# A combined space discrete algorithm with a Taylor series by time for solution of the non-stationary CFD problems 

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Abstract: - The first order by time partial differential equation (PDE) is used as models in applications such as fluid flow, heat transfer, solid deformation, electromagnetic waves, and many others. In this paper we propose the new numerical method to solve a class of the initial-boundary value problems for the PDE using any known space discrete numerical schemes and a Taylor series expansion by time. Derivatives by time are got from the outgoing PDE and its further differentiation (for second and higher order derivatives by time).

By numerical solution of the PDE and PDE arrays normally a second order discretization by space is applied while a first order by time is sometimes satisfactory too. Nevertheless, in a number of different problems, discretization both by temporal and by spatial variables is needed of highest orders, which complicates the numerical solution, in some cases dramatically. Therefore it is difficult to apply the same numerical methods for the solution of some PDE arrays if their parameters are varying in a wide range so that in some of them different numerical schemes by time fit the best for precise numerical solution.

The Taylor series based solution strategy for the non-stationary PDE in CFD simulations has been proposed here that attempts to optimise the computation time and fidelity of the numerical solution. The proposed strategy allows solving the non-stationary PDE with any order of accuracy by time in the frame of one algorithm on a single processor, as well as on a parallel cluster system. A number of examples considered in this paper have shown applicability of the method and its efficiency.

Key-Words: - Non-stationary, First Order by Time; Navier-Stokes Equations; Taylor Series; Numerical; Fractional Derivative

## 1 Introduction

The second order PDE have found extensive applications in the study of problems in fluid mechanics, flow in porous media, heat conduction, etc. [1-4]. A large number of numerical methods have been proposed for solving the second order PDE, which are mainly the first order in time and second order in space, in a CFD simulation.

A key issue is the need to effectively use the high performance numerical methods [5-12] and computers including parallel clusters [13] to complete analysis in time frames. In designing CFD software tools the author has attempted to build an essentially open single software framework, that enables arbitrarily complex non-stationary 3-D PDE array to be represented, which run efficiently on modern computers and allow simple increasing of accuracy in numerical simulation by time.

The features of the above approach are that it employs Taylor series expansions to compute solution of the PDE by time without temporal
discretization of the PDE. The idea of using the Taylor series expansions for numerical solution of non-stationary boundary problems has arisen from original use of a Taylor series described in [14, 15] as an efficient procedure for parametric study in complex problems where a number of typical computations was replaced by Taylor series approximations.

## 2 Problem Formulation

### 2.1 Strategy for solution of the nonstationary PDE

The strategy for the numerical solution of the nonstationary 3-D PDE (or PDE array) using Taylor series by time has been proposed as follows:

- Numerical solution starts as usually with a spatial discretization of the numerical domain and with development of an appropriate numerical grid
- Discretization of PDE by space is done by one of the known methods, which fits the best to the PDE and physical domain given
- The temporal derivatives are computed up to the desired order by time for the numerical solution sought based on the outgoing PDE and its differentiation by time
- Using computed temporal derivatives the numerical solution sought is found from the Taylor series using the temporal derivatives at each spatial point of the numerical grid.


### 2.2 The idea of the combined numerical method using a Taylor series by time

Conventionally numerical solution of any initialboundary problem for PDE or PDE array with one of the known numerical methods is going as follows:

1. Discretization of the numerical domain and development of the appropriate numerical grid.
2. Discretization of the PDE by space and time with further transformation of the outgoing PDE to its approximation, for example algebraic finite-difference equations by space and time.
3. Numerical solution of the approximate (e.g. algebraic finite-difference) equations by space and time by explicit or implicit algorithm.
4. Testing the numerical solution obtained and validation of it against the known data (the other numerical solutions, analytical solutions for limit cases of the PDE stated, experimental results, etc.).

### 2.2.1 Peculiarity of the method

Highly important peculiarity of the above described any numerical method is discretization of the PDE (step 2) performed according to the accuracy of the numerical solution by space and time required as far as this predetermines further steps and methods selected for the numerical solution.

If any changes to the requirements of solution accuracy are requested, then a step 2 changes, thus, the numerical algorithm changes totally.

Our strategy proposed here replaced the steps 2 , 3 of the above common algorithm, which trasforms to the following one:

1. Spatial discretization of the numerical domain and development of the appropriate numerical grid.
2. Discretization of the PDE by space with further transformation of the outgoing PDE to its approximation, for example, algebraic finitedifference equations by space.
3. Numerical solution of the approximate (e.g. algebraic finite-difference) equations by space.
4. Computing the temporal derivatives using the outgoing PDE or PDE array with further calculation of the numerical solution in time based on the Taylor series expansion of the solution sought by time.
5. Testing the numerical solution obtained and its validation against the known data (other numerical solutions, analytical solutions for limit cases of the PDE stated, experimental results, etc.).
Thus, in short, it may be said that the strategy proposed is based on replacement of the nonstationary boundary-value problem for PDE by consecutive stationary problems, calculation from PDE the temporal derivatives up to desired order and finally computation of the Taylor series by time.

The algorithm starts with the initial data and goes step-by-step by time as mentioned.

### 2.2.2 Examples illustrating the idea of method

Let us start with a few simple examples showing the idea of the proposed strategy. For this, first consider the following one-dimensional non-stationary equation (describing, for example 1-D flow) along with the corresponding initial and boundary conditions:

$$
\begin{gather*}
\frac{\partial U}{\partial t}+U \frac{\partial U}{\partial x}=\frac{\partial^{2} U}{\partial x^{2}}  \tag{1}\\
t=0, U=U_{0}(x), x \in \Gamma, U=U_{\Gamma}(t) . \tag{2}
\end{gather*}
$$

We do not specify the boundary condition (2) yet because it is no matter for explanation of the proposed method.

Supposed $\frac{\partial^{2} U}{\partial x \partial t}=\frac{\partial^{2} U}{\partial t \partial x}$, the equation (1) is rewritten in a more convenient form

$$
\begin{equation*}
\frac{\partial U}{\partial t}=-U \frac{\partial U}{\partial x}+\frac{\partial^{2} U}{\partial x^{2}}=f(x), \tag{3}
\end{equation*}
$$

where from, differentiating the last equation by time, results

$$
\begin{align*}
& \frac{\partial^{2} U}{\partial t^{2}}=\frac{\partial}{\partial t}\left(\frac{\partial^{2} U}{\partial x^{2}}-U \frac{\partial U}{\partial x}\right)=\frac{\partial^{2}}{\partial x^{2}}\left(\frac{\partial U}{\partial t}\right)+  \tag{4}\\
& -\frac{\partial U}{\partial t} \frac{\partial U}{\partial x}-U \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial t}\right)=\frac{\partial^{2} f}{\partial x^{2}}-\frac{\partial}{\partial x}(f U)
\end{align*}
$$

### 2.2.3 Taylor series by time

Now a Taylor series with a second order accuracy by time (introduce $\Delta t$ as the time step in numerical solution), accounted (3), (4) yields the following approximation of the solution to the equation (1)
$U=U_{0}+f_{0} \Delta t+\frac{1}{2!}\left(\frac{\partial^{2} f}{\partial x^{2}}-\frac{\partial(f u)}{\partial x}\right)_{0}(\Delta t)^{2}+o\left((\Delta t)^{2}\right)$,

Where $U_{0}$ in eq. (5) is the known initial data, and
$f_{0}=\left(\frac{\partial^{2} U}{\partial t^{2}}\right)_{0}-U_{0}\left(\frac{\partial U}{\partial x}\right)=\frac{\partial^{2} U_{0}}{\partial x^{2}}-U_{0} \frac{\partial U_{0}}{\partial x}+o\left((\Delta t)^{2}\right)$,
is easily computed by the known function $U_{0}(x)$. In a similar way,

$$
\begin{align*}
& \left(\frac{\partial^{2} f}{\partial x^{2}}-\frac{\partial(f U)}{\partial x}\right)_{0}=\frac{\partial^{2} f_{0}}{\partial x^{2}}-\frac{\partial\left(f_{0} U_{0}\right)}{\partial x}=\frac{\partial^{4} U_{0}}{\partial x^{4}}+ \\
& -\frac{\partial^{2}}{\partial x^{2}}\left(U_{0} \frac{\partial U_{0}}{\partial x}\right)-\frac{\partial}{\partial x}\left(U_{0} \frac{\partial^{2} U_{0}}{\partial x^{2}}-U_{0}{ }^{2} \frac{\partial U_{0}}{\partial x}\right) \tag{7}
\end{align*}
$$

Then substitution of the equations (6), (7) into a Taylor series (5) results in

$$
\begin{align*}
& U=U_{0}+\left(\frac{\partial^{2} U}{\partial x^{2}}-U \frac{\partial U}{\partial x}\right)_{0} \Delta t+ \\
& \frac{1}{2}\left[\begin{array}{l}
\frac{\partial^{4} U_{0}}{\partial x^{4}}-2 U_{0} \frac{\partial^{3} U_{0}}{\partial x^{3}}+ \\
+\left(U_{0}^{2}-4 \frac{\partial U_{0}}{\partial x}\right) \frac{\partial^{2} U_{0}}{\partial x^{2}}+2 U_{0}\left(\frac{\partial U_{0}}{\partial x}\right)^{2}
\end{array}\right](\Delta t)^{2}+  \tag{8}\\
& +o\left((\Delta t)^{2}\right),
\end{align*}
$$

and so on. Evidently, one can continue this procedure to get any desired order of accuracy by time. The transformation of the procedure from any n -th layer by time to the $(\mathrm{n}+1)$-th layer by time is similar to the stated in the equations (5), (8) above:

$$
\begin{equation*}
U_{n+1}=U_{n}+f_{n} \Delta t+\frac{1}{2}\left(\frac{\partial^{2} f}{\partial x^{2}}-\frac{\partial(f U)}{\partial x}\right)_{n}(\Delta t)^{2}+o\left((\Delta t)^{2}\right) \tag{9}
\end{equation*}
$$

where $\mathrm{n}=0,1,2, \ldots, \mathrm{~N}$.

### 2.3 Substitution of the temporal derivatives with the spatial ones

Thus, the right hand of the equations (5), (9) is always a function of the coordinates at the current moment of time. Thus, neither explicit, nor implicit approximations by time are applied; no difference equations by time are needed!

As the equation (9) shows, the second order by time approximation in the equation (1) requests all spatial derivatives of the function sought up to the 4-th order. Adding the next term in the time series requires the corresponding term twice differentiated by space. If computing the highest order derivatives $\partial^{i} f / \partial t^{i}$ is analytically complicated, it is done numerically.

