## 3 Analysis of the solution obtained for some limit cases

Further analysis of the analytical solution obtained is easier performed for the limit cases when the solution is substantially simplified.

If $\rho_{2 / 1} \ll 1,\left(1-\rho_{2 / 1}\right) h_{0} \ll F r$, then (12), (23), (26), (28) result in the following solution:

$$
\begin{align*}
& h_{0} \approx \frac{H}{2}, \quad t_{0} \approx H, \quad \alpha \approx 2.85 \sqrt{\frac{H}{F r}},  \tag{34}\\
& e^{k \tau} \approx\left(\frac{t}{H}+1\right)^{\alpha},
\end{align*}
$$

which can be easily analysed. Here should be noted that this approximation satisfies a wide range of parameters because many practical situations correspond to the large Froude numbers. Accounting (16), (24), from (29), (30) yields the following approximate solution for the depth of a jet penetration, as well as for its velocity and acceleration:

$$
\begin{aligned}
& h=\left(\frac{3}{2}\right)^{2 / 3(1-c h k \tau)}\left(\frac{H}{2}\right)^{c h k \tau} e^{\frac{1}{1,43}} \sqrt{\frac{F r}{H}} \operatorname{sht}, \\
& v_{1}=\frac{d h}{d t}=2.85 \frac{h}{H} \sqrt{\frac{H}{F r}}\left(\frac{t}{H}+1\right)^{-1} . \\
& \cdot\left\{\ln \left[\frac{H}{2}\left(\frac{2}{3}\right)^{2 / 3}\right] s h k \tau+\frac{1}{1.43} \sqrt{\frac{F r}{H}} c h k \tau\right\} \\
& a_{1}=\frac{d^{2} h}{d t^{2}}=2.85 \frac{1}{H} \sqrt{\frac{H}{F r}}\left(\frac{t}{H}+1\right)^{-1} . \\
& \cdot\left(\left\{\ln \left[\frac{H}{2}\left(\frac{2}{3}\right)^{2 / 3}\right] s h k \tau+\frac{1}{1.43} \sqrt{\frac{F r}{H}} c h k \tau\right\} .\right. \\
& \cdot\left[\frac{d h}{d t}-\frac{h}{H}\left(\frac{t}{H}+1\right)^{-1}\right]+2.85 \frac{h}{H} \sqrt{\frac{H}{F r}}\left(\frac{t}{H}+1\right)^{-1} . \\
& \left.\left.\cdot\left\{\ln \left[\frac{H}{2}\left(\frac{2}{3}\right)^{2 / 3}\right] \operatorname{chk\tau }\right] \frac{1}{1.43} \sqrt{\frac{F r}{H}} s h k \tau\right\}\right)
\end{aligned}
$$

or, with explicit expression for $d h / d t$, (35) is

$$
\begin{gathered}
v_{1}=\left(\frac{3}{2}\right)^{2 / 3}\left[\frac{H}{2}\left(\frac{2}{3}\right)^{2 / 3}\right]^{c h k \tau} e^{\frac{1}{1,33}} \frac{\sqrt[F r]{H}}{\operatorname{shk\tau }} \\
\frac{2.85}{H} \sqrt{\frac{H}{F r}}\left(\frac{t}{H}+1\right)^{-1} . \\
\cdot\left\{\ln \left[\frac{H}{2}\left(\frac{2}{3}\right)^{2 / 3}\right] \operatorname{shk} \tau+\frac{1}{1.43} \sqrt{\frac{F r}{H}} \operatorname{chk} \tau\right\}
\end{gathered}
$$

where are:

$$
\begin{align*}
& c h k \tau \approx \frac{1}{2}\left[\left(\frac{t}{H}+1\right)^{2.85 \sqrt{H / F r}}+\left(\frac{t}{H}+1\right)^{-2.85 \sqrt{H / F r}}\right] \\
& s h k \tau \approx \frac{1}{2}\left[\left(\frac{t}{H}+1\right)^{2.85 \sqrt{H / F r}}-\left(\frac{t}{H}+1\right)^{-2.85 \sqrt{H / F r}}\right] . \tag{36}
\end{align*}
$$

### 3.1 Influence of the Froude number and the initial length of a jet

Analysis of the expressions (35), (36) shows the solution dependence on parameters $\sqrt{H / F r}, t / H$. A key feature of a jet penetration is determined by the Froude number and initial jet length, e.g. for $H / F_{r} \ll 1$ :

$$
\begin{aligned}
& \left(\frac{t}{H}+1\right)^{2.85 \sqrt{\frac{H}{F r}}} \approx 1+2.85 \sqrt{\frac{H}{F r}} \ln \left(\frac{t}{H}+1\right), \\
& \operatorname{shk} \tau \approx 2.85 \sqrt{\frac{H}{F r}} \ln \left(\frac{t}{H}+1\right), \quad \operatorname{ch} k \tau \approx 1
\end{aligned}
$$

up to a limit $t / H \sim 1$ and even higher. For example, $10^{0,1} \approx 1,23, \quad 1000^{0,1} \approx 2, \quad$ therefore the approximations used here satisfy a wide range of the varying parameters. By such assumptions, linearization of the solution (35) by the parameter $H / F r$ yields

$$
\begin{gather*}
h \approx \frac{H}{2}\left(\frac{t}{H}+1\right)^{2}, \\
\frac{d h}{d t} \approx\left\{4,06 \frac{H}{F r} \ln \left[\frac{H}{2}\left(\frac{2}{3}\right)^{2 / 3}\right] \ln \left(\frac{t}{H}+1\right)+1\right\}\left(\frac{t}{H}+1\right), \\
\frac{d^{2} h}{d t^{2}} \approx 2\left\{\frac{1}{H}\left[\left(\frac{t}{H}+1\right)^{-1} \frac{d h}{d t}-\frac{1}{2}\right]+\right. \tag{37}
\end{gather*}
$$

$$
+\frac{1}{F r} \ln \left[\frac{H}{2}\left(\frac{2}{3}\right)^{2 / 3}\left(\frac{t}{H}+1\right)^{2}\right]
$$

With an order of the term $\ln \left[H / 2(2 / 3)^{2 / 3}\right] \ln (t / H+1)$ restricted by 1, a further simplification is as follows

$$
\begin{gathered}
\frac{d h}{d t} \approx \frac{t}{H}+1 \\
\frac{d^{2} h}{d t^{2}} \approx \frac{1}{H}+\frac{2}{F r} \ln \left[\frac{H}{2}\left(\frac{2}{3}\right)^{2 / 3}\left(\frac{t}{H}+1\right)^{2}\right] \approx \frac{1}{H}
\end{gathered}
$$

Here $H \sim 1$ or $H \gg 1$ were considered because by $H \ll 1$ there is actually no jet (a length of a jet supposed to be at least larger than its diameter). But this case might be also considered using the solution obtained.

### 3.2 The case of a long finite jet or a jet coming from a nozzle

The case of $H \gg 1$ is considered separately due to its most practicality. It corresponds to a long jet or to a jet coming from the nozzle. For this case, the equation (11) yields

$$
\begin{gather*}
\left(h_{0} \rho_{2 / 1} / H+1\right)^{2}=1 / u_{p}^{2} \\
\left(1+\rho_{2 / 1}\right) u_{p}^{2}=1+2 h_{0}\left(1-\rho_{2 / 1}\right) / F r \tag{38}
\end{gather*}
$$

where $u_{p} \approx 1$, and the last equation (38) gives the approximate initial depth of a jet penetration:

$$
\begin{equation*}
h_{0}=\frac{\rho_{2 / 1}}{2\left(1-\rho_{2 / 1}\right)} F r . \tag{39}
\end{equation*}
$$

But the formula (39) according to (38) is justified only for $\rho_{2 / 1} h_{0} \ll 1$, therefore $H \gg \rho_{2 / 1}^{2} F r / 2$ is required. For example, if $\rho_{2 / 1}=0.1$, and $\operatorname{Fr}=10^{2}$, then $H \gg 0.5$ has to be, and $h_{0} \approx 5, \mathrm{u}_{\mathrm{p}} \approx 1$. By $F r=10^{4}$, there are $H \gg 50$, and $h \sim 500$, respectively. Thus, the assumption made is reasonable.

