Asymptotic Solutions of a Fifth Order Ordinary Differential Equation

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Abstract: The objective of this paper is to construct the asymptotic solutions of a fifth order model equation for steady capillary-gravity waves over a smooth compact bump with the Froude number near 1 and the Bond number near 1/3.

Key-Words: Steady capillary-gravity wave, fifth order model equation, bump, Green's function, asymptotic solution.

1 Introduction

Progressive capillary-gravity waves on an irrotional incompressible inviscid fluid of constant density with surface tension in a two-dimensional channel of finite depth have been studied since nineteen century. Assume that a coordinate system moving with the wave at a speed is chosen so that in reference to it the wave motion is steady. Let H be the depth of water, g the acceleration of gravity, T the coefficient of surface tension, and ρ the constant density of the fluid. Then there are two nondimensional numbers which are important and defined as $F = c^2/(gH)$, the Froude number, and $\tau = T/(\rho g H^2)$, the Bond number.

When F is not close to 1, the linear theory of water waves is applicable. But when F approaches to 1, the solutions of linearized equations of water waves will grow to infinity (Peters and Stoker [11]). Therefore for F close to 1 nonlinear effect must be taken into account and thus $F = 1$ is a critical value. The first study of a solitary wave on water with surface tension is due to Korteweg and DeVries [10] after whom the K-dV equation with surface tension effect is named. A stationary K-dV equation with Bond number not near $1/3$ can also be formally derived by different approaches. However, if τ is close to 1, the formal derivation of the stationary K-dV equation fails. Thus $\tau = 1/3$ is also a critical value.

It becomes apparent that the problems for F near 1 and for τ near $1/3$ depend on each other and are difficult because they are not only strongly nonlinear, but also very delicate. Since the full nonlinear equations for the water waves are too complicated to study, it is of interest to study model equations. In Hunter and Vanden-Broeck's work [8], a fifth order ordinary differential equation considered as a perturbed stationary K-dV equation was obtained with the assumption that $F = 1 + F_2\epsilon^2$, $\tau = 1/3 + \tau_1\epsilon$ and ϵ is a small positive parameter. By integrating the fifth order ordinary differential equation once and set the con-stant of integration to be zero, then the model equation becomes

$$2F_2\eta-\frac{3}{2}\eta^2+\tau_1\eta_{xx}-\frac{1}{45}\eta_{xxxx}=0 \quad (1)$$

Equation (1) has been studied extensively by many authors [1-7, 9] and several types of solutions have been found, such as periodic solutions [1, 5, 6, 7], solitary wave solutions [2-7], generalized solitary wave solutions (solitary waves with osciallatory tails at infinity) in the parameter region $\tau_1<0$ and $F_2>0$ [1,9], etc.

We add a bump $y=b(x)$ at the bottom of the two-dimensional ideal fluid flow and then derive a forced model equation (56),

$$2F_2\eta-\frac{3}{2}\eta^2+\tau_1\eta_{xx}-\frac{1}{45}\eta_{xxxx}=\mathbf{b}\cdot$$

Equation (56) has been studied in [12-16] and several types of solutions have been found, such as unsymmetric solitary wave solutions [14], solitary wave solutions [15], and generalized solitary wave solutions [16].

However, for **Case 0**,**2**,**3** and **4** in section 3, the proof of existence of bounded solutions of (56) is not available at present; we construct asymptotic solutions for these four **Cases** by assuming η and bump b to be small.

2 Derivation of the model equation

We consider the two-dimensional flow of an irrotional incompressible inviscid fluid of constant density ρ^* with surface tension T^* in a two-dimensional channel of finite depth. A rectangular coordinate system $(x^*,\ y^*)$ is chosen such that the flow is bounded above by the free surface $y^*=\eta^*(x^*,t^*)$ and below by the rigid horizontal bottom with a bump $y^*=-H+\mathbf{b}^*(x^*)$.

The governing equations are:

In $-\infty<x^*<\infty, -H+\mathbf{b}^*(x^*)<y^*<\eta^*$

$$\phi^*_{x^*x^*}+\phi^*_{y^*y^*}=0, \quad (2)$$

at the free surface, $y^*=\eta^*$

$$\eta^*_{t^*}+\phi^*_{x^*}\eta^*_{x^*}-\phi^*_{y^*}=0, \quad (3)$$

$$\phi^*_t+\frac{1}{2}(\phi^{*\,2}_{x^*}+\phi^{*\,2}_{y^*})+g\eta^*-\frac{T^*\eta^*_{x^*x^*}}{\rho^*(1+\eta^{*\,2}_{x^*})^{\frac{3}{2}}}=\frac{B^{*2}}{2}. \quad (4)$$