Consequently, instead of a solution of a difference (or any discrete) equation, computation of the spatial derivatives with the accuracy stated is proposed. Then the numerical solution sought is computed from the Taylor series by time.

### 2.4 The examples of application of a Taylor series by time in numerical solutions

### 2.4.1 Example 1

In case of a simple wave equation with the following initial and boundary conditions

$$
\begin{gather*}
\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x} \\
t=0, u=u_{0}(x)=x ; x=0, u=u(t)=t \tag{10}
\end{gather*}
$$

an analytical solution of the boundary problem (10) is known as a wave spreading with the velocity 1 countercurrent to the axis $x, u=f(x+t)$.

According to our strategy, the numerical solution of the boundary problem (10) is done with a first order accuracy by time as follows: $u=u_{0}+(\partial u / \partial t)_{0} \Delta t+O\left((\Delta t)^{2}\right)$, where from with account of the above-mentioned yields $u=x+\Delta t$. The solution obtained does not change with an increase of accuracy because the first order solution coincides here with the exact analytical solution.

### 2.4.2 Example 2

The one-dimensional non-linear equation

$$
\begin{equation*}
\frac{\partial U}{\partial t}+U \frac{\partial U}{\partial x}=v \frac{\partial^{2} U}{\partial x^{2}} \tag{11}
\end{equation*}
$$

with the following initial and boundary conditions

$$
\begin{equation*}
t=0, U=1 ; \quad x=0, \quad \frac{\partial U}{\partial x}=0, \frac{\partial^{2} U}{\partial x^{2}}=0 \tag{12}
\end{equation*}
$$

is solved according to the proposed strategy as follows

$$
\begin{aligned}
& \frac{\partial U}{\partial t}=v \frac{\partial^{2} U}{\partial x^{2}}-U \frac{\partial U}{\partial x} \\
& \frac{\partial^{2} U}{\partial t^{2}}=\frac{\partial}{\partial t}\left(\frac{\partial U}{\partial t}\right)=v \frac{\partial^{2}}{\partial x^{2}}\left(\frac{\partial U}{\partial t}\right)-\frac{1}{2} \frac{\partial}{\partial t} \frac{\partial U^{2}}{\partial x}= \\
& =v \frac{\partial^{2}}{\partial x^{2}}\left(\frac{\partial U}{\partial t}\right)-\frac{\partial}{\partial x}\left(U \frac{\partial U}{\partial t}\right)= \\
& =v \frac{\partial^{2}}{\partial x^{2}}\left(v \frac{\partial^{2} U}{\partial x^{2}}-U \frac{\partial U}{\partial x}\right)-\frac{\partial}{\partial x}\left(U\left(v \frac{\partial^{2} U}{\partial x^{2}}-U \frac{\partial U}{\partial x}\right)\right)
\end{aligned}
$$

where from is got

$$
U=U_{0}+\left(v \frac{\partial^{2} U}{\partial x^{2}}-U \frac{\partial U}{\partial x}\right)_{0} \Delta t
$$

in a first order approach by time, or

$$
\begin{aligned}
& U=U_{0}+\left(v \frac{\partial^{2} U}{\partial x^{2}}-U \frac{\partial U}{\partial x}\right) \Delta t+ \\
& +\frac{1}{2}\left[\begin{array}{c}
v \frac{\partial^{2}}{\partial x^{2}}\left(v \frac{\partial^{2} U}{\partial x^{2}}-U \frac{\partial U}{\partial x}\right)+ \\
-\frac{\partial}{\partial x}\left(U\left(v \frac{\partial^{2} U}{\partial x^{2}}-U \frac{\partial U}{\partial x}\right)\right)
\end{array} . \begin{array}{l}
(\Delta t)^{2}
\end{array} .\right.
\end{aligned}
$$

in a second order approach by time.
The first order approach by time gives here the same result as any higher order approaches, which is: $U=1$.

## 3 Solution of the Navier-Stokes equations with the proposed method

### 3.1 Numerical algorithm for NSE

Consider numerical algorithm for the solutions to the equations of 3-D non-stationary motion of heatconducting incompressible viscous fluids:

$$
\begin{align*}
& \operatorname{div} \vec{v}=0, \quad \frac{\partial \vec{v}}{\partial t}+\vec{v} \nabla \vec{v}=-\frac{1}{\rho} \nabla p+v \Delta \vec{v}+\vec{f} \\
& \frac{\partial T}{\partial t}+\vec{v} \nabla T=\frac{1}{\rho c} \operatorname{div}(\lambda \nabla T)+\Phi \tag{13}
\end{align*}
$$

where $\vec{v}=\left\{v_{x}, v_{y}, v_{z}\right\}, \vec{f}=\left\{f_{x}, f_{y}, f_{z}\right\}$ are the velocity and external force vectors, respectively.

The Cartesian coordinates $x, y, z$ are implied here, then $\rho, \mu, \lambda$ are the density, dynamic viscosity coefficient, and heat conductivity coefficient, correspondingly, $v$ is the coefficient of kinematic viscosity and $c$ is the specific heat capacity, and $\nabla, \Delta$ denote the gradient and Laplace operators, respectively. Finally, $\Phi$ is the dissipate function,

$$
\Phi=\frac{\mu}{\rho c}\left(2|\nabla \vec{v}|^{2}+\left|\frac{\tau}{\mu}\right|^{2}\right)
$$

$$
\left|\frac{\tau}{\mu}\right|^{2}=\left(\frac{\partial v_{x}}{\partial y}+\frac{\partial v_{y}}{\partial x}\right)^{2}+\left(\frac{\partial v_{x}}{\partial z}+\frac{\partial v_{z}}{\partial x}\right)^{2}+\left(\frac{\partial v_{y}}{\partial z}+\frac{\partial v_{z}}{\partial y}\right)^{2}
$$

The partial differential equation array (13) thus obtained has to be supplemented with the corresponding initial conditions:

$$
\begin{gather*}
t=0, \vec{v}=\vec{v}_{0}(x, y, z), p=p_{0}(x, y, z) \\
T=T_{0}(x, y, z), \quad(x, y, z) \in \Omega \tag{14}
\end{gather*}
$$

as well as with the corresponding boundary conditions (Dirichlet, Neumann, mixed, etc.) at the boundary $(x, y, z) \in \Gamma$. They are not specified here because it has no matter for the proposed strategy of the numerical solution, which is applicable by any boundary conditions.

To apply the above described numerical strategy to the Navier-Stokes equation (NSE) array (13) with the initial conditions (14), rewrite these equations in the form:

$$
\begin{aligned}
& \operatorname{div} \vec{v}=0, \quad \frac{\partial \vec{v}}{\partial t}=\vec{F}(\vec{v}, \nabla \vec{v}, \vec{f})-\frac{1}{\rho} \nabla p \\
& \frac{\partial T}{\partial t}=F_{T}(\vec{v}, T, \nabla T, \nabla \vec{v})
\end{aligned}
$$

where is

$$
\begin{equation*}
\vec{F}(\vec{v}, \nabla \vec{v}, \vec{f})=v \Delta \vec{v}-\vec{v} \nabla \vec{v}+\vec{f} \tag{16}
\end{equation*}
$$

$F_{T}(\vec{v}, T, \nabla T, \nabla \vec{v})=\Phi+\frac{1}{\rho c} \operatorname{div}(\lambda \nabla T)-\vec{v} \nabla T$,
$\vec{F}=\left(F_{x}, F_{y}, F_{z}\right)$ is the vector of the right hand of the momentum equation excluding the pressure gradient.

Thus, all right hands of the equations (16) are known at the initial moment of time from the initial data (14), and then all temporal derivatives of the velocity vector and the temperature are computed from the PDE array (15).

Consequently, the velocity and the temperature fields are calculated at the next time step from the Taylor series:

$$
\begin{align*}
& \vec{v}=\vec{v}_{0}(x)+\left(\frac{\partial \vec{v}}{\partial t}\right)_{0} \Delta t+\frac{1}{2!}\left(\frac{\partial^{2} \vec{v}}{\partial t^{2}}\right)_{0}(\Delta t)^{2}+ \\
& +\frac{1}{3!}\left(\frac{\partial^{3} \vec{v}}{\partial t^{3}}\right)_{0}(\Delta t)^{3}+O\left((\Delta t)^{4}\right) \\
& T=  \tag{17}\\
& =T_{0}(x)+\left(\frac{\partial T}{\partial t}\right)_{0} \Delta t+\frac{1}{2!}\left(\frac{\partial^{2} T}{\partial t^{2}}\right)_{0}(\Delta t)^{2}+ \\
& \quad+\frac{1}{3!}\left(\frac{\partial^{3} T}{\partial t^{3}}\right)_{0}(\Delta t)^{3}+O\left((\Delta t)^{4}\right) .
\end{align*}
$$

Here $\Delta t$ is the temporal step chosen for the generated numerical grid.

### 3.2 Accuracy of the approximate numerical solution by time and space for the NSE

The approximate numerical solution (17) is computed with a required order of accuracy by time (here it is up to the third order terms for example).

It is very important that at the first temporal step computed by equations (17) the pressure distribution is unknown and continuity equation has not been used yet.

Surprisingly, the velocity and temperature fields do not depend on the pressure distribution at the first time step.

This numerical scheme in a first order by $\Delta t$ (when only the terms up to $\Delta t$ are kept in (17)) completely coincide with the simplest first order explicit numerical scheme. But these two methods differ completely afterwards.

For example, the well-known numerical schemes of a second order by time are very time-consuming and cumbersome while, in the strategy proposed here, the numerical solution procedure in a second order by time (as well as in any higher order by time) is nearly the same as in the first order by time.

Moreover, any highest order numerical solution is got similarly and, what is very important, does not request more computer resources than the first order solution. All what is needed for this procedure is just an easy computation by equations (15), (16) with a further substitution of the results into the Taylor series (17).