It should be noted that this case is absolutely different from the case considered in $[1,5,6,13]$.

### 3.3 Parameters of the jet's penetration into the pool

The formula (39) expresses $h_{0}$ through two parameters, the density ratio and the Froude number, e.g. $h_{0}$ does not depend on $H$. Substitution of (39) into (23), (26)-(29) results for $\rho_{2 / 1} \ll 1$ in the following:

$$
\begin{align*}
& h \approx\left(\frac{3}{2}\right)^{2 / 3}\left[\left(\frac{2}{3}\right)^{2 / 3} \frac{\rho_{2 / 1}}{2} F r\right]^{\text {chk }} \frac{1}{\sqrt{\rho_{2 / 1}}} \operatorname{shk\tau } \\
& \frac{d h}{d t} \approx \frac{2 h}{\sqrt{\rho_{2 / 1}} F r}\left(\frac{t}{\rho_{2 / 1} F r}+1\right)^{-1}\left\{\ln \left[\left(\frac{2}{3}\right)^{\frac{2}{3}} \frac{\rho_{2 / 1}}{2} F r\right] \operatorname{shk} \tau+\frac{\operatorname{chk\tau }}{\sqrt{\rho_{2 / 1}}}\right\} \\
& \frac{d^{2} h}{d t^{2}} \approx \frac{4 h}{\rho_{2 / 1} F r^{2}}\left(\frac{t}{\rho_{2 / 1} F r}+1\right)^{-2} \cdot  \tag{40}\\
& \cdot\left\{\left\{\ln \left[\left(\frac{2}{3}\right)^{2 / 3} \frac{\rho_{2 / 1}}{2} F r\right] \cdot \operatorname{shk} \tau+\frac{\operatorname{chk\tau }}{\sqrt{\rho_{2 / 1}}}\right\}^{2}+\right. \\
&+ \ln \left[\left(\frac{2}{3}\right)^{\frac{2}{3}} \frac{\rho_{2 / 1}}{2} F r\right] \operatorname{chk\tau }+\frac{\operatorname{shk} \tau}{\sqrt{\rho_{2 / 1}}}+ \\
&-\frac{1}{2 \sqrt{\rho_{2 / 1}}}\left\{\ln \left[\left(\frac{2}{3}\right)^{\frac{2}{3}} \frac{\rho_{2 / 1}}{2} F r\right] \operatorname{shk\tau +\frac {\operatorname {ch}k\tau }{\sqrt {\rho _{2/1}}}\} \} }\right\}
\end{align*}
$$

where are:

$$
\begin{align*}
& t_{0}=2 h_{0}=\rho_{2 / 1} F r, \quad h_{0} \approx \frac{\rho_{2 / 1}}{2} F r, \\
& e^{k \tau} \approx\left(\frac{t}{\rho_{2 / 1} F r}+1\right)^{2 \sqrt{\rho_{2 / 1}}} . \tag{41}
\end{align*}
$$

The equations (41) yield for $t \ll \rho_{2 / 1} F r$ the following approximations:

$$
\operatorname{shk\tau } \approx 2 t /\left(\rho_{2 / 1}^{3 / 2} / F r^{2}\right), \operatorname{ch} k \tau \approx 1,
$$

therefore solution of the problem in a form (40) goes to the following simplified expressions:

$$
\begin{equation*}
h \approx h_{0}+\frac{t}{\rho_{2 / 1} F r}, \quad v_{1} \approx \frac{1}{\rho_{2 / 1} F r} . \tag{42}
\end{equation*}
$$

Analysis of the simple partial limit solution (42) shows that at the beginning of the jet penetration, the depth of penetration is a linear function of time, and the velocity of penetration is nearly constant being inversely proportional to the density ratio and to the Froude number.

### 3.4 The approximate solution for the extended time interval

Similar approximation for the extended time $t \gg \rho_{2 / 1} F r$ is the following:

$$
\begin{gather*}
h \approx\left(\frac{3}{2}\right)^{2 / 3}\left[\left(\frac{2}{3}\right)^{2 / 3} \frac{\rho_{2 / 1}}{2} F r\right]^{\frac{1}{2}\left(\frac{t}{\rho_{2 / 1} F r}\right)^{2 / \sqrt{\rho_{2 / 1}}}} .  \tag{43}\\
\cdot e^{\frac{1}{\sqrt{\rho_{2 / 1}}}\left(\frac{t}{\rho_{2 / 1} F_{r} r}\right) \sqrt{\rho_{2 / 1}}}
\end{gather*}
$$

with the depth of a jet penetration growing in time.
Analysis of the solution (40) reveals an interesting feature with a jet velocity, which can be decreased if and only if

$$
\begin{equation*}
\ln \left[\left(\frac{2}{3}\right)^{2 / 3} \frac{\rho_{2 / 1}}{2} F r\right]<0, \Rightarrow F r<\frac{2}{\rho_{2 / 1}}\left(\frac{3}{2}\right)^{2 / 3} \approx \frac{2,62}{\rho_{2 / 1}} . \tag{44}
\end{equation*}
$$

The condition (44) is necessary but not satisfactory. Actually one needs to know when the jet acceleration is negative. A full penetration is determined by the condition of $v_{1}=0$, where from

$$
\operatorname{shk} \tau_{*} \ln \left[\left(\frac{2}{3}\right)^{2 / 3} \frac{\rho_{2 / 1}}{2} F r\right]=-\frac{\operatorname{chk} \tau_{*}}{\sqrt{\rho_{2 / 1}}}
$$

with a time and a depth of penetration, $\tau_{*}, h_{*}$, respectively.

Solving this equation with (41) yields

$$
h_{x}=\left(\frac{3}{2}\right)^{2 / 3}\left[\left(\frac{2}{3}\right)^{2 / 3} \frac{\rho_{2 / 1} F r}{2}\right]^{\text {chhts }} \frac{\text { chltss }}{\rho_{221} \ln [2 / 3)^{\left.2 \beta^{2} \frac{\rho_{21}}{2} F r\right]}},
$$

$$
\begin{equation*}
\gamma=-\sqrt{\rho_{2 / 1}} \ln ^{-1}\left[\left(\frac{2}{3}\right)^{2 / 3} \frac{\rho_{2 / 1}}{2} F r\right], \tag{45}
\end{equation*}
$$

$$
\begin{aligned}
& \left(\frac{t_{*}}{\rho_{2 / 1} F r}+1\right)^{2 \sqrt{\rho_{2 / 1}}}-\left(\frac{t_{*}}{\rho_{2 / 1} F r}+1\right)^{-2 \sqrt{\rho_{2 / 1}}}= \\
& =\gamma\left[\left(\frac{t_{*}}{\rho_{2 / 1} F r}+1\right)^{2 \sqrt{\rho_{2 / 1}}}+\left(\frac{t_{*}}{\rho_{2 / 1} F r}+1\right)^{-2 \sqrt{\rho_{2 / 1}}}\right]
\end{aligned}
$$

and further goes for the penetration time:

$$
\begin{gather*}
t_{*}=\left[\left(\frac{1+\gamma}{1-\gamma}\right)^{0.25 / \sqrt{\rho_{2 / 1}}}-1\right] \rho_{2 / 1} F r, \\
\left(\frac{t_{*}}{\rho_{2 / 1} F r}+1\right)^{2 \sqrt{\rho_{2 / 1}}}=\left(\frac{1+\gamma}{1-\gamma}\right)^{0.5}, \tag{46}
\end{gather*}
$$