at the bottom, $y^*=-H+b^*(x^*)$

$$\phi^*_{y^*}-\phi^*_{x^*}\mathbf{b}^*_{x^*}=0 \quad (5)$$

Where $\phi^*(x^*,y^*,t^*)$ is the potential function, B^* is an arbitrary constant, and H is the depth when the bump $\mathbf{b}^*$ is zero. In order to investigate long waves and derive asymptoyic solutions, it is conventient to introduce the following dimensionless variables:

$$\left.\begin{aligned} &x=\tfrac{x^*}{L}, y=\tfrac{y^*}{H}, t=(\tfrac{H}{L})^4\tfrac{(gH)^{\frac{1}{2}}}{L}t^*,\\ &\eta(x,t)=\tfrac{\eta^*(x^*,t^*)}{A}, B=\tfrac{B^*}{(gH)^{\frac{1}{2}}}\\ &\phi(x,y,t)=\tfrac{H}{LA(gH)^{\frac{1}{2}}}\phi^*(x^*,y^*,t^*),\\ &\tau=\tfrac{T^*}{\rho^* gH^2}, \mathbf{b}(x)=\tfrac{(H/L)^{-2M}}{H}\mathbf{b}^*(x^*), \end{aligned}\right\} \quad (6)$$

where M is a positive integer to be chosen later. In terms of the nondimensional variables (6), (2)-(5) become:

In $-\infty<x<\infty, -1+\beta^M\mathbf{b}(x)<y<\alpha\eta$

$$\beta\phi_{xx}+\phi_{yy}=0, \quad (7)$$

At the free surface, $y=\alpha\eta$

$$\beta^2\eta_t+\alpha\phi_x\eta_x-\beta^{-1}\phi_y=0, \quad (8)$$

$$\beta^2\phi_t+\frac{\alpha}{2}(\phi_x^2+\beta^{-1}\phi_y^2)+\eta$$

$$-\frac{\beta\tau\eta_{xx}}{(1+\alpha^2\beta\eta_x^2)^{\frac{3}{2}}}=\frac{B^2}{2\alpha}, \quad (9)$$

at the bottom, $y=-1+\beta^M\mathbf{b}(x)$

$$\phi_y - \beta^{M+1}\phi_x \mathbf{b}_x = 0. \quad (10)$$

In (7)-(10), α, β , and τ are nondimensional parameters

$$\alpha = \frac{A}{H}, \quad \beta = (\frac{H}{L})^2, \quad \tau = \frac{T^*}{\rho^* g H^2}. \quad (11)$$

We seek solutions for periodic water waves of wavelength λ^*, and introduce the dimensionless wavelength

$$\lambda = \frac{\lambda^*}{L}, \quad (12)$$

The Froude number F is defined as

$$F = \frac{c}{(gH)^{\frac{1}{2}}} = \frac{\alpha}{\lambda}\int_0^{\lambda} \phi_x dx. \quad (13)$$

Since we are interested in small amplitude and shallow-water waves with τ near $\frac{1}{3}$, in (7)-(10), we take

$$\alpha = \epsilon^2, \quad \beta = \epsilon. \quad (14)$$

and expand η, ϕ, τ , and B as

$$\left.\begin{aligned} \eta &= \eta_0 + \epsilon\eta_1 + \epsilon^2\eta_2 + \cdots \\ \phi &= \tfrac{Bx}{\epsilon^2} + \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + \cdots \\ \tau &= \tfrac{1}{3} + \epsilon\tau_1 + \epsilon^2\tau_2 + \cdots \\ B &= B_0 + \epsilon B_1 + \epsilon^2 B_2 + \cdots \\ F &= F_0 + \epsilon F_1 + \epsilon^2 F_2 + \cdots \end{aligned}\right\} \quad (15)$$

Substituting (14) and (15) into (7)-(10), taking M = 4 in (10), and expanding at the boundary condition $y = 0$ and $y = -1$, we obtain in $-\infty < x < \infty, \quad -1 < y < 0$

$$\epsilon(\phi_{0_{xx}} + \epsilon\phi_{1_{xx}} + \epsilon^2\phi_{2_{xx}} + O(\epsilon^3)) + (\phi_{0_{yy}} + \epsilon\phi_{1_{yy}} + \epsilon^2\phi_{2_{yy}} + O(\epsilon^3)) = 0, \quad (16)$$