### 3.3 Numerical solution of a second order accuracy by time for the NSE

The numerical solution of a second order accuracy by time requests, in contrast with the first order approximation, calculation of the derivatives of pressure, because the equation array (13) transforms to the following one

$$
\begin{gather*}
\frac{\partial}{\partial t}(d i v \vec{v})=0, \quad \frac{\partial^{2} T}{\partial t^{2}}=\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial t}\right)=\frac{\partial F_{T}}{\partial t}, \\
\frac{\partial}{\partial t}\left(\frac{\partial \vec{v}}{\partial t}\right)=\frac{\partial^{2} \vec{v}}{\partial t^{2}}=\frac{\partial \vec{F}}{\partial t}-\frac{1}{\rho} \frac{\partial}{\partial t} \nabla p, \tag{18}
\end{gather*}
$$

Where is
$\frac{\partial \vec{F}}{\partial t}=\nu \Delta\left(\frac{\partial \vec{v}}{\partial t}\right)+\frac{\partial \vec{f}}{\partial t}-\left(\left(\frac{\partial \vec{v}}{\partial t}\right) \nabla \vec{v}+\vec{v} \nabla\left(\frac{\partial \vec{v}}{\partial t}\right)\right)$,
$\frac{\partial F_{T}}{\partial t}=\Phi_{1}+\frac{1}{\rho c} d i v\left(\lambda \nabla\left(\frac{\partial T}{\partial t}\right)\right)-\vec{v} \nabla\left(\frac{\partial T}{\partial t}\right)-\nabla T\left(\frac{\partial \vec{v}}{\partial t}\right)$.

Here are
$\Phi_{1}=\frac{\partial \Phi}{\partial t}=\frac{\mu}{\rho c}\left(2|\nabla \vec{F}|^{2}+\left|\frac{\tau_{1}}{\mu}\right|^{2}\right)$,
$\left|\frac{\tau_{1}}{\mu}\right|^{2}=\left(\frac{\partial F_{x}}{\partial y}+\frac{\partial F_{y}}{\partial x}\right)^{2}+\left(\frac{\partial F_{x}}{\partial z}+\frac{\partial F_{z}}{\partial x}\right)^{2}+\left(\frac{\partial F_{y}}{\partial z}+\frac{\partial F_{z}}{\partial y}\right)^{2}$.

And then it results in a calculation of the following next expressions by the right hands of the outgoing PDE similar to the previous ones:

$$
\begin{aligned}
& \vec{F}_{1}=\frac{\partial \vec{F}}{\partial t}=v \Delta\left(\vec{F}-\frac{1}{\rho} \nabla p\right)+\frac{\partial \vec{f}}{\partial t}+ \\
& -\left(\left(\vec{F}-\frac{1}{\rho} \nabla p\right) \nabla \vec{v}+\vec{v}\left(\nabla \vec{F}-\frac{1}{\rho} \Delta p\right)\right)^{\prime} \\
& F_{T}^{1}=\frac{\partial F_{T}}{\partial t}=\Phi_{1}-\left(\vec{v} \nabla F_{T}+\nabla T\left(\vec{F}-\frac{1}{\rho} \nabla p\right)\right)+ \\
& +\frac{1}{\rho c} \operatorname{div}\left(\lambda \nabla F_{T}\right)
\end{aligned}
$$

Then transformation of the first equation in the equation array (18) to the form $\operatorname{div}\left(\frac{\partial \vec{v}}{\partial t}\right)=0$, with further substitution of the components $\frac{\partial \vec{v}}{\partial t}$ from the equations (15), yields

$$
\operatorname{div}\left(\frac{\partial \vec{v}}{\partial t}\right)=\operatorname{div} \vec{F}-\frac{1}{\rho} \operatorname{div}(\nabla p)=0
$$

where from finally goes the following equation for calculation of the pressure distribution inside the numerical domain

$$
\begin{equation*}
\Delta p=\rho d i v \vec{F} . \tag{20}
\end{equation*}
$$

### 3.3.1 The Poisson equation for the pressure

The Poisson equation (20) thus obtained allows computing the pressure distribution in the numerical domain by the known values of the vector $\vec{F}$, which have been computed from the equation (16) based on the first order approximation for the velocity field (described above).

This equation (20) is solved comparably easy and further numerical algorithm does not request solution of any equation, because all needed is computed of the spatial derivatives from the functions in the numerical domain. The second order approximation to the numerical solution is got afterwards simply from the Taylor series (17).

### 3.3.2 Final closed equation array

Finally, the closed system of the equations (18)-(20) has been got here for the computation of the second order by time numerical approximation to the outgoing Navier-Stokes equation. Importantly, any
difference equations are absent here, except the one well-known Poisson equation (20) for the pressure.

Only the spatial derivatives are to be computed with the required accuracy, which is much easier than solution of the difference equations and it does not complicate the algorithm in a second order approach.

### 3.4 Numerical solution of a third order accuracy by time for NSE

Now differentiating the partial differential equations (18) by time, accounting to (19), the third order accuracy by time is got:

$$
\begin{gather*}
\frac{\partial^{3} \vec{v}}{\partial t^{3}}=\frac{\partial^{2} \vec{F}}{\partial t^{2}}-\frac{1}{\rho} \frac{\partial^{2}}{\partial t^{2}} \nabla p \\
\frac{\partial^{3} T}{\partial t^{3}}=\frac{\partial^{2} F_{T}}{\partial t^{2}} \tag{21}
\end{gather*}
$$

where $\frac{\partial^{2}}{\partial t^{2}} \nabla p$ is computed based on the first and second approximations by time, which were got for the pressure and its derivative by time including the initial data as described above.

Obviously, the second order derivative for the pressure by time can be computed only with the first order accuracy because the first order derivative has been computed with the first order accuracy, and afterwards the second order derivatives were computed also with the first order accuracy as derivative from the first order derivative. This important question is subject for a separate detail investigation.

Here are

$$
\begin{aligned}
& \frac{\partial^{2} \vec{F}}{\partial t^{2}}=\frac{\partial}{\partial t}\left(\frac{\partial \vec{F}}{\partial t}\right)=v \Delta\left[\left(\frac{\partial \vec{F}}{\partial t}\right)-\frac{1}{\rho} \frac{\partial}{\partial t} \nabla p\right]+\frac{\partial^{2} \vec{f}}{\partial t^{2}}+ \\
& -\left\{\left[\left(\frac{\partial \vec{F}}{\partial t}\right)-\frac{1}{\rho} \frac{\partial}{\partial t} \nabla p\right] \nabla \vec{v}+\vec{v}\left[\nabla\left(\frac{\partial \vec{F}}{\partial t}\right)-\frac{1}{\rho} \Delta\left(\frac{\partial p}{\partial t}\right)\right]+\right. \\
& \left.+\left(\vec{F}-\frac{1}{\rho} \nabla p\right) \nabla\left(\vec{F}-\frac{1}{\rho} \nabla p\right)+\left(\vec{F}-\frac{1}{\rho} \nabla p\right) \nabla\left(\vec{F}-\frac{1}{\rho} \nabla p\right)\right\} \\
& =v \Delta\left[\left(\frac{\partial \vec{F}}{\partial t}\right)-\frac{1}{\rho} \frac{\partial \nabla p}{\partial t}\right]+\frac{\partial^{2} \vec{f}}{\partial t^{2}}-\left\{\left[\left(\frac{\partial \vec{F}}{\partial t}\right)-\frac{1}{\rho} \frac{\partial \nabla p}{\partial t}\right] \nabla \vec{v}+\right.
\end{aligned}
$$

$$
\begin{aligned}
& \left.2\left(\vec{F}-\frac{\nabla p}{\rho}\right)\left(\nabla \vec{F}-\frac{\Delta p}{\rho}\right)+\vec{v}\left[\nabla\left(\frac{\partial \vec{F}}{\partial t}\right)-\frac{1}{\rho} \Delta\left(\frac{\partial p}{\partial t}\right)\right]\right\} \\
& \frac{\partial^{2} F_{T}}{\partial t^{2}}=\frac{\partial}{\partial t}\left(\frac{\partial F_{T}}{\partial t}\right)=\Phi_{2}+\frac{1}{\rho c} \operatorname{div}\left(\lambda \nabla F_{T}^{1}\right)+ \\
& -\left(\vec{F}_{1} \nabla T+2 \vec{F} \nabla F_{T}+\vec{v} \nabla F_{T}^{1}\right)
\end{aligned}
$$

Where are:

$$
\begin{gathered}
\Phi_{2}=\frac{\partial \Phi_{1}}{\partial t}=\frac{\mu}{\rho c}\left(2\left|\nabla \vec{F}_{1}\right|^{2}+\left|\frac{\tau_{2}}{\mu}\right|^{2}\right) \\
\left|\frac{\tau_{2}}{\mu}\right|^{2}=\left(\frac{\partial F_{x}^{1}}{\partial y}+\frac{\partial F_{y}^{1}}{\partial x}\right)^{2}+\left(\frac{\partial F_{x}^{1}}{\partial z}+\frac{\partial F_{z}^{1}}{\partial x}\right)^{2}+\left(\frac{\partial F_{y}^{1}}{\partial z}+\frac{\partial F_{z}^{1}}{\partial y}\right)^{2}
\end{gathered}
$$

$\vec{F}_{1}=\left(F_{x}^{1}, F_{y}^{1}, F_{z}^{1}\right), F_{T}^{1}$ are the new right hands of the equations according to (19).

### 3.4.1 Final third order equation array

Finally the equations for the third order accuracy by time (21) result in

$$
\begin{gather*}
\frac{\partial^{3} \vec{v}}{\partial t^{3}}=\vec{F}_{2}-\frac{1}{\rho} \frac{\partial^{2}}{\partial t^{2}} \nabla p, \\
\frac{\partial^{3} T}{\partial t^{3}}=F_{T}^{2} \tag{22}
\end{gather*}
$$

where are:

$$
\begin{aligned}
& F_{T}^{2}=\Phi_{2}+\frac{1}{\rho c} \operatorname{div}\left(\lambda \nabla F_{T}^{1}\right)-\left(\vec{F}_{1} \nabla T+2 \vec{F} \nabla F_{T}+\vec{v} \nabla F_{T}^{1}\right), \\
& \vec{F}_{2}=v \Delta\left(\vec{F}_{1}-\frac{1}{\rho} \frac{\partial \nabla p}{\partial t}\right)+\frac{\partial^{2} \vec{f}}{\partial t^{2}}-\left[\left(\vec{F}_{1}-\frac{1}{\rho} \frac{\partial \nabla p}{\partial t}\right) \nabla \vec{v}+\right. \\
& \left.+2\left(\vec{F}-\frac{\nabla p}{\rho}\right)\left(\nabla \vec{F}-\frac{\Delta p}{\rho}\right)+\vec{v}\left(\nabla \vec{F}_{1}-\frac{1}{\rho} \Delta\left(\frac{\partial p}{\partial t}\right)\right)\right] .
\end{aligned}
$$

Here the temporal derivative $\frac{\partial^{2} \vec{f}}{\partial t^{2}}$ has to be known.