$\operatorname{ch} k \tau_{*}=\frac{1}{2}\left\{\begin{array}{l}{\left[\frac{\sqrt{\rho_{2 / 1}} \ln \left[(2 / 3)^{2 / 3} 0.5 \rho_{2 / 1} F r\right]-1}{\sqrt{\rho_{2 / 1}} \ln \left[(2 / 3)^{2 / 3} 0.5 \rho_{2 / 1} F r\right]+1}\right]^{0.5}+} \\ {\left[\frac{\sqrt{\rho_{2 / 1}} \ln \left[(2 / 3)^{2 / 3} 0.5 \rho_{2 / 1} F r\right]-1}{\sqrt{\rho_{2 / 1}} \ln \left[(2 / 3)^{2 / 3} 0.5 \rho_{2 / 1} F r\right]+1}\right]^{-0.5}}\end{array}\right\}$

If
$(2 / 3)^{2 / 3} 0.5 \rho_{2 / 1} F r \ll 1 \Rightarrow-\ln \left[(2 / 3)^{2 / 3} 0.5 \rho_{2 / 1} F r\right] \gg 1$, $\operatorname{chk} \tau_{*} \approx 1$, then from (45) yields:

$$
\begin{equation*}
h_{*} \approx 0.5 F r \rho_{2 / 1} e^{\left.-\frac{\rho_{1 / 2}}{\ln \left[(2 / 3)^{2 / 3} 0.5 \rho_{2 / 1} F r\right.}\right]} \tag{47}
\end{equation*}
$$

## 4 Peculiarities of the jet penetration by different parameters

### 4.1 Accelerating jet ( $a_{1}>0$ )

By $t \gg \rho_{2 / 1} F r$, a simple condition for $a_{1}>0$ (positive acceleration of a jet, velocity is growing) follows from (40). Due to the correlations $\operatorname{ch} k \tau \approx \operatorname{shk} \tau \approx 0.5 e^{k \tau}$, the condition $a_{1}<0$ results as:

$$
\begin{align*}
& \ln ^{2} \beta+\left(1+1.5 / \sqrt{\rho_{2 / 1}}\right) \ln \beta+ \\
& +1 / \sqrt{\rho_{2 / 1}}\left(1+0.5 / \sqrt{\rho_{2 / 1}}\right)<0 \tag{48}
\end{align*}
$$

where $\beta=(2 / 3)^{2 / 3} 0.5 \rho_{2 / 1} F r$. The solution of the equation (48) is the following:

$$
\begin{align*}
& \quad-1-0.5 / \sqrt{\rho_{2 / 1}}<\ln \beta<-1 / \sqrt{\rho_{2 / 1}}, \Rightarrow \\
& 2(3 / 2)^{2 / 3} / \rho_{2 / 1} e^{-1 / \sqrt{\rho_{2 / 1}}}<F r<2(3 / 2)^{2 / 3} / \rho_{2 / 1} e^{-1-1 / \sqrt{\rho_{2 / 1}}} \tag{49}
\end{align*}
$$

For the density ratio $\rho_{2 / 1}=0.1$, from the inequalities (49) yields approximately:

$$
F r \in(1.07 ; 1.95)
$$

By $t \gg \rho_{2 / 1} F r$ there is a narrow range of the Froude numbers where a jet velocity may decrease. Normally velocity is growing in time if the density ratio is small because the gravity force exceeds the drag force.

### 4.1.1 Condition for the jet's velocity decrease

In general, the condition of velocity decrease follows from (40):

$$
\begin{align*}
& A^{2}(\tau)-\left(2-1 / \sqrt{\rho_{2 / 1}}\right) e^{k \tau} A(\tau)+ \\
& -4\left(1 / \sqrt{\rho_{2 / 1}}-\ln \beta\right) e^{k \tau}<0  \tag{50}\\
& A(\tau)=\left(\ln \beta+1 / \sqrt{\rho_{2 / 1}}\right) e^{2 k \tau}+ \\
& +\left(2-1 / \sqrt{\rho_{2 / 1}}\right) e^{k \tau}+1 / \sqrt{\rho_{2 / 1}}-\ln \beta \tag{51}
\end{align*}
$$

Solving the quadratic inequality (50) for the function $A(\tau)$ results in

$$
\begin{equation*}
A_{1}(\tau)<A(\tau)<A_{2}(\tau) \tag{52}
\end{equation*}
$$

where the limits of the interval are:

$$
\begin{align*}
& A_{1,2}=\left(1-0.5 / \sqrt{\rho_{2 / 1}}\right) e^{k \tau}+  \tag{53}\\
& \mp \sqrt{\left(1-0.5 / \sqrt{\rho_{2 / 1}}\right)^{2}+4\left(1 / \sqrt{\rho_{2 / 1}}-\ln \beta\right)} e^{k \tau}
\end{align*}
$$

Required $A_{1,2}(\tau)$ be the real functions, with account of (53) and (49), after some simple transformations, one can get the following condition for the Froude number:

$$
\begin{equation*}
F r \leq 2(3 / 2)^{2 / 3} e^{0.25\left(1+3 / \sqrt{\rho_{2 / 1}}+0.25 / \rho_{2 / 1}\right)} / \rho_{2 / 1} \tag{54}
\end{equation*}
$$

Thus, for $\rho_{2 / 1}=0.1$, the condition (54) gives $F r \leq 785$. Then, putting (51), (53) into (52) yields:

$$
\begin{gather*}
\left(\ln \beta+1 / \sqrt{\rho_{2 / 1}}\right) e^{2 k \tau}+\left(\gamma_{1}+\gamma_{2}\right) e^{k \tau}+ \\
+1 / \sqrt{\rho_{2 / 1}}-\ln \beta \geq 0  \tag{55}\\
\left(\ln \beta+1 / \sqrt{\rho_{2 / 1}}\right) e^{2 k \tau}+\left(\gamma_{1}-\gamma_{2}\right) e^{k \tau}+ \\
+1 / \sqrt{\rho_{2 / 1}}-\ln \beta \leq 0
\end{gather*}
$$

where are:

$$
\begin{align*}
& \gamma_{1}=1-0.5 / \sqrt{\rho_{2 / 1}} \\
& \gamma_{2}=\sqrt{\left(1-0.5 / \sqrt{\rho_{2 / 1}}\right)^{2}+4\left(1 / \sqrt{\rho_{2 / 1}}-\ln \beta\right)} \tag{56}
\end{align*}
$$

Both conditions (55) must be satisfied simultaneously (not separately!). The first one in case of

$$
\begin{equation*}
\ln \beta+1 / \sqrt{\rho_{2 / 1}}>0 \tag{57}
\end{equation*}
$$

which corresponds to the left side of (49), gives the following two solutions:

$$
\begin{gather*}
e^{k \tau} \leq B_{1}, \quad e^{k \tau} \geq B_{2},  \tag{58}\\
B_{1,2}=\frac{-\left(\gamma_{1}+\gamma_{2}\right) \mp \sqrt{\left(\gamma_{1}+\gamma_{2}\right)^{2}+4\left(\ln ^{2} \beta-\rho_{1 / 2}\right)}}{2\left(\ln \beta+\sqrt{\rho_{1 / 2}}\right)} .
\end{gather*}
$$

As far as in (58) $B_{1}<0$ is, only the second solution supposed to be real. Similarly, the other inequality in (55) has the following solution:

$$
\begin{align*}
& D_{1} \leq e^{k \tau} \leq D_{2}  \tag{59}\\
& D_{1,2}=\frac{\left(\gamma_{2}-\gamma_{1}\right) \mp \sqrt{\left(\gamma_{2}-\gamma_{1}\right)^{2}+4\left(\ln ^{2} \beta-\rho_{1 / 2}\right)}}{2\left(\ln \beta+\sqrt{\rho_{1 / 2}}\right)}
\end{align*}
$$

### 4.1.2 Conditions for the Froude number

When $|\ln \beta|>\sqrt{\rho_{1 / 2}}$, both $B_{1,2}$ and $D_{1,2}$ are the real values. And this is the sufficient but not the necessary condition. It is satisfied by small, as well as by large values of the Froude number:

$$
\begin{align*}
& F r>2(3 / 2)^{2 / 3} e^{1 / \sqrt{\rho_{2 / 1}}} / \rho_{2 / 1}, \quad \text { or } \\
& F r<2(3 / 2)^{2 / 3} e^{-1 / \sqrt{\rho_{2 / 1}}} / \rho_{2 / 1} \tag{60}
\end{align*}
$$