It is just a derivative from an external force stated. The last one is most often constant (e.g. gravitation) or known (stated electromagnetic force in a conductive media, vibration acceleration, etc).

Consequently, the equations (22) give the thirdorder accuracy solution by time for the outgoing equations (15) using equations (17). The pressure distribution in third-order accuracy is got from a solution of the Poisson equation (20) after substitution of the computed velocity field.

Obviously, there is no problem to implement the proposed method to the case of variable physical properties of continua and to some other more general cases.

It must be noted that an increase of accuracy of the approximations by time requires computing the temporal derivatives of the pressure, which needs to keep a few temporal layers in a computer memory. Therefore this question still needs some separate deep study more in detail.

Nevertheless, this preliminary analysis has shown high efficiency and simplicity of the strategy proposed for the numerical solution of the nonstationary non-isothermal equations of the NavierStokes type, as well as many other partial differential equation arrays of the first-order by time.

### 3.4.2 Application of the algorithm at every time step by Taylor series

The same strategy described above is applied consequently at the each and every time step, so that the transformation from any $n$-th layer by time to the ( $\mathrm{n}+1$ )-th layer by time ( $t_{n+1}=t_{n}+\Delta t$ ) is similar to the stated in the equations (17):

$$
v_{n+1}=v_{n}(x)+\left(\frac{\partial v}{\partial t}\right)_{n} \Delta t+\frac{1}{2!}\left(\frac{\partial^{2} v}{\partial t^{2}}\right)_{n}(\Delta t)^{2}+
$$

$$
+\frac{1}{3!}\left(\frac{\partial^{3} v}{\partial t^{3}}\right)_{n}(\Delta t)^{3}+O\left((\Delta t)^{4}\right)
$$

$$
\begin{equation*}
T_{n+1}=T_{n}(x)+\left(\frac{\partial T}{\partial t}\right)_{n} \Delta t+\frac{1}{2!}\left(\frac{\partial^{2} T}{\partial t^{2}}\right)_{n}(\Delta t)^{2}+ \tag{23}
\end{equation*}
$$

$$
+\frac{1}{3!}\left(\frac{\partial^{3} T}{\partial t^{3}}\right)_{n}(\Delta t)^{3}+O\left((\Delta t)^{4}\right)
$$

where is $n=0,1,2, \ldots, N$.

Certainly, the time steps $\Delta t$ and the number of iterations at the each time step may vary from one time step to another time step; therefore there is the subject for additional investigation concerning optimization of the numerical algorithm proposed here.

## 4 Comparison of the proposed strategy with method of fractional differentials

Let us consider an example from [16] on computation of a heat flux by the known temperature distribution using an analytical method of the fractional differentials.

The method of fractional differentials allows obtaining an analytical solution for the heat flux at the boundary for any non-linear PDE; therefore it gives good possibility for the validation of our algorithm.

### 4.1 Equation of heating of the semi-infinite domain

The task on heating of the semi-infinite domain is modeled by the following boundary-value problem for PDE:

$$
\begin{align*}
& \left(\frac{\partial}{\partial t}-\frac{\partial^{2}}{\partial x^{2}}\right) T=0, \quad 0<x<\infty, \quad 0<t<\infty  \tag{24}\\
& T_{x=0}=T_{s}(t), \quad T_{x=\infty}=0, T_{t=0}=0
\end{align*}
$$

The heat flux is computed here as $q_{s}=(\partial T / \partial X)_{x=0}$.

According to [16], the differential operator in the PDE (24) is represented in the form:

$$
\begin{equation*}
\left(\frac{\partial^{1 / 2}}{\partial t^{1 / 2}}-\frac{\partial}{\partial x}\right)\left(\frac{\partial^{1 / 2}}{\partial t^{1 / 2}}+\frac{\partial}{\partial x}\right) T=0, \tag{25}
\end{equation*}
$$

supposed that $\frac{\partial^{1 / 2} T}{\partial t^{1 / 2}} \frac{\partial T}{\partial x}=\frac{\partial}{\partial x} \frac{\partial^{1 / 2} T}{\partial t^{1 / 2}}$.
Now consider the equation presented by the right multiplayer of (25):

$$
\begin{equation*}
\left(\frac{\partial^{1 / 2}}{\partial t^{1 / 2}}+\frac{\partial}{\partial x}\right) T=0 \tag{26}
\end{equation*}
$$

### 4.2 Analytical solution of the non-linear equation

The solutions to the equation (26) are also solutions to the equations (25) because the operator applied to a zero results in zero. Thus, solutions to the equation (26) are solutions to the equation (24) as well.

The equation (4) written for $x=0$ gives immediately the solution of the task stated, namely the temperature gradient at the boundary of the domain:
$q_{s}=-\frac{\partial T}{\partial x /_{x=0}}=\frac{d^{1 / 2} T_{s}(t)}{\alpha t^{1 / 2}}=\frac{1 d}{\sqrt{\pi} d t} \int_{0} \frac{T_{s}(t)}{\sqrt{t-\tau}} d \tau$.

Note that the temperature gradient (27) has been found without solution of the task for temperature distribution (24). This is why the method of the fractional differentials is also called the non-field method. It allows computing analytically a heat flux at the boundary of domain directly through such comparably simple transformation of the outgoing differential equation.

### 4.3 Comparison of the methods for simple linear equations

Let us analyze the simple linear equation to compare solutions of the boundary problems by the method of fractional differentials with solution by the method proposed here. For this, except solution (27), consider also a general solution to the boundary problem (24) [17]:

$$
\begin{align*}
& T(x, t)=-\int_{0}^{t} T_{s}(\tau) \frac{\partial T_{\tau}}{\partial \tau} d \tau  \tag{28}\\
& -\frac{\partial T_{\tau}}{\partial \tau}=\frac{\partial}{\partial \tau} \Theta\left(\frac{\mathrm{x}}{2 \sqrt{\mathrm{t}-\tau}}\right)=\frac{\partial}{\partial \tau} \frac{2}{\sqrt{\pi}} \int_{0}^{\frac{\mathrm{x}}{2 \sqrt{\mathrm{t}-\tau}} e^{-\alpha^{2}}} d \alpha= \\
& =\frac{\mathrm{x}}{2 \sqrt{\pi(\mathrm{t}-\tau)^{3 / 2}}} e^{-\frac{\mathrm{x}^{2}}{4(\mathrm{t}-\tau)}} \tag{29}
\end{align*}
$$

where the temperature $T_{s}(t)$ is kept by $x=0$ for all $\tau$ from 0 till $t$.

Then from (28), (29) follows

$$
\begin{equation*}
T(x, t)=\frac{x}{2 \sqrt{\pi}} \int_{0}^{t} \frac{T s(\tau)}{(t-\tau)^{3 / 2}} e^{\frac{\mathrm{x}^{2}}{4(\mathrm{t}-\tau)}} d \tau \tag{30}
\end{equation*}
$$

Introducing in (30) the new variable $\xi=\frac{x}{2 \sqrt{t-\tau}}$ results in

$$
\begin{equation*}
T(x, t)=\frac{2}{\sqrt{\pi}} \int_{\frac{x}{2 \sqrt{t}}}^{\infty} T_{s}\left(t+\frac{x^{2}}{4 \xi^{2}}\right) e \xi^{2} d \xi \tag{31}
\end{equation*}
$$

which gives solution to the task (24). By $x=0$, the solution (31) satisfies to the boundary condition (24).

Now the value of $-q_{s}=-\left(\frac{\partial T}{\partial X}\right)_{x=0}$ can be got from (31) or directly from (30) using the rule of differentiation of the integral by parameter [18]:

$$
\begin{gathered}
\frac{\partial T}{\partial X}=\frac{2}{\sqrt{\pi}} \int_{0}^{\infty} \frac{T_{s}(\tau)}{(t-\tau)^{3 / 2}} e^{-\frac{x^{2}}{4(t-\tau)}} d \tau+\frac{x}{2 \sqrt{\pi}} \cdot \\
\cdot \int_{0}^{t} \frac{T_{s}(\tau)}{(t-\tau)^{3 / 2}} e^{-\frac{x^{2}}{4(t-\tau)}} \cdot\left(-\frac{2 x}{4(t-\tau)}\right) d \tau= \\
=\frac{1}{2 \sqrt{\pi}} \int_{0}^{t} \frac{T_{s}(\tau)}{(t-\tau)^{3 / 2}} e^{-\frac{x^{2}}{4(t-\tau)}} d \tau+\frac{x}{2 \sqrt{\pi}} \int_{0}^{t} \frac{T_{s}(\tau)}{(t-\tau)^{3 / 2}} \cdot \\
\cdot e^{-\frac{x^{2}}{4(t-\tau)}} \cdot\left(-\frac{x}{2(t-\tau)}\right) d \tau=\frac{1}{2 \sqrt{\pi}} \int_{0}^{t} \frac{T_{s}(\tau)}{(t-\tau)^{3 / 2}} . \\
\cdot e^{-\frac{x^{2}}{4(t-\tau)}}\left(1-\frac{x^{2}}{2(t-\tau)}\right) d \tau,
\end{gathered}
$$

where from is got

$$
\begin{align*}
& -q_{s}=-\left(\frac{\partial T}{\partial X}\right)_{x=0}=\frac{-1}{2 \sqrt{\pi}} \int_{0}^{t} \frac{T_{s}(\tau)}{(t-\tau)^{3 / 2}} e^{0} \cdot(1-0) d \tau= \\
& =\frac{-1}{2 \sqrt{\pi}} \int_{0}^{t} \frac{T_{s}(\tau)}{(t-\tau)^{3 / 2}} d \tau \tag{32}
\end{align*}
$$

Comparing the equations (32) and (27), one can observe that they completely coincide because from the equation (27) follows

$$
-q_{s}=-\left(\frac{\partial T}{\partial X}\right)_{x=0}=\frac{1}{\sqrt{\pi}} \frac{d}{d t} \int_{0}^{t} \frac{T_{s}(\tau)}{\sqrt{t-\tau}} d \tau=
$$

$$
\begin{align*}
& =\frac{1}{\sqrt{\alpha \pi}}\left[\int_{0}^{t}\left(-\frac{1}{2}\right) \frac{T_{s}(\tau)}{(t-\tau)^{3 / 2}} d \tau+1 \cdot\left(\frac{T_{s}(\tau)}{\sqrt{t-\tau}}\right)_{\tau=t}\right]= \\
& =-\frac{1}{2 \sqrt{\pi}} \int_{0}^{t} \frac{T_{s}(\tau)}{(t-\tau)^{3 / 2}} d \tau \tag{33}
\end{align*}
$$

here $T_{s}(\tau)=0$ until $t=\tau$. Thus, both solutions, (32) and (33), coincide.

With the above, it was proven that the solutions obtained by the method of fractional differentials and the exact analytical solution of the boundary problem (24) completely coincide.

### 4.3.1 Heat flux at the boundary by our numerical method

Now this heat flux at the boundary will be got ones more following to our algorithm. For this, first the temperature profile is computed through the Taylor series:

$$
\begin{equation*}
T=T_{0}+\left(\frac{\partial T}{\partial t}\right)_{0} \Delta t+\frac{1}{2!}\left(\frac{\partial^{2} T}{\partial t^{2}}\right)_{0}(\Delta t)^{2}+\ldots \tag{34}
\end{equation*}
$$

where $T_{0}=T(x, 0)=0$ according to Therefore from (34) follows

$$
\begin{equation*}
T=\left(\frac{\partial T}{\partial t}\right)_{0} \Delta t+\frac{1}{2!}\left(\frac{\partial^{2} T}{\partial t^{2}}\right)_{0}(\Delta t)^{2}+\frac{1}{3!}\left(\frac{\partial^{3} T}{\partial t^{3}}\right)_{0}(\Delta t)^{3}+\ldots \tag{35}
\end{equation*}
$$

Replacing here $\frac{\partial T}{\partial t}$ through $\frac{\partial^{2} T}{\partial x^{2}}$ in accordance with (24) results further on from the equation (35)

$$
\begin{align*}
& T=\left(\frac{\partial^{2} T}{\partial x^{2}}\right)_{0} \Delta t+\frac{1}{2!}\left(\frac{\partial^{4} T}{\partial x^{4}}\right)_{0}(\Delta t)^{2}+\ldots .= \\
& =\frac{\partial^{2} T_{s}}{\partial x^{2}} \Delta t+\frac{1}{2!} \frac{\partial^{4} T_{s}}{\partial x^{4}}(\Delta t)^{2}+\ldots . \tag{36}
\end{align*}
$$

Then the Taylor series expansion (35) with account of the (36) can be rewritten up to arbitrary order by $\Delta t$ :
$\Delta t: \quad T_{1}=\sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{\partial^{2 n} T}{\partial x^{2 n}}\right)_{0}(\Delta t)^{n}$,
$(2 \Delta t):$

$$
T_{2}=\sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{\partial^{2 n} T}{\partial x^{2 n}}\right)_{0}(\Delta t)^{n}+\sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{\partial^{2 n} T}{\partial x^{2 n}}\right)_{1}(\Delta t)^{n}
$$

$$
\begin{align*}
& R \cdot \Delta t: \quad T_{R}=T\left(t_{R}\right)=  \tag{37}\\
& =\sum_{n=1}^{\infty} \frac{1}{n!}\left[\left(\frac{\partial^{2 n} T}{\partial x^{2 n}}\right)_{0}+\left(\frac{\partial^{2 n} T}{\partial x^{2 n}}\right)_{1}+\ldots .+\left(\frac{\partial^{2 n} T}{\partial x^{2 n}}\right)_{R}\right](\Delta t)^{n}
\end{align*}
$$

for the constant time step $\Delta t$ starting from $t=0$ (actually, time step may be chosen variable, which does not affect the proposed method).

The Taylor series (37) represents the algorithm for numerical solution of the boundary problem (24) using approximations by time up to the desired accuracy. All needed for this is spatial derivatives by $x$ in the domain $x$. According to (35):
$-q=-\left(\frac{\partial T}{\partial x}\right)_{x=0}=-\left(\frac{\partial^{3} T}{\partial x^{3}}\right)_{0} t-\frac{1}{2!}\left(\frac{\partial^{5} T}{\partial x^{5}}\right)_{x=0} t^{2}+\ldots$.

Now compute derivatives by $x$ using the exact analytical solution (30) of the outgoing boundary problem (24) and substitute it into (38).

Differentiating by $x$ yields from (30):

$$
\begin{gathered}
\frac{\partial^{2} T}{\partial x^{2}}=\frac{1}{2 \sqrt{\pi}} \int_{0}^{t} \frac{T_{s}(\tau)}{(t-\tau)^{3 / 2}} e^{-\frac{x^{2}}{4(t-\tau)}} \cdot \\
\cdot\left[\frac{x}{2(t-\tau)}\left(\frac{x^{2}}{2(t-\tau)}-3\right)\right] d \tau, \\
\frac{\partial^{3} T}{\partial x^{3}}=\frac{1}{2 \sqrt{\pi}} \int_{0}^{t} \frac{T_{s}(\tau)}{(t-\tau)^{3 / 2}} e^{-\frac{x^{2}}{4(t-\tau)}} \cdot \\
\cdot\left[\frac{x^{2}}{4(t-\tau)^{2}}\left(3-\frac{x^{2}}{2(t-\tau)}\right)-\frac{3}{4(t-\tau)}\left(\frac{x^{2}}{(t-\tau)}-2\right)\right] d \tau \\
\cdot\left\{\frac{x}{8(t-\tau)}\left[\frac{\partial^{4} T}{(t-\tau)^{2}}\left(\frac{x^{2}}{2(t-\tau)}-3\right)+\frac{3}{2 \sqrt{\pi}} \int_{0}^{t} \frac{T_{s}(\tau)}{(t-\tau)^{3 / 2}} e^{-\frac{x^{2}}{4(t-\tau)}}\left(2-\frac{x^{2}}{(t-\tau)}\right)\right]+\right.
\end{gathered}
$$

$$
\left.-\frac{x^{3}}{(t-\tau)^{3}}+\frac{3 x}{2(t-\tau)^{2}}+\frac{3 x}{2(t-\tau)^{2}}\right\} d \tau
$$

$$
\left.-\frac{x^{3}}{(t-\tau)^{3}}+\frac{3 x}{2(t-\tau)^{2}}+\frac{3 x}{2(t-\tau)^{2}}\right\}+\frac{5 x^{4}}{(t-\tau)^{4}}+
$$

$$
-\frac{3}{8(t-\tau)} d+\frac{6}{8(t-\tau)^{2}}-\frac{9 x^{2}}{(t-\tau)^{3}}-\frac{3 x^{2}}{2(t-\tau)^{3}}+
$$

$$
\left.+\frac{3}{(t-\tau)^{2}}\right\} d \tau
$$

$$
\frac{\partial^{6} T}{\partial x^{6}}=\frac{1}{2 \sqrt{\pi}} \int_{0}^{t} \frac{T_{s}(\tau)}{(t-\tau)^{3 / 2}} e^{-\frac{x^{2}}{4(t-\tau)}} \frac{x}{4(t-\tau)^{3}}
$$

$$
\left[\frac{x^{6}}{16(t-\tau)^{3}}-\frac{21 x^{4}}{8(t-\tau)^{2}}+\frac{105 x^{2}}{4(t-\tau)}-45\right] d \tau
$$

$$
\frac{\partial^{7} T}{\partial x^{7}}=\frac{1}{2 \sqrt{\pi}} \int_{0}^{t} \frac{T_{s}(\tau)}{(t-\tau)^{3 / 2}} e^{-\frac{x^{2}}{4(t-\tau)}}
$$

$$
\left\{\begin{array}{l}
-\frac{x^{8}}{128(t-\tau)^{7}}+\frac{28 x^{6}}{64(t-\tau)^{6}}-\frac{210 x^{4}}{32(t-\tau)^{5}}+ \\
+\frac{405 x^{2}}{16(t-\tau)^{4}}+\frac{45}{4(t-\tau)^{3}}
\end{array}\right\} d \tau
$$

$$
\operatorname{sgn} 0=0, \quad \operatorname{sgn} x=1, \quad x>0
$$

The approximate numerical solution is given by the recurrent formulae (37), therefore the analytic
expression for derivatives (39) in equation (38) are satisfactory only for an initial small time step:

$$
\begin{align*}
& \left(\frac{\partial^{3} T}{\partial x^{3}}\right)_{0}=-\frac{3}{4 \sqrt{\pi}} \int_{0}^{t} \frac{T_{s}(\tau)}{(t-\tau)^{5 / 2}} d \tau  \tag{40}\\
& \left(\frac{\partial^{5} T}{\partial x^{5}}\right)_{0}=-\frac{3}{8 \sqrt{\pi}} \int_{0}^{t} \frac{T_{s}(\tau)}{(t-\tau)^{5 / 2}}\left(\frac{1}{2}-\frac{5}{t-\tau}\right) d \tau
\end{align*}
$$

### 4.3.2 First and second order accuracy by time

Now the heat fluxes in a first and second approach by time are, respectively:

$$
\begin{align*}
& -q=-\left(\frac{\partial T}{\partial x}\right)_{0}=-\frac{3 t}{4 \sqrt{\pi}} \int_{0}^{t} \frac{T_{s}(\tau)}{(t-\tau)^{5 / 2}} d \tau \\
& -q=-\left(\frac{\partial T}{\partial x}\right)_{0}=-\frac{3 t}{4 \sqrt{\pi}} \int_{0}^{t} \frac{T_{s}(\tau)}{(t-\tau)^{5 / 2}} d \tau-  \tag{41}\\
& \frac{3 t^{2}}{16 \sqrt{\pi}} \int_{0}^{t} \frac{T_{s}(\tau)}{(t-\tau)^{5 / 2}}\left(\frac{1}{2}-\frac{5}{t-\tau}\right) d \tau
\end{align*}
$$

where $t$ is small.