For $\rho_{2 / 1} \ll 1$ considered here, $\gamma_{1}<0$, therefore it goes to $\left(\gamma_{1}+\gamma_{2}\right)^{2}<\left(\gamma_{2}-\gamma_{1}\right)^{2}$. When $D_{2}$ is real value, $B_{2}$ is always real. That is why more simple condition than (60) is considered when $D_{2}$ is real value: $\left(\gamma_{1}+\gamma_{2}\right)^{2}+4\left(\ln ^{2} \beta-\rho_{1 / 2}\right) \geq 0$. Then it goes to the simpler condition than (60):
$2\left(\ln ^{2} \beta-\ln \beta+0.5 \sqrt{\rho_{1 / 2}}-0.875 \rho_{1 / 2}+0.5\right) \geq$
$\geq\left(0.5 \sqrt{\rho_{1 / 2}}-1\right) \sqrt{\left(1-0.5 \sqrt{\rho_{1 / 2}}\right)^{2}+4\left(\sqrt{\rho_{1 / 2}}-\ln \beta\right)}$

For $\rho_{1 / 2} \geq 4$ the right side of the inequality is positive. The left side is positive if

$$
\ln ^{2} \beta-\ln \beta+0.5 \sqrt{\rho_{1 / 2}}-0.875 \rho_{1 / 2}+0.5 \geq 0
$$

where from following $\ln \beta<(\ln \beta)_{1}$ or $\ln \beta>(\ln \beta)_{2}$,

$$
(\ln \beta)_{1,2}=0.5 \mp \sqrt{0.875 \rho_{1 / 2}-0.5 \sqrt{\rho_{1 / 2}}-0.25}
$$

which is real by $\rho_{1 / 2} \geq 4$. Therefore, taking into account the previous condition $\ln \beta>-\sqrt{\rho_{1 / 2}}$, one can come to the requirements:

$$
\ln \beta<0.5-\sqrt{0.875 \rho_{1 / 2}-0.5 \sqrt{\rho_{1 / 2}}-0.25}
$$

or
$\ln \beta>0.5+\sqrt{0.875 \rho_{1 / 2}-0.5 \sqrt{\rho_{1 / 2}}-0.25}$,
where from:

$$
\begin{gather*}
(7 / 36)^{2} \approx 0,04 \leq \rho_{2 / 1} \leq 0,25 \\
F r<2 \rho_{1 / 2}(1.5)^{2 / 3} e^{0.5-\sqrt{0.875 \rho_{1 / 2}-0.5 \sqrt{\rho_{1 / 2}}-0.25}}, \tag{61}
\end{gather*}
$$

or

$$
F r>2 \rho_{1 / 2}(1.5)^{2 / 3} e^{0.5+\sqrt{0.875 \rho_{1 / 2}-0.5 \sqrt{\rho_{1 / 2}}-0.25}}
$$

For $\rho_{2 / 1}=0.1$, from (61) yields solution $F r<\sim 3,12$, or $F r>\sim 600$. Comparing the last condition with the request of real values $A_{1,2}$, one can get: $600<F r<785$. It is very narrow gap by the Froude numbers (except the low Froude numbers) when the velocity decreases with time.

When (58) is not satisfied, the case is not interesting because it requires too small Froude numbers determined by the last condition (61), e.g., for $\rho_{2 / 1}=0.1$ there is $F r<\sim 1$.

Due to $D_{1}<0$, the solution (60) changes to the following one: $0<e^{k \tau} \leq D_{2}$, where the left side is always satisfied. Therefore the common solution (56) yields: $B_{2} \leq e^{k \tau} \leq D_{2}$, where from with account of (57)-(60) and the last correlation of (41), as well as the expressions for $\beta$ from above, one can compute the temporal interval $t_{1} \leq t \leq t_{2}$ corresponding to the case of a jet velocity decrease ( $a_{1}<0$, decelerating jet flow):

$$
\begin{aligned}
& t_{1}=\rho_{2 / 1} F r\{-1+ \\
& \left.+\left[\frac{\left(\gamma_{2}-\gamma_{1}\right)+\sqrt{\left(\gamma_{2}-\gamma_{1}\right)^{2}+4\left(\ln ^{2} \beta-\rho_{1 / 2}\right)}}{2\left(\ln \beta+\sqrt{\rho_{1 / 2}}\right)}\right]^{0.5} \sqrt{\rho_{\rho_{1 / 2}}}\right\} \\
& t_{2}=\rho_{2 / 1} F r\{-1+ \\
& \left.\left[\frac{-\left(\gamma_{2}+\gamma_{1}\right)+\sqrt{\left(\gamma_{2}+\gamma_{1}\right)^{2}+4\left(\ln ^{2} \beta-\rho_{1 / 2}\right)}}{2\left(\ln \beta+\sqrt{\rho_{1 / 2}}\right)}\right]^{0.5 \sqrt{\rho_{1 / 2}}}\right\}
\end{aligned}
$$

### 4.2 The bifurcation points of the jet

The non-linear solution thus obtained is an exact analytical solution for a solid rod penetration into the pool and for some initial part of a jet penetration before remarkable growing of its radius. It might be used as approximate step-by-step solution for a jet penetration into a pool for small temporal intervals correcting the jet radius from one to another one. Therefore it is crucial to estimate an evolution of the jet's radius to get an idea how to correct solution aiming at good correspondence with the experimental data. With this purpose, the Bernoulli equation and the mass conservation equation are considered for the jet in the following form:

$$
\begin{gathered}
S_{1}\left[\left(\rho_{1}-\rho_{2}\right) h g+0,5 \rho_{1} v_{1}^{2}\right]=0,5 \rho_{1} u_{0}^{2} S_{0} \\
\rho_{1} v_{1} S_{1}=\rho_{1} u_{0} S_{0}
\end{gathered}
$$

where $S$ is the area of the jet's cross section. Index 0 denotes the initial state while the index 1 denotes some current state afterwards.

### 4.2.1 dimensionless conservation equations

In a dimensionless form, retaining the same symbols:

$$
\begin{gather*}
S_{1}\left[2 h\left(1-\rho_{2 / 1}\right) / F r+v_{1}^{2}\right]=1 \\
S_{1} v_{1}=1 \tag{63}
\end{gather*}
$$

The equation array (63) has the following solution:

$$
\begin{gather*}
S_{1}=\frac{F r}{4 h\left(1-\rho_{2 / 1}\right)}\left[1 \pm \sqrt{1-8 h\left(1-\rho_{2 / 1}\right) / F r}\right] \\
v_{1}=1 / S_{1} \tag{64}
\end{gather*}
$$

### 4.2.2 Bifurcation point

There are two possible solutions for the jet radius with the point of bifurcation:

$$
h=\frac{F r}{8\left(1-\rho_{2 / 1}\right)}
$$

After this point the solution (64) does not exist anymore in real numbers, therefore the jet can change its solution abruptly between these two available solutions.

The jet starts penetration into the pool with initial cross-sections, thus, $S_{1}=1$. Analysing the equation (64) one can note that for a small penetration depth or, more common, $8 h\left(1-\rho_{2 / 1}\right) \ll F r$, it goes to:

$$
S_{1} \approx 1 \text { or } S_{1} \approx \frac{F r}{2 h\left(1-\rho_{2 / 1}\right)} \gg 1
$$

There is no reason for a jet to become abruptly from the section area 1 to the bigger one because the jet momentum directs mainly along its axis. But further on, due to instability causing by the free surface perturbations and due to a loss of momentum, the jet area may change at any moment.

Strictly saying, it requires complete instability and bifurcation analysis, therefore it is a subject of a separate paper. Here only some estimation has been done for the moment.