### 4.3.3 Comparison of the numerical solution with the exact solution

To compare (41) with the exact solution, let us take also the third order approximation by $t$
$-q=-\left(\frac{\partial T}{\partial x}\right)_{0}=-\frac{3 t}{4 \sqrt{\pi}} \int_{0}^{t} \frac{T_{s}(\tau)}{(t-\tau)^{5 / 2}} d \tau+$
$\frac{3 t^{2}}{16 \sqrt{\pi}} \int_{0}^{t} \frac{T_{s}(\tau)}{(t-\tau)^{5 / 2}}\left(\frac{1}{2}-\frac{5}{t-\tau}\right) d \tau+\frac{t^{3}}{64 \sqrt{\pi}} \int_{0}^{t} \frac{T_{s}(\tau)}{(t-\tau)^{7 / 2}}\left(1-\frac{42}{t-\tau}\right) d \tau$

The approximate solution (35) with accuracy by third order terms by time is as follows:

$$
\begin{aligned}
& T(x, t)=T_{s}(t)(1-\operatorname{sgn} x) \frac{x}{2 \sqrt{\pi}} \cdot \\
& \cdot \int_{0}^{t} \frac{T_{s}(\tau)}{(t-\tau)^{3 / 2}} e^{-\frac{x^{2}}{4(t-\tau)}} \frac{1}{2}\left(\begin{array}{l}
\frac{t}{(t-\tau)}\left(\frac{x^{2}}{2(t-\tau)}-3\right)+ \\
\frac{t^{2}}{4(t-\tau)^{2}}\left(\frac{x^{4}}{4(t-\tau)^{2}}-\frac{5 x^{2}}{t-\tau}+15\right)^{+}
\end{array} .\right.
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{t^{2}}{4(t-\tau)^{2}}\left(\frac{x^{4}}{4(t-\tau)^{2}}-\frac{5 x^{2}}{t-\tau}+15\right)+\frac{t^{3}}{12(t-\tau)^{3}} . \\
& \cdot\left[\frac{x^{6}}{16(t-\tau)^{3}}+-\frac{21 x^{4}}{8(t-\tau)^{2}}+\frac{105 x^{2}}{4(t-\tau)}-45\right] d r .
\end{aligned}
$$

To estimate the approximate solution (43) obtained by our strategy and compare it with the exact analytical solution of the problem, the new variable $\xi=\frac{x}{2 \sqrt{t-\tau}}$ is introduced in (43)

$$
\begin{align*}
& \Delta T(x, t)=\frac{2}{\sqrt{\pi}} \int_{\frac{x}{2 \sqrt{t}}}^{\infty} T\left(t+\frac{\left.x^{2}\right)}{4 \xi^{2}}\right) e^{-\xi^{2}}\left[\begin{array}{l}
0-2 \frac{t \xi^{2}}{x^{2}}\left(2 \xi^{2}-3\right)- \\
-2 \frac{t^{2} \xi^{4}}{x^{4}}\left(4 \xi^{4}-2 \xi^{2}+15\right)+ \\
\left.\quad+\frac{8 t^{3} \xi^{6}}{3 x^{6}}\left(4 \xi^{6}-42 \xi^{4}+105 \xi^{2}-45\right)\right] d \xi
\end{array}, \$\right. \text {, }
\end{align*}
$$

Where is: $x=0,1-\operatorname{sgg} 0=1 ; x>0,1-\operatorname{sgnx}=1-1=0$, $\Delta T(x, t)=T(x, t)-\widetilde{T}(x, t), \widetilde{T}(x, t)$ is an approximate solution for the equation (43). In a first order approach by time from the equation (44) follows

$$
\begin{align*}
& \Delta T=[1-(1-\operatorname{sgn} x)] T_{s}(t)+\ldots= \\
& =(\operatorname{sgn} x) T_{s}(t)+\ldots=-\frac{x}{4 \sqrt{t}} e^{\frac{x^{2}}{4 t}} \tag{45}
\end{align*}
$$

with a small exponential inaccuracy. By $x=0$, $\Delta T=0$.

If the multiplayer $\frac{2}{\sqrt{\pi}} T_{S}^{*}$ is got out of integral $\left(T_{S}^{*}=\max T_{S}\right):$ then

$$
\begin{align*}
& \int_{\frac{x}{2 \sqrt{t}}}^{\infty} e^{-\xi^{2}}\left[-2 \frac{t \xi^{2}}{x^{2}}\left(2 \xi^{2}-3\right)-\right. \\
& \left.2 \frac{t^{2} \xi^{4}}{x^{4}}\left(4 \xi^{4}-2 \xi^{2}+15\right)-\frac{8 t^{3} \xi^{6}}{3 x^{6}} \cdot\right] d \xi  \tag{46}\\
& \cdot\left(4 \xi^{6}-42 \xi^{4}+105 \xi^{2}-45\right)
\end{align*}
$$

In a first order approach by time $t$ :

$$
\begin{align*}
& \operatorname{sgn} x-\int_{0}^{\frac{x}{2 \sqrt{t}}} e^{-\xi^{2}} d \xi-\frac{x}{4 \sqrt{t}} e^{-\frac{x^{2}}{4 t}}= \\
& -\int_{0}^{\frac{x}{2 \sqrt{t}}} e^{-\xi^{2}} d \xi-\frac{x}{4 \sqrt{t}} e^{-\frac{x^{2}}{4 t}}, \tag{47}
\end{align*}
$$

where from by $x=0, t=0$ it results exactly zero: $0-0$ $0=0$ and $1-1-0=0$.

Accounting the equations (46) and (45), in a first order approach by time the following inaccuracy is got:
$\Delta T=\left(\int_{0}^{\infty} e^{-\xi^{2}} d \xi-\int_{0}^{\frac{x}{2 \sqrt{t}}} e^{-\xi^{2}} d \xi\right) \operatorname{sgn} x-\frac{x}{4 \sqrt{t}} e^{-\frac{x^{2}}{4 t}}$.
In a second order approach by $t$, accordingly:

$$
\begin{align*}
& \Delta T=\left(\int_{0}^{\infty} e^{-\xi^{2}} d \xi-\int_{0}^{\frac{x}{2 \sqrt{t}}} e^{-\xi^{2}} d \xi\right) \operatorname{sgn} x-\frac{x}{4 \sqrt{t}} e^{-\frac{x^{2}}{4 t}}+ \\
& -\frac{x}{16 \sqrt{t}}\left(5+\frac{x^{2}}{2 t}\right) e^{-\frac{x^{2}}{4 t}} \tag{49}
\end{align*}
$$

where from follows that in a first order approach by time inaccuracy of numerical solution is $\sim \frac{x}{\sqrt{t}} e^{-\frac{x^{2}}{t}}$, which exceeds by small $t$ an order of $t$ dramatically, $\frac{t \sqrt{t}}{x} e^{\frac{x^{2}}{t}}$ times, exponentially.

For instance, by $t=10^{-2}$ it is

$$
\frac{10^{-2} \sqrt{10^{-2}}}{x} e^{\frac{x^{2}}{10^{-2}}}=\frac{10^{-3}}{x} e^{10^{2} x^{2}}
$$

or $\frac{e^{100 x^{2}}}{10^{3} x}$, huge value, except small $x$, where $e^{\frac{x^{2}}{4 t}} \approx 1-\frac{x^{2}}{4 t}$ and then $\Delta T=-\frac{x}{4 \sqrt{t}}\left(1-\frac{x^{2}}{4 t}\right)$, small value due to $\operatorname{small} x$.

### 4.3.4 The case of complete coincide with the exact analytical solution

By $x=0, \Delta T=0$, then by $t=0, \Delta T=0$, and we got complete coincide with the exact solution. The second approach (49) decreases accuracy by $t$ adding the term $\frac{x^{3}}{t \sqrt{t}} e^{-\frac{x^{2}}{t}}$, of order $x^{2} / t$ comparing to the first order solution.

Obviously, by fixed $x$ and small $t$ this can decrease accuracy, which is good in a first order approach. In (46) the following integrals were computed:

$$
\begin{aligned}
& 2 \int_{\frac{x}{2 \sqrt{t}}}^{\infty} \xi^{2} e^{-\xi^{2}} d \xi= \\
& =-\int_{\frac{x}{2 \sqrt{t}}}^{\infty} \xi e^{-\xi^{2}} d\left(\xi^{2}\right)=-\binom{\left.\xi e^{-\xi^{2}}\right|_{\frac{x}{x}} ^{2 \sqrt{t}}}{\int_{x}^{\infty} e^{-\xi^{2}} d \xi}= \\
& \frac{x}{2 \sqrt{t}} e^{-\frac{x^{2}}{4 t}}+\int_{\frac{x}{2 \sqrt{t}}}^{\infty} e^{-\xi^{2}} d \xi, \\
& 2 \int_{\frac{x}{2 \sqrt{t}}}^{\infty} e^{-\xi^{2}} \xi^{8} d \xi=\int_{\frac{x}{2 \sqrt{t}}}^{\infty} e^{-\xi^{2}} \xi^{3} d \xi= \\
& =\frac{x^{3}}{8 t \sqrt{t}} e^{-\frac{x^{2}}{4 t}}+\frac{3 x}{4 \sqrt{t}} e^{-\frac{x^{2}}{4 t}}+\frac{3}{2} \int_{\frac{x}{2 \sqrt{t}}}^{\infty} e^{-\xi^{2}} d \xi, \\
& 2 \int_{\frac{x}{2 \sqrt{t}}}^{\infty} e^{-\xi^{2}} \xi^{8} d \xi=-\left[\begin{array}{l}
\xi^{7} e^{-\xi^{2}}+ \\
\frac{7}{2}\left(\xi^{7} e^{-\xi^{2}} 5 \int e^{-\xi^{2}} \xi^{4} d \xi\right)
\end{array}\right] \frac{\left.\right|_{x} ^{\infty} x \sqrt{2 \sqrt{t}}=}{} \\
& =\left(\frac{x^{7}}{2^{7} t^{7 / 2}}+\frac{7 x^{5}}{2^{6} t^{5 / 2}}+\frac{35 x^{3}}{32 t^{3 / 2}}+\frac{105 x}{16 \sqrt{t}}\right) e^{-\frac{x^{2}}{4 t}}+ \\
& +\frac{105}{8} \int_{\frac{x}{2 \sqrt{t}}}^{\infty} e^{-\xi^{2}} d \xi
\end{aligned}
$$

$$
\begin{aligned}
& 2 \int_{\frac{x}{2 \sqrt{t}}}^{\infty} e^{-\xi^{2}} \xi^{6} d \xi=\int_{\frac{x}{2 \sqrt{t}} \xi^{5} d e^{-\xi^{2}}=}^{-\left(\frac{x^{5}}{2^{5} t^{5 / 2}}+\frac{5 x^{3}}{16 t^{3 / 2}}+\frac{15 x}{8 \sqrt{t}}\right) e^{-\frac{x^{2}}{4 t}}-\frac{15}{4} \int_{\frac{x}{2 \sqrt{t}}}^{\infty} e^{-\xi^{2}} d \xi ;} \\
& -\left(\frac { 4 t ^ { 2 } } { x ^ { 4 } } \left(\frac{x^{7}}{2^{7} t^{1 / 2}}+\frac{7 x^{5}}{2^{6} t^{5 / 2}}+\frac{35 x^{3}}{32 t^{2}} \frac{t^{2}}{x^{4}} e^{-\xi^{2}}\left(4 \xi^{4}-20 \xi^{2}+15\right) d \xi=e^{-\frac{x^{2}}{4 t}}+\right.\right. \\
& +\frac{4 t^{2}}{x^{4}} \frac{105 x}{8} \\
& \frac{\int_{x}}{2 \sqrt{t}} e^{-\xi^{2}} d \xi+20 \frac{t^{2}}{x^{4}} e^{-\frac{x^{2}}{4 t}} \cdot \\
& \cdot\left(-\frac{x^{5}}{2^{5} t^{5 / 2}}-\frac{5 x^{3}}{16 t^{3 / 2}}-\frac{15 x}{8 \sqrt{t}}\right)-20 \frac{t^{2}}{x^{4}} \frac{15}{4} \int_{\frac{x}{2 \sqrt{t}}}^{\infty} e^{-\xi^{2}} d \xi+ \\
& +15 \frac{t^{2}}{x^{4}}\left(\frac{x^{3}}{8 t^{3 / 2}}+\frac{3 x}{4 \sqrt{t}}\right) e^{-\frac{x^{2}}{4 t}}+45 \frac{t^{2}}{x^{4}} \int_{\frac{x}{2 \sqrt{t}}}^{\infty} e^{-\xi^{2}} d \xi= \\
& =\frac{x}{2^{4} t^{1 / 2}}\left(5+\frac{x^{2}}{2 t}\right) e^{-\frac{x^{2}}{4 t}} .
\end{aligned}
$$

Surprisingly all terms of a negative power by $x$ a mutually omitted, the same as integrals, then on . the exponent and arguments $\frac{x}{\sqrt{t}}, \frac{x^{2}}{t}$ are kept.

By $x=0$ the numerical solution by our method completely coincides with the exact analytical solution in a first order approach.

By fixed $x$ the accuracy is high (deficiency decreases exponentially by time). By small $x$ inaccuracy can grow, therefore it is important to choose right steps by $x$ and $t$ nearby the boundary $x=0$.

## 5 Numerical simulation

The proposed numerical algorithm described and studied in detail above was applied for the numerical simulation of the heat transfer process around a particle in a fluid flow.

### 5.1 Heat transfer for a particle in a flow

The process of a heat transfer for a particle in a flow is described by the boundary problem presented below.

Let us consider some particle in a fluid flow using the polar coordinate system $\rho, \vartheta$, where $\rho$ and $\vartheta$ are respectively coordinates by radius and angle, $\rho=0$ is the center of the moving particle and $\rho=1$ is its surface, $\tau$ is time. Then twodimensional energy equation is written as follows [16]:

$$
\begin{align*}
& \frac{\partial T}{\partial \tau}-\frac{\partial^{2} T}{\partial \rho^{2}}-\left[\frac{2}{\rho}-\frac{P e}{2} \cos \vartheta\left(1-\frac{3}{2 \rho}+\frac{1}{2 \rho^{3}}\right)\right] \frac{\partial T}{\partial \rho}-  \tag{51}\\
& \frac{\partial^{2} T}{\rho^{2} \partial \vartheta^{2}}-\left[\frac{\operatorname{ctg} \vartheta}{\rho^{2}}+\frac{P e \sin \vartheta}{2 \rho}\left(1-\frac{3}{4 \rho}-\frac{1}{4 \rho^{3}}\right)\right] \frac{\partial T}{\partial \vartheta}=0
\end{align*}
$$

where $\quad \rho=\frac{r}{R}, \quad \tau=\frac{a t}{R^{2}} \quad$ and $\quad P e=\frac{2 U R}{a} \quad$ are dimensionless coordinate, time and Peclet number, correspondingly, $T$ is a temperature, $R$ is a radius of a particle, $U, a$ - flow velocity and heat diffusivity coefficient in a fluid flow.

The domain of fluid flow is considered as $1<\rho>\infty, \quad 0 \leq \vartheta \leq \pi, \quad 0<\tau<\infty$. The next boundary and initial conditions for the partial differential equation (51) are stated

$$
\begin{gather*}
\tau=0, \quad T=T_{0}  \tag{52}\\
\rho=1, \quad T=T_{s}(\vartheta, \tau) ; \quad \rho=\infty, \quad T=0 \tag{53}
\end{gather*}
$$

A few selected simulation results to show the efficacy of the method in a first order approach by time are given in the Figs 1-4 below.

### 5.2 The results of numerical simulation

By the method described above the boundary problem for PDE (51)-(53) was solved numerically and the results are presented here. For example, solution of the problem for the uniform initial
( $\tau=0$ ) temperature distribution shown in Fig. 1 is presented in Fig. 2- Fig. 4 for $\mathrm{Pe}=10$ :


Fig. 1. Initial temperature distribution in the domain


Fig. 2. Temperature distribution in the particle and fluid flow around it for $\tau=2.41$

The fidelity of computations was taken here as $\varepsilon=10^{-4}$. A picture around a sphere is symmetrical. For the initial-boundary value problem (51)-(53) there were not found any remarkable difference between computation results in the second and in the third order by time (for higher orders as well) despite diverse boundary conditions proven in the numerical simulation. In this case even the first and second order calculations are very close as seen in Fig. 4 and Fig. 5.

It was considered a case when particle has higher temperature than fluid flow in surroundings around particle, for example, radioactive fuel drop or particle during severe accident at NPP [1, 2]. As shown in Figs 1-5, first fluid flow does not heat up
from particle, then temperature behind the particle is slightly growing and afterwards temperature is front of particle increases.


Fig.3. Temperature distribution in the particle and fluid flow around it for $\tau=10.0$


Fig.4. Temperature distribution in the particle and fluid flow around it for $\tau=15.0$


Fig.5. Temperature for $\tau=15.0$, second order approximation by time

Numerical simulation performed for a number of a different boundary temperature distribution $T_{s}(\theta, \tau)$ has shown that the second and the third order by time solutions nearly coincide so that normally for this problem no higher than a second orders by time is needed. A computation time among the first-fifth order by time does not reveal
remarkable difference but by the sixth order by time it starts to grow substantially.

A few selected simulation results to support the efficacy of the method are given in the Table below:

Table. Computation of a temperature distribution $T(r, \theta, \tau)$ in a Stokes flow around a sphere for a time dependent temperature on the sphere $(\mathrm{r}=1) T_{s}(\theta, \tau)=|\cos \theta| \exp (100 \tau)$

| $1^{\text {st }}$ | $\tau=0.03$ |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| order | $\theta=$ |  |  |  |  |  |  |  |
| $\mathrm{r}=$ | 0.0000 | 0.6283 | 1.2566 | 1.8850 | 2.5133 | 3.1416 | 3.7699 | 4.3982 |
| 1 | 20.08554 | 16.24954 | 6.206772 | 6.206772 | 16.24954 | 20.08554 | 16.24954 | 6.206772 |
| 2 | 0.081860 | 0.115846 | 0.061748 | 0.060168 | 0.105751 | 0.178708 | 0.105759 | 0.059840 |
| 5 | 0.012018 | 0.012017 | 0.012013 | 0.012007 | 0.012003 | 0.012000 | 0.012003 | 0.012007 |
| 8 | 0.007506 | 0.007506 | 0.007504 | 0.007501 | 0.007499 | 0.007498 | 0.007499 | 0.007501 |
| $2^{\text {nd }}$ | $\theta$ |  |  |  |  |  |  |  |
| 1 | 20.08554 | 16.24954 | 6.206772 | 6.206772 | 16.24954 | 20.08554 | 16.24954 | 6.206772 |
| 2 | 0.081728 | 0.115994 | 0.061897 | 0.060317 | 0.105899 | 0.179139 | 0.105907 | 0.059987 |
| 5 | 0.012078 | 0.012077 | 0.012073 | 0.012067 | 0.012063 | 0.012060 | 0.012063 | 0.012067 |
| 8 | 0.007544 | 0.007543 | 0.007541 | 0.007539 | 0.007537 | 0.007536 | 0.007537 | 0.007539 |
| $3^{\text {rd }}$ | $\theta$ |  |  |  |  |  |  |  |
| 1 | 20.08554 | 16.24954 | 6.206772 | 6.206772 | 16.24954 | 20.08554 | 16.24954 | 6.206772 |
| 2 | 0.081728 | 0.115994 | 0.061897 | 0.060318 | 0.105900 | 0.179140 | 0.105907 | 0.059988 |
| 5 | 0.012078 | 0.012077 | 0.012073 | 0.012068 | 0.012063 | 0.012060 | 0.012063 | 0.012068 |
| 8 | 0.007544 | 0.007543 | 0.007541 | 0.007539 | 0.007537 | 0.007536 | 0.007537 | 0.007539 |

A picture around the sphere is symmetrical, therefore the three points after $\theta=4.3982$ are omitted just to save a place in the Table.

For the initial-boundary value problem considered there any remarkable difference between
computation results in the second and in the third order by time (for higher orders as well) were not found despite the diverse boundary conditions has been proven in the numerical simulation.

Many different cases were tested in these computational experiments including oscillating
temperature distribution on the particle surface, huge gradients, etc. In all cases the methods wrked fine.

### 5.3 Numerical experiment by particle in a fluid flow

Many parameters were varied for the boundary problem studied. An influence of Peclet number, initial time step, fidelity of calculation, etc. were experienced [19-21] for different orders of accuracy by time, diverse temperature distributions on the particle's surface, and so on.