### 4.2.3 Parameters of a jet with bifurcation

From $S_{1}=1$ the jet should become to $S_{1}=2$ at the point

$$
h_{1}=\frac{F r}{8\left(1-\rho_{2 / 1}\right)}=\frac{1}{8 R i},
$$

when further existence of the two possible jet's radiuses is impossible. Here $R i$ is the Richardson number (the ratio between the momentum and buoyancy forces of a jet).

Substituting $S_{1}=2$ into the last expression (63) gives $v_{1}=0,5$. The jet is going from $h=h_{0}$ to $h_{1}=\frac{1}{8 R i}$ and during this time its radius is growing from 1 to $r_{1}=\sqrt{2}$, when the jet velocity becomes $v_{1}=0,5$, e.g. for the density ratio 0.1 the total depth of a jet penetration into a pool up to this point is computed as

$$
h_{0}+h_{1} \approx 5,5+13,9 \approx 19,4
$$

From the equation (64) a jet cross-section at the depth of penetration of $h=h_{0}$ is as follows:

$$
\begin{equation*}
S_{1}=0,5 \rho_{1 / 2}\left(1 \pm \sqrt{1-4 \rho_{2 / 1}},\right) \tag{65}
\end{equation*}
$$

where from for the density ratio 0.1 follows

$$
S_{1} \approx 1,15, \quad r_{1} \approx 1,07, \quad v_{1} \approx 0,87
$$

or

$$
\begin{equation*}
S_{1} \approx 8,87, \quad r_{1} \approx 2,98, \quad v_{1} \approx 0,11, \tag{66}
\end{equation*}
$$

so that the first set of parameters (66) is close to the assumptions made above, while the other set of parameters is a possible solution, which may occur abruptly at the point of bifurcation $h=h_{1}$ due to an instability of the jet when any regular solution, as it is shown by (64), does not exist.

### 4.2.4 Basic features of a jet penetration into a pool

The phenomenon of a jet penetration in a pool accounting the results obtained and the experimental data presented in Fig. 1 seems to be as follows. First a jet penetrates into a pool at the distance $h_{0}$ determined by the initial length of a jet, the Froude number and the density ratio. In case of a long jet (as well as the jet permanently spreading out of the nozzle) the initial penetration length is determined by the Froude number and the density ratio.

Then jet is going with a slight increase of its radius till $h_{1}$, which represents the bifurcation point.. After this bifurcation point, the jet is sharply enlarged and goes further with a nearly constant radius. Applying the solution obtained to those parts with their own initial data, the whole jet might be computed based on the analytical solution got here.

## 5 Correspondence of the model to experimental data

To validate the model developed and the analytical solution obtained, the computed penetration depth of a jet had been compared to experimental data from the literature.

The maximum penetration depth $h$ from the nonlinear analytical model for a continuous jet and for a finite jet of the length $H$ compared to the experimental data [8, 17] are given in Fig. 3 and Fig. 4, correspondingly.

The dark bands (trust regions) in the Figs 3, 4 include the region between the upper line corresponding to the experimental data [8] and the bottom line corresponding to the experimental data of the work [17].

### 5.1 The results by the model without account of bifurcation point

The data obtained by model for the continuous jet are presented in Fig. 3, while the data by the finite
jet are drawn in Fig. 4. First of all Fig. 3 illustrates that the penetration depth increases with a decrease of the pool-to-jet density ratio.


Fig. 3. Maximum penetration depth $h$ by the non- linear analytical model vs experimental data $[8,17]$ for continuous jet


Fig. 4. Maximum penetration depth $h$ by the non- linear analytical model vs experimental data $[8,17]$ for finite jet

Although the idealistic assumptions were employed in the analytical model, for the continuous jet, the solution showed reasonable match with the experimental data until the Froude numbers up to 300 , in the wide range of the density ratio (up to ten times).

However, the solution strongly overpredicted the penetration depth after the Froude numbers over 300
(approximately), the higher density ratio was, the more inconsistency with experimental data was observed. For instance for the density ratio $\rho_{2 / 1}=1.9$ the results by the model obtained were out of the trust region approximately at $\mathrm{Fr}=100$ while for the $\rho_{2 / 1}=9.4$ the results by the model leaved the trust region approximately at $\mathrm{Fr}=300$.

The correspondence of the presented results and experimental data was good despite of the model that was not accounted for the jet radius evolution with a penetration depth, which would decrease the penetration depth due to increase of the drag force. Then increase of the velocity (correspondingly increase of the Froude number) caused the air entrainment after the velocity threshold, which also was not taken into account (and might be an additional reason for the jet expansion and consequently for the growing drag force). This caused an additional shortening of a penetration depth. Therefore an account of the above-mentioned additional factors would improve the model. Presently analytical solution showed good results important for the model validation.

For the finite jet, the Fig. 4 shows the jet penetration depths estimated from the analytical solution in the terms of a jet length $H$ and of a density ratio. As expected, the solutions demonstrated the increase of a penetration depth with a jet length $H^{*}$.

For $\rho_{2 / 1}=9.4$, the estimated penetration depth remained nearly constant at a longer jet. However, the Fr-term dominated at higher Fr. As expected, a jet penetrated deeper into a liquid pool for the lower pool/jet density ratio $\rho_{2 / 1}\left(\rho_{\mathrm{p}} / \rho_{\mathrm{j}}\right)$.

The asymptotic solution shown in Fig. 4 took over the prediction by the Fr-dominant penetration when $h_{*} \gg H^{*}$. It was noted that for the low Fr , the estimated penetration depths was longer than those estimated by the continuous jets (Fig. 3). It was resulted from the definition of the continuous jet, i.e., $H^{*}=\mathrm{Fr} / 2$. Therefore for low Fr the actual jet penetration depth for the continuous jet was smaller than those for $H^{*}=25$ and 50.

A dimensionless mean jet penetration depth scaled with the jet radius can be expressed in terms of the Richardson number $R i: h=C / R i^{b}$, where $C, b$ are the constants. The correlation is normally used in the form $h=C / \sqrt{R i}$, where $C=4$ corresponds to the closest fit with the Turner's results [3].

Fig. 5 shows the comparison of the experimental results [3, 23] with analytical solution represented
as a curve 1 for the $\rho_{2 / 1}=9.4$ and 2 - for $\rho_{2 / 1}=1.9$, respectively. Evidently analytical solution 1 the best fitted the Turner's data [3] and two groups of the data [23] for the range of the Richardson numbers approximately $R i \approx 0.03-1.0$.

For $R i<0.03$ the analytical solution went far away from the data. Another solution $2\left(\rho_{2 / 1}=1.9\right)$ fitted well both experimental results [3,23] only in a narrow region by Richardson number around $R i \approx 0.004$.

For smaller $R i$ it overpredicted the experimental data - the less $R i$, the higher overprediction. In the range $R i=0.004-0.01$ the analytical solution differed from the experimental data [3, 23] mostly less than $30 \%$ (only close to $R i=0.01$ it is about $70 \%$ ) while the maximal measurement errors of the experimental data exceeded $100 \%$ in some points.


Fig. 5. Maximum penetration depth $h$ against the Richardson number by analytical model against experimental data [3, 23].

### 5.2 The results by the model with account of bifurcation point

As it was discussed before and shown in Fig. 1, real jet is penetrating a pool similar to the peculiarities got by the model with account of the bifurcation point.

The results of computations by the model described are presented in Fig. 6, where from may be clearly observed that after $\mathrm{Fr}=100$ the calculation with account of bifurcation point give much shorter length of a jet penetration into a pool, so that by $\mathrm{Fr}=300$ the difference is nearly $50 \%$.


Fig. 6. Comparison of the model calculations with account of bifurcation

Another illustration of these peculiarities is given in Fig. 7 and Fig. 8, Fig. 9 presenting the numerical solution of the full Navier-Stokes equation array for the jet penetrating pool and experimental data borrowed from [17]:


Fig. 7. Jet penetration with initial velocity $4 \mathrm{~m} / \mathrm{s}$ : experimental and numerical data

Numerical simulation was performed with the computer code Casper [17]. Both data, experimental
and numerical, are presented in Figs 7-9 for the corresponding moments of time (in ms).


Fig. 8. Experimental data by the jet penetration into the pool: with initial velocity $4 \mathrm{~m} / \mathrm{s}, 6 \mathrm{~m} / \mathrm{s}, 9 \mathrm{~m} / \mathrm{s}$, respectively, from the top to the bottom picture

Presented data have evidently shown that the jet penetration is really going according to the model developed here, with one critical bifurcation point.


Fig. 9. Numerical simulation by computer code Casper [17].

Further investigations are needed for a clarification of the diverse factors influencing the features of a jet penetration into a pool in the different flow regimes in a wide range of parameters.

A general conclusion is that all experimental data of different authors and analytical solution suffer on restricted ranges of parameters where they are valid, therefore presently the results cannot be generalized.

## 6 Bending instability and breakup length of the thin jets

Special case of a jet penetration into a pool, namely bending instability and breakup phenomenon of a thin liquid jet penetrating into another fluid has been studied in [16, 24].

Although remarkable progress has been made in explaining the effects of surface tension (Weber number) and viscosity (Reynolds number) on jet breakup behaviour, little attention has been given to the thin jets (i.e. the ratio of the characteristic transverse size to the longitudinal is small) penetrating into pool of fluid. Especially, in the case
when inertia, drag and buoyancy are the dominating fluid forces [25].

Several analytical and experimental studies have been conducted in the past to obtain the breakup length of buoyant jet penetrating into another fluid. Most of these studies were related to the injection of gas jet into another fluid as in fluidized bed, e.g. Yang and Kearins [26] and Blake et al [27].

The mechanism associated with the interaction of jet and ambient fluid is predominant for highvelocity laminar jets, which breakup as a result of growing bending disturbances on the jet axis.

Theoretical studies on the dynamics of bending disturbances of liquid jets were initiated by Weber [28] and continued further by other investigators (see for example [14], where is also bibliography on the subject).

Quasi-one-dimensional equations were obtained by Entov and Yarin [29] for an arbitrary parameterisation of a jet and successfully applied to predict breakup length of a buckling jet.

The objectives of the works $[16,24]$ were to determine the penetration behaviours and breakup length of a thin jet penetrating into a fluid pool. There are evidences that the buoyant jet bending mechanism seem to occur and eventually dominate the breakup behaviour. The jet breakup behaviour and the jet breakup length was obtained and compared to the experimental data available in the literature.

### 6.1 Mathematical model's formulation

Study of the bending jet decay during its penetration into a pool was performed according to the scheme shown in Fig. 10:


Fig. 10. Scheme of a bending jet in a pool

The flow of fluid jet into another fluid can be characterized by a set of dimensionless numbers. The density ratio ( $\rho_{21}=\rho_{2} / \rho_{1}$ ) is important as it determines the penetration rate of the jet head and plays an important role in the jet instability (here $\rho_{2}, \rho_{1}$ is the density of the fluid pool and the jet, respectively).

In the case of large Reynolds and Weber numbers, the inertia force is dominating compared to viscosity and capillary forces. For a jet discharging vertically under the influence of gravity into a fluid pool, the density ratio and Froude number may be dominant parameters, which determine the jet instability. In such condition the moment and moment of momentum equations for the jet can be written on the following form [28, 29]:

$$
\begin{align*}
& \rho_{1} f_{0} \partial V_{n} / \partial t=\partial Q_{n} / \partial s-\kappa Q_{b}+q_{n}+\left(\rho_{1}-\rho_{2}\right) g_{n}, \\
& \rho_{1} f_{0} \partial V_{b} / \partial t=\partial Q_{b} / \partial s+\kappa Q_{n}+q_{b}+\left(\rho_{1}-\rho_{2}\right) g_{b}, \\
& \rho_{1} I \partial / \partial t\left(\partial V_{b} / \partial s+\kappa V_{n}\right)=-\partial M_{n} / \partial s+  \tag{67}\\
& +\kappa M_{b}+Q_{b}+k\left(\rho_{2}-\rho_{1}\right) g_{\tau} I, \\
& \quad \rho_{1} I \partial / \partial t\left(\partial V_{n} / \partial s-\kappa V_{b}\right)=\partial M_{b} / \partial s+ \\
& +\kappa M_{n}+Q_{n}+k\left(\rho_{2}-\rho_{1}\right) g_{b} I
\end{align*}
$$

where the hydrodynamic $\left(q_{i}\right)$ and buoyancy $\left(g_{i}\right)$ forces are:
$q_{n}=-\rho_{2} U_{0}^{2} f_{0} \chi^{2} / a_{0}^{2} \exp (\gamma t)$.
$\cdot\left[A^{2} \cos ^{2}\left(\chi s / a_{0}\right)+B^{2} \sin ^{2}\left(\chi s / a_{0}\right)\right]^{\frac{1}{2}}$,
$g_{n}=f_{0} g \chi / a_{0} \exp (\gamma t)$.
$\cdot\left[A^{2} \sin ^{2}\left(\chi s / a_{0}\right)+B^{2} \cos ^{2}\left(\chi s / a_{0}\right)\right]^{\frac{1}{2}}$,
$g_{b}=q_{b}=0, \quad \chi=2 \pi a_{0} / l$.
Here, $a_{o}$ and $U_{o}$ are the initial radius and velocity of the jet, respectively, $f_{0}=\pi a_{0}^{2}$ is the area and $I=1 / 4 \pi a_{0}^{4}$ is the moment of inertia of the jet.

The variables $Q, M$ and $\chi$ represent the shearing force, the moment of stresses in the jet cross-section and the jet perturbation length, respectively. The variable $k$ is the curvature and $\kappa$ is the torsion of the jet. The position of a point (or a
liquid particle) in the jet is determined by three parameters: y, z, and s, which serve as the coordinates in a moving frame of curvilinear (nonorthogonal in the case $\kappa \neq 0$ ) coordinate system.

### 6.2 Modelling of the bending jet's perturbations

The distributed forces $q_{i}$ and $g_{i}$ are calculated and applied to the axis in a similar way as given in references [28, 29]. The jet's axes are parametrized and the equations along the axes are written in the following form

$$
\begin{equation*}
\eta=H(s, t), \quad \zeta=Z(s, t), \tag{69}
\end{equation*}
$$

where H and Z are the displacements of the axis in the directions $O_{1} \eta$ and $O_{1} \zeta$, respectively at $\xi=s$.

Only projection of the momentum and the moment-of-momentum equations to the normal and binormal axis are retained from equation (67), i.e. only the bending perturbations are considered. Those equations describe small bending perturbations of the liquid jet penetrating into the fluid pool, neglecting the change in jet radius. These disturbances have a growth rate $\gamma$.

A projection of equation (67) to the tangent of the axis describes the growth of small axially symmetric disturbances of the jet [28, 29]. The perturbations on the jet axis are considered to have following form [28]:

$$
\begin{align*}
& H=A \exp (\gamma t) \cos \left(\chi S / a_{0}\right), \\
& Z=B \exp (\gamma) \sin \left(\chi S / a_{0}\right), \tag{70}
\end{align*}
$$

where the curvature and torsion are respectively:
$k=\chi^{2} / a_{0}^{2} \exp (\gamma t)\left[A^{2} \cos ^{2}\left(\chi s / a_{0}\right)+\right.$
$\left.+B^{2} \sin ^{2}\left(\chi S / a_{0}\right)\right]^{\frac{1}{2}}$
$\kappa=\chi / a_{0} A B\left[A^{2} \cos ^{2}\left(\chi S / a_{0}\right)+B^{2} \sin ^{2}\left(\chi S / a_{0}\right)\right]^{\frac{1}{2}}$
The bending jet perturbation and the moment $M_{b}$ can be written as:

$$
\begin{aligned}
& \sqrt{H^{2}+Z^{2}} \\
& M_{b}=I\left[\rho_{2} U_{0}^{2} k+\left(\rho_{2}-\rho_{1}\right) g \sqrt{H_{s}^{2}+Z_{s}^{2}}\right]
\end{aligned}
$$

where are:

$$
\begin{aligned}
H_{s} & =-A \chi / a_{0} \exp (\gamma t) \sin \left(\chi s / a_{0}\right), \\
Z_{s} & =B \chi / a_{0} \exp (\gamma t) \cos \left(\chi s / a_{0}\right) .
\end{aligned}
$$