For example, the results of computer experiment by $\mathrm{Pe}=0.1, \mathrm{Dt}=1$ are presented in Fig. 6- Fig. 9 with respective matrices of computations to see the small differences in computations:
$\mathrm{r}=1.000000 \quad \mathrm{q}=0.0000 \quad \mathrm{q}=0.6283 \quad \mathrm{q}=1.2566 \quad \mathrm{q}=1.8850$ $\mathrm{q}=2.5133 \mathrm{q}=3.1416 \mathrm{q}=3.7699 \mathrm{q}=4.3982 \mathrm{q}=5.0265 \mathrm{q}=5.6549$ $\begin{array}{lllll}\mathrm{r}=1.000000 & 1.000000 & 1.000000 & 1.000000 & 1.000000\end{array}$ 1.0000001 .0000001 .0000001 .0000001 .0000001 .000000 $\begin{array}{lllll}\mathrm{r}=2.000000 & 0.100363 & 0.100363 & 0.100363 & 0.100363\end{array}$ 0.1003630 .1003630 .1003630 .1003630 .1003630 .100363 $\begin{array}{llllll}\mathrm{r}=3.000000 & 0.007364 & 0.007364 & 0.007364 & 0.007364\end{array}$ 0.0073640 .0073640 .0073640 .0073640 .0073640 .007364 $\begin{array}{llllll}\mathrm{r}=4.000000 & 0.000396 & 0.000396 & 0.000396 & 0.000396\end{array}$ 0.0003960 .0003960 .0003960 .0003960 .0003960 .000396 $\begin{array}{llllll}\mathrm{r}=5.000000 & 0.000016 & 0.000016 & 0.000016 & 0.000016\end{array}$ 0.0000160 .0000160 .0000160 .0000160 .0000160 .000016 $\begin{array}{llllll}\mathrm{r}=6.000000 & 0.000001 & 0.000001 & 0.000001 & 0.000001\end{array}$ 0.0000010 .0000010 .0000010 .0000010 .0000010 .000001 $\begin{array}{lllll}\mathrm{r}=7.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000\end{array}$ 0.0000000 .0000000 .0000000 .0000000 .0000000 .000000 $\begin{array}{lllll}\mathrm{r}=8.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000\end{array}$ 0.0000000 .0000000 .0000000 .0000000 .0000000 .000000 $\begin{array}{lllll}\mathrm{r}=9.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000\end{array}$ 0.0000000 .0000000 .0000000 .0000000 .0000000 .000000


Fig. 6. Results for accuracy $0.01, \mathrm{n}=1, \mathrm{t}=1.250$
$\mathrm{r}=1.000000 \quad \mathrm{q}=0.0000 \quad \mathrm{q}=0.6283 \quad \mathrm{q}=1.2566 \quad \mathrm{q}=1.8850$ $\mathrm{q}=2.5133 \mathrm{q}=3.1416 \mathrm{q}=3.7699 \mathrm{q}=4.3982 \mathrm{q}=5.0265 \mathrm{q}=5.6549$ $\begin{array}{lllll}\mathrm{r}=1.000000 & 1.000000 & 1.000000 & 1.000000 & 1.000000\end{array}$ 1.0000001 .0000001 .0000001 .0000001 .0000001 .000000 $\begin{array}{lllll}\mathrm{r}=2.000000 & 0.014893 & 0.014893 & 0.014893 & 0.014893\end{array}$ 0.0148930 .0148930 .0148930 .0148930 .0148930 .014893 $\begin{array}{lllll}\mathrm{r}=3.000000 & 0.000163 & 0.000163 & 0.000163 & 0.000163\end{array}$ 0.0001630 .0001630 .0001630 .0001630 .0001630 .000163 $\begin{array}{lllll}\mathrm{r}=4.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000\end{array}$ 0.0000000 .0000000 .0000000 .0000000 .0000000 .000000 $\begin{array}{lllll}\mathrm{r}=5.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000\end{array}$ 0.0000000 .0000000 .0000000 .0000000 .0000000 .000000 $\begin{array}{lllll}\mathrm{r}=6.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000\end{array}$ 0.0000000 .0000000 .0000000 .0000000 .0000000 .000000 $\begin{array}{llllll}\mathrm{r}=7.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000\end{array}$ 0.0000000 .0000000 .0000000 .0000000 .0000000 .000000 $\begin{array}{lllll}\mathrm{r}=8.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000\end{array}$ 0.0000000 .0000000 .0000000 .0000000 .0000000 .000000 $\begin{array}{llllll}\mathrm{r}=9.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000\end{array}$ 0.0000000 .0000000 .0000000 .0000000 .0000000 .000000


Fig. 7. Results for accuracy $0.01, \mathrm{n}=2, \mathrm{t}=1.031$
$\mathrm{r}=1.000000 \quad \mathrm{q}=0.0000 \quad \mathrm{q}=0.6283 \quad \mathrm{q}=1.2566 \quad \mathrm{q}=1.8850$ $\mathrm{q}=2.5133 \mathrm{q}=3.1416 \mathrm{q}=3.7699 \mathrm{q}=4.3982 \mathrm{q}=5.0265 \mathrm{q}=5.6549$ $\begin{array}{lllll}\mathrm{r}=1.000000 & 1.000000 & 1.000000 & 1.000000 & 1.000000\end{array}$ 1.0000001 .0000001 .0000001 .0000001 .0000001 .000000 $\begin{array}{lllll}\mathrm{r}=2.000000 & 0.066300 & 0.066300 & 0.066300 & 0.066300\end{array}$ 0.0663000 .0663000 .0663000 .0663000 .0663000 .066300 $\begin{array}{lllll}\mathrm{r}=3.000000 & 0.003277 & 0.003277 & 0.003277 & 0.003277\end{array}$ 0.0032770 .0032770 .0032770 .0032770 .0032770 .003277 $\begin{array}{lllll}\mathrm{r}=4.000000 & 0.000125 & 0.000125 & 0.000125 & 0.000125\end{array}$ 0.0001250 .0001250 .0001250 .0001250 .0001250 .000125 $\begin{array}{lllll}\mathrm{r}=5.000000 & 0.000004 & 0.000004 & 0.000004 & 0.000004\end{array}$ 0.0000040 .0000040 .0000040 .0000040 .0000040 .000004 $\begin{array}{lllll}\mathrm{r}=6.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000\end{array}$ 0.0000000 .0000000 .0000000 .0000000 .0000000 .000000 $\begin{array}{lllll}\mathrm{r}=7.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000\end{array}$ 0.0000000 .0000000 .0000000 .0000000 .0000000 .000000 $\begin{array}{lllll}\mathrm{r}=8.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000\end{array}$ 0.0000000 .0000000 .0000000 .0000000 .0000000 .000000 $\begin{array}{lllll}\mathrm{r}=9.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000\end{array}$ 0.0000000 .0000000 .0000000 .0000000 .0000000 .000000


Fig. 8. Results for accuracy $0.01, \mathrm{n}=3, \mathrm{t}=1.156$

$$
\begin{array}{lllllc}
\mathrm{r}=1.000000 & \mathrm{q}=0.0000 & \mathrm{q}=0.6283 & \mathrm{q}=1.2566 & \mathrm{q}=1.8850 \\
\mathrm{q}=2.5133 & \mathrm{q}=3.1416 & \mathrm{q}=3.7699 & \mathrm{q}=4.3982 & \mathrm{q}=5.0265 & \mathrm{q}=5.6549 \\
\mathrm{r}=1.000000 & 1.000000 & 1.000000 & 1.000000 & 1.000000 \\
1.000000 & 1.000000 & 1.000000 & 1.000000 & 1.000000 & 1.000000 \\
\mathrm{r}=2.000000 & 0.115972 & 0.115972 & 0.115972 & 0.115972 \\
0.115972 & 0.115972 & 0.115972 & 0.115972 & 0.115972 & 0.115972 \\
\mathrm{r}=3.000000 & 0.010848 & 0.010848 & 0.010848 & 0.010848 \\
0.010848 & 0.010848 & 0.010848 & 0.010848 & 0.010848 & 0.010848 \\
\mathrm{r}=4.000000 & 0.000801 & 0.000801 & 0.000801 & 0.000801 \\
0.000801 & 0.000801 & 0.000801 & 0.000801 & 0.000801 & 0.000801 \\
\mathrm{r}=5.000000 & 0.000048 & 0.000048 & 0.000048 & 0.000048 \\
0.000048 & 0.000048 & 0.000048 & 0.000048 & 0.000048 & 0.000048 \\
\mathrm{r}=6.000000 & 0.000002 & 0.000002 & 0.000002 & 0.000002 \\
0.000002 & 0.000002 & 0.000002 & 0.000002 & 0.000002 & 0.000002 \\
\mathrm{r}=7.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 \\
0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 \\
\mathrm{r}=8.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 \\
0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 \\
\mathrm{r}=9.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 \\
0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000
\end{array}
$$



Fig. 9. Results for accuracy $0.01, \mathrm{n}=4, \mathrm{t}=1.313$

It is clearly observed that by small time intervals calculations with any order of accuracy by time
from $\mathrm{n}=1$ to $\mathrm{n}=4$ are nearly the same. Exact comparison is difficult to perform due to application of the automatically corrected time step.

The results in Figs 6-9 and Figs 10-12 differ from the ones presented in Figs $1-5$ because the Peclet number is ten time lower ( 1 comparing to the previous 10), so that convective and conductive heat transfer is of the same order here.

There are revealed some questions to investigate more in detail, e.g. growing of the inaccuracy of computation by large time values (after 100-200 depending on the parameters) when unphysical data are got, e.g. temperature far from particle exceeds the particle temperature, which is impossible.

Also by increasing the temporal approximations, the number of the grid points by space is growing (every additional term in Taylor series by time means increase of the order of spatial derivatives by 2 ), and the difference between spatial derivatives inside the numerical domain and outside of it becomes more and more substantial.


Fig. 10. Results for accuracy $0.01, \mathrm{n}=1, \mathrm{t}=50,9$


Fig. 11. Results for accuracy $0.01, \mathrm{n}=2, \mathrm{t}=56,0$


Fig. 12. Results for accuracy $0.01, n=4, t=48,3$
Figs 10-12 show ones more that for this task all four first approximations by time give close results.

## 6 Conclusion

The examples considered here have clearly demonstrated an effectiveness and simplicity of the strategy proposed for the numerical solution of the non-stationary non-isothermal Navier-Stokes equations, as well as any other first-order by time partial differential equation array. An order of the equations by spatial variables has no matter.

The strategy is based on application of a Taylor series by time for the computation of the solution sought by its temporal derivatives. These temporal derivatives of the functions are expressed from the outgoing equations through their spatial derivatives.

The highest-order temporal derivatives used in the Taylor series for computation of the approximate solution are computed by differentiating the outgoing partial differential equations by time.

Only the one Poisson equation has to be solved numerically in case of the full Navier-Stokes equation array. All the other computation are just based on the computation of the spatial derivatives and corresponding temporal derivatives through them directly using the equations obtained in the strategy proposed here.

The strategy of such numerical solution of the boundary problems for the partial differential equations has been considered for a few diverse examples.

The efficiency and simplicity of the method is achieved through a substitution of a solution of the finite-difference (or finite-element, etc.) equations in the known numerical methods with a simpler procedure of the spatial differentiation and further computation of the Taylor series.

Proposed strategy is going to be proved on different boundary problems for the partial differential equations to study all the pros and cons for its further implementation into a practice.

Important to underline that the proof of the method proposed was successfully done also through the comparison of the numerical solution with the exact analytical solution got for one case using the method of fractional differentials.

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