The velocity normal and binormal to the jet axis can be obtained by differentiating (70) with respect to a time:

$$
\begin{align*}
& V_{n}=\sqrt{H_{t}^{2}+Z_{t}^{2}}=-\gamma \exp (\gamma t) . \\
& \cdot\left[A^{2} \cos ^{2}\left(\chi s / a_{0}\right)+B^{2} \sin ^{2}\left(\chi s / a_{0}\right)\right]^{\frac{1}{2}}  \tag{72}\\
& V_{b}=0 .
\end{align*}
$$

Substituting (68)-(72) into the equation array (67), leads to the following set of equations for small perturbations on the thin jet:

$$
\begin{aligned}
& Q_{n}=\rho_{1} I \partial^{2} V_{n} / \partial s \partial t-\partial M_{b} / \partial s \\
& Q_{b}=\rho_{1} I \partial\left(\kappa V_{n}\right) / \partial t-\kappa V_{b} \\
& \partial Q_{b} / \partial s=-\kappa Q_{n} \\
& \rho_{1} f_{0} \partial V_{n} / \partial t=\partial Q_{n} / \partial s-\kappa Q_{b}+q_{n}+g_{n}\left(\rho_{1}-\rho_{2}\right) .
\end{aligned}
$$

Here in this case the momentum normal to the axis, $M_{n}$, is assumed to be negligible. Inserting $Q_{n}$ and $Q_{b}$ from the first two equations of (73) into the third and fourth, makes third equation an identity and the forth equation for the moment of momentum in the normal direction becomes

$$
\begin{align*}
& \rho_{1} I \partial^{2}\left(\kappa V_{n}\right) / \partial s \partial t-\partial\left(\kappa M_{b}\right) / \partial s+  \tag{74}\\
& +\rho_{1} \kappa I \partial^{2} V_{n} / \partial s \partial t-\kappa \partial M_{b} / \partial s=0 .
\end{align*}
$$

### 6.3 Bending instability analysis

Stability of the fluid jet due to linear perturbations on the jet's axis was examined and a jet breakup length was derived that relates the jet coherent length with dimensionless parameters.

Substituting equation (71), (72) and $M_{b}$ into (74) and retaining only terms of first order in $\chi$, the following equation for the perturbation growth rate is obtained:

$$
\begin{equation*}
\gamma^{2}=\left(\rho_{21}-1\right) g\left(\chi / a_{0}\right)(B / A) \tag{75}
\end{equation*}
$$

Equation (75) shows that bending jet instability is observed if and only if the density of the pool is larger than density of the jet ( $\rho_{21}>1$ ).

Assuming that the bending jet perturbations are of the order of few diameters of the jet: $\sqrt{Z^{2}+H^{2}}=\Delta a_{0}$, for long wave perturbations $\cos \left(\chi S / a_{0}\right) \approx 1$ we obtain $\gamma_{*} t_{*}=\ln \left(\Delta a_{0} / A\right)$, $t_{*}=1 / \gamma_{*} \ln \left(\Delta a_{0} / A\right)$. The jet breakup length is then,

$$
L_{*} / a_{0}=U_{0} t_{*}=\ln \left(\frac{\Delta a_{0}}{A}\right) \frac{U_{0}}{a_{0} \sqrt{B / A\left(\rho_{21}-1\right) g \chi_{*} / a_{0}}}
$$

which can be written as

$$
\begin{equation*}
L_{*} / a_{0}=\delta_{1} \sqrt{F r} / \sqrt{\delta_{2}\left(\rho_{21}-1\right)} \tag{76}
\end{equation*}
$$

Here $F r=U_{0}^{2} /\left(g a_{0}\right) \quad$ is the Froude number, $\delta_{1}=\ln \left(\Delta a_{0} / A\right)$ is a constant, which depends on the initial perturbation level, $\delta_{2}=\chi_{*} B / A$ is a constant with regard to the wavelength and initial perturbation level of the two coordinates in the plane perpendicular to the jet curvilinear axis.

Now, taking into account the terms of second order in $\chi$ in equation (75), the growth rate $\gamma$ of the bending jet perturbations is got as follows:

$$
\begin{equation*}
\gamma= \pm \frac{4 U_{0} \chi}{a_{0} \sqrt{4+\chi^{2}}} \sqrt{\rho_{21}+\frac{B}{A} \frac{\rho_{21}-1}{\chi F r}} \tag{77}
\end{equation*}
$$

where positive growth rate $\gamma$ corresponds to a jet instability mode and negative to a stable perturbation mode.

From the equation (77) the optimal bending perturbation length can be found, which determines the jet's decay on the fragments. Differentiating the expression (77) as a function of the jet's perturbation length $\chi$ yields to the extreme point at

$$
\begin{equation*}
\chi_{*}=4\left[\frac{F r}{1-\rho_{12}} \frac{B}{A}+\sqrt{\left(\frac{F r}{1-\rho_{12}}\right)^{2}\left(\frac{A}{B}\right)^{2}+\frac{1}{4}}\right] \tag{78}
\end{equation*}
$$

where $\rho_{12}=1 / \rho_{21}\left(\chi_{*}\right.$ is the most unstable perturbation length). Equation (78) was obtained by making the following assumptions:

$$
U_{0} \neq 0, \quad \chi_{*} \neq 0, \quad \rho_{21}+\frac{\rho_{21}-1}{\chi_{*} F r} \frac{B}{A} \neq 0
$$

Investigation of the equations (77) and (78) showed that bending jet's decay is possible in all cases for $\rho_{21}>1$. When $\rho_{21}<1$, the bending jet perturbations are growing in time if and only if $\chi>\frac{\rho_{12}-1}{F r} \frac{B}{A}$. The other ones do not have to grow with time (oscillations but not jet decay). Here it should be marked that $\chi$ is considered small. Taking into account the above, the jet breakup length can be obtained on the following form:

$$
\begin{equation*}
\frac{L_{*}}{a_{0}}=\frac{\delta_{1}}{2 \sqrt{\chi}} \frac{\sqrt{F r}}{\sqrt{\rho_{21} \chi F r+\left(\rho_{21}-1\right) B / A}} . \tag{79}
\end{equation*}
$$

In case of $\rho_{21}=1$ (equal densities of the jet and pool) a perturbation length does not exist by this model. In this case solution of the higher order terms in $\chi$ should be considered.

Using equations (77) and (78) for the most unstable bending jet perturbations the jet breakup length was obtained as

$$
\begin{equation*}
\frac{L_{*}}{a_{0}}=\frac{\delta_{1}}{4} \frac{\sqrt{4+\chi_{*}^{2}}}{\sqrt{\left[\rho_{21} \chi_{*}+B / A\left(\rho_{21}-1\right) / F r\right] \chi_{*}}} \tag{80}
\end{equation*}
$$

### 6.4 Discussion of the results by bending jet's instability analysis

A few limiting cases of the equations (79) and (80) were considered and analyzed for the governing parameters, density ratio and Froude number.

In a case of a small Froude number $F r \ll 1$ (small jet velocity or large jet diameter) the following estimation for an optimal perturbation length was obtained from the equation (79), $\chi_{*} \approx 2$. In this case the jet breakup length is

$$
\begin{equation*}
\frac{L_{*}}{a_{0}}=\frac{\delta_{1}}{2} \frac{\sqrt{F r}}{\sqrt{\left(\rho_{21}-1\right) B / A+2 \rho_{21} F r}} \tag{81}
\end{equation*}
$$

and it follows for $\rho_{21} \gg 1: \frac{L_{*}}{a_{0}}=\delta_{2} \sqrt{\rho_{12} F r}$ where $\quad \delta_{2}=\frac{\delta_{1}}{2} \sqrt{\frac{A}{B}}$. The other limiting case,
( $\left.\rho_{21}-1\right) \sim 1$ gives $L_{*} / a_{0}=\delta_{2} \sqrt{F r /\left(\rho_{21}-1\right)}$. The length of bending jet decay is predominantly determined by the square root of the Froude number. This is in good agreement with Blake et al. [27] and Yang and Keairns [26] and some other studies analyzed in $[16,24]$ who obtained the following experimental correlation for the breakup length of jet penetrating into a fluidized bed:

$$
\begin{equation*}
\frac{L_{*}}{a_{0}} \approx 9.16 \sqrt{\frac{F r}{1-\rho_{21}}}=\frac{9.16}{\sqrt{R i}} \tag{82}
\end{equation*}
$$

where $R i=\frac{1-\rho_{21}}{F r}$ is the Richardson number. Comparing the equations (81) and (82) one can obtain the constant in the theoretical solution (81) as shown in Figs 11, 12. In case (82) the jet decay depends only on the Richardson number.


Fig. 11. The results of computation by (81) against experimental data [26]


Fig. 12. The results of computation by (81) against experimental data [27]

It is important to note that well-known experimental Saito correlation [8] is got here as one limit case from the analytical solution obtained by mathematical model.

## 7 Conclusion

Analyses on the penetration phenomena of a jet into another liquid with various densities at isothermal condition were performed and compared with the data from literature.

The non-linear analytical models for the continuous and finite jets to predict the maximum penetration of the plunging jet were developed and reasonably described the characteristics of the penetration behaviours.

The general behaviours of a jet consisted of a surface cavity of a pool liquid by the initial mechanical impact of the jet, air pocket formation during the penetration, radial bottom spreading of the jet and entrained air and interfacial instability between the pool liquid and entrained air must be taken into account to further improvement of the model.

Presently, an analytical solution obtained was accurate for the solid rod penetration into a liquid pool and is approximate for the jet penetration into a pool of other liquid.

## References:

[1] X1. R. J. Eichelberger, Experimental test of the theory of penetration by metallic jets, Journal of Applied Physics, Vol.27, No.1, 1956, pp. 6368.
[2] X2. M. I. Gurevich, The theory of ideal liquid jets, Moscow: Nauka, 1961 (In Russian).
[3] X3. J. S. Turner, Jets and plumes with negative or reversing buoyancy, Journal of Fluid Mechanics, Vol.26, No.4, 1966, pp. 779-792.
[4] X4. M.A. Lavrent'ev and B. V. Shabat, The problems of hydrodynamics and their mathematical models, Moscow, Nauka, 1973, (In Russian).
[5] X5. J. Carleone, R. Jameson and P. C. Chou, The tip origin of a shaped charge jet, Propellants and Explosives, Vol.2, No.6, 1977, pp. 126-130.
[6] X6. E. Hirsch, A formula for the shaped charge jet breakup-time, Propellants and Explosives, Vol.4, No.5, 1979, pp. 89-94.
[7] X7. M.L. Corradini, B.J. Kim, M.D. Oh, Vapor explosions in light water reactors: A review of theory and modeling, Progress in Nuclear Energy, Vol.22, No.1, 1988, pp. 1-117.
[8] X8. M. Saito, et al. Experimental study on penetration behaviors of water jet into Freon-11 and Liquid Nitrogen, ANS Proceedings, Natl. Heat Transfer Conference, Houston, Texas, USA, July 24-27, 1988.
[9] X9. R.W. Cresswell, R.T. Szczepura, Experimental investigation into a turbulent jet with negative buoyancy, Physics of Fluids, A5, No.11, 1993, pp. 86-107.
[10] X10. M. Epstein, H.K. Fauske, Steam film instability and the mixing of core-melt jets and water. ANS proceedings. National heat transfer conference. Aug. 4-7. Denver, Colorado, 1985.
[11] X11. D. F. Fletcher, The particle distribution of solid melt debris from molten fuel-coolant interaction experiments. Nuclear Engineering and Design, No.105, 1988.
[12] X12. D.F. Fletcher, A. Thyagaraja, The CHYMES mixing model, Progress in Nuclear Energy, Vol.26, No.1, 1991, pp. 3161.
[13] X13. S.A. Kinelovsky and K.K. Maevsky, On the cumulative jet penetration into hard plate, Journal of Applied Mathematics and Technical Physics, No.2, 1989, pp. 97-105 (In Russian).
[14] X14. A.L. Yarin, Free liquid jets and films: hydrodynamics and rheology, Longman Scientific \& Technical, Haifa, 1993.
[15] X15. D. Goldman, Y. Jaluria, Effect of opposing buoyancy on the flow in free and wall jets, Journal of Fluid Mechanics, Vol.166, 1986, pp. 41-56.
[16] X16. H.O. Haraldsson, I.V. Kazachkov, T.N. Dinh and B.R. Sehgal, Analysis of thin jet breakup depth in immiscible fluids, Abstracts of the $3^{\text {rd }}$ International Conference on Advances in Fluid Mechanics, 24-26 May Montreal, Canada, 2000.
[17] X17. H.S. Park, I.V. Kazachkov, B.R. Sehgal, Y. Maruyama and J. Sugimoto, Analysis of Plunging Jet Penetration into Liquid Pool in Isothermal Conditions, ICMF 2001: Fourth International Conference on Multiphase Flow, New Orleans, Louisiana, U.S.A., May 27 - June 1, 2001.
[18] X18. F. Bonetto, D. Drew and R.T. Lahey, Jr., The analysis of a plunging liquid jet-air entrainment process, Chemical Engineering Communications, Vol.130, 1994, pp. 11-29.
[19] X19. A.M. Lezzi, A. Prosperetti, The stability of an air film in a liquid flow, Journal Fluid Mechanics, Vol.226, 1991, pp. 319-347.
[20] X20. P. Lara, Onset air entrainment for a water jet impinging vertically on a water surface, Chemical Engineering Sciences, Vol.34, 1979, pp. 1164-1165.
[21] X21. G.K. Batchelor, An Introduction to Fluid Dynamics, New York: Cambridge University Press, 1967.
[22] X22. P.L. Sachdev, Non-linear ordinary differential equations and their applications, Marcel Dekker, Inc., 1991.
[23] X23. J. Dahlsveen, R. Kristoffersen and L. Saetran, Jet mixing of cryogen and water. In Turbulence and Shear Flow Phenomena, $2^{\text {nd }}$ International Symposium (Lindborg, Johansson, Eaton, Humphrey, Kasagi, Leschziner and Sommerfeld, eds.), KTH Stockholm, Sweden, June 27-29, 2, 2001.
[24] X24. I.V. Kazachkov, A.H. Moghaddam, Modeling of thermal hydraulic processes during severe accidents at nuclear power plants, National Technical University of Ukraine "KPI", Kyiv, 2008 (in Russian).
[25] X25. Lin, S.P. and Reitz, R.D., Drop and spray formation from a liquid jet, Annual Review of Fluid Mechanics, Vol. 30, 1998, pp. 85-105.
[26] X26. W.C. Yang and D.L. Keairns, in Fluidization, D.L. Keairns and J.F. Davidson (eds.), Cambridge Univ. Press, Cambridge, 1978.
[27] X27. T.R. Blake, H. Webb and P.B. Sunderland, The Nondimensionalization of Equations Describing Fluidization with Application to the Correlation of Jet Penetration Height, Chemical Engineering Society, Vol.45, No.2, 1990, pp. 365-371.
[28] X28. C.Z. Weber, Zum Zerfall eines Fluessigkeitsstrahles, Zeitschrift fuer Angewandte Mathematik und Mechanik, Vol.11, 1931, pp. 136-154.
[29] X29. V.M. Entov and A.L. Yarin, The dynamics of thin liquid jets in air, Journal of Fluid Mechanics, Vol.140, 1984, pp. 91-111.