at $y = 0$,

$$\begin{aligned} &\epsilon^2(\eta_{0t} + \epsilon\eta_{1t} + O(\epsilon^2)) \\ &+ \epsilon^2\{(\frac{B_0 + \epsilon B_1 + \epsilon^2 B_2 + O(\epsilon^3)}{\epsilon^2} + \phi_{0_x}(x,0,t) + O(\epsilon))\} \\ &(\eta_{0x} + \epsilon\eta_{1x} + \epsilon^2\eta_{2x} + O(\epsilon^3)) \\ &- \epsilon^{-1}\{(\phi_{0y}(x,0,t) + \epsilon^2(\eta_0 + \epsilon\eta_1 + O(\epsilon^2))\phi_{0yy}(x,0,t) \\ &+ \epsilon(\phi_{1y}(x,0,t) + \epsilon^2(\eta_0 + \epsilon\eta_1 + O(\epsilon^2))\phi_{1yy}(x,0,t) \\ &+ O(\epsilon^4)) + \epsilon^2(\phi_{2y}(x,0,t) + O(\epsilon^2)) + \epsilon^3(\phi_{3y}(x,0,t) \\ &+ O(\epsilon^2)) + O(\epsilon^4))\} = 0, \end{aligned} \quad (17)$$

$$\begin{aligned} &\epsilon^2(\phi_{0t}(x,0,t) + \epsilon\phi_{1t}(x,0,t) + O(\epsilon^2)) \\ &+ \frac{\epsilon^2}{2}\{\{\frac{B_0 + \epsilon B_1 + \epsilon^2 B_2 + O(\epsilon^3)}{\epsilon^2} \\ &+ (\phi_{0x}(x,0,t) + \epsilon^2\eta_0\phi_{0xy}(x,0,t)) + \epsilon\phi_{1x}(x,0,t) \\ &+ \epsilon^2\phi_{2x}(x,0,t) + O(\epsilon^3)\}^2 + \epsilon^{-1}\{(\phi_{0y} + \epsilon^2\phi_{0yy}) + \epsilon\phi_{1y} \\ &+ \epsilon^2\phi_{2y} + O(\epsilon^3)\}^2\} + (\eta_0 + \epsilon\eta_1 + \epsilon^2\eta_2 + O(\epsilon^3)) \\ &- \epsilon(\frac{1}{3} + \epsilon\tau_1 + \epsilon^2\tau_2 + O(\epsilon^3))(\eta_{0xx} + \epsilon\eta_{1xx} + \epsilon^2\eta_{2xx} \\ &+ O(\epsilon^3))(1 + O(\epsilon^{10})) \\ &= \frac{(B_0 + \epsilon B_1 + \epsilon^2 B_2 + \epsilon^3 B_3 + \epsilon^4 B_4 + O(\epsilon^5))^2}{2\epsilon^2}, \end{aligned} \quad (18)$$

at $y = -1$

$$\begin{aligned} &(\phi_{0y}(x,-1,t) + \epsilon\phi_{1y}(x,-1,t) + \epsilon^2\phi_{2y}(x,-1,t) \\ &+ \epsilon^3\phi_{3y}(x,-1,t) + O(\epsilon^4)) - \epsilon^5(\frac{B_0 + \epsilon B_1 + \epsilon^2 B_2 + O(\epsilon^3)}{\epsilon^2}) \\ &+ \phi_{0x}(x,-1,t) + O(\epsilon))\mathbf{b}_x = 0. \end{aligned} \quad (19)$$

From (16) to (19), we have

$O(\epsilon^{-1})$:

$$\phi_{0y}(x,0,t) = 0. \quad (20)$$

$O(1)$:

$$\phi_{0yy}(x,y,t) = 0, \quad (21)$$

$$B_0\eta_{0x} - \phi_{1y}(x,0,t) = 0, \quad (22)$$

$$B_0\phi_{0x}(x,0,t) + \eta_0 = 0, \quad (23)$$

$$\phi_{0y}(x,-1,t) = 0. \quad (24)$$

From (21) and by (22) or (24), it follows that

$$\phi_{0y} = 0, \quad \phi_0(x,y,t) = \phi_0(x,t). \quad (25)$$

$O(\epsilon)$:

$$\phi_{0xx}(x,t) + \phi_{1yy}(x,y,t) = 0, \quad (26)$$

$$B_0\eta_{1x} + B_1\eta_{0x} - \phi_{2y}(x,0,t) = 0, \quad (27)$$

$$B_0\phi_{1x}(x,0,t) + B_1\phi_{0x}(x,0,t) + \frac{\phi_{0y}^2(x,0,t)}{2} + \eta_1 - \tfrac{1}{3}\eta_{0xx} = 0, \quad (28)$$

$$\phi_{1y}(x,-1,t) = 0. \quad (29)$$

From (26) and by (29), we found that

$$\phi_{1y}(x,y,t) = -\phi_{0xx}(x,t)(y+1), \quad (30)$$

and

$$\phi_{1x}(x,y,t) = -\phi_{0xxx}(x,t)(\frac{y^2}{2}+y) + R_{1x}(x,t), \quad (31)$$

From (22), (23), and by (30), we obtain

$$B_0 = 1, \quad (32)$$

$$\phi_{0x} = -\eta_0. \quad (33)$$

From (28) and by (25), (31), and (32), it follows that

$$\phi_{1xx}(x,0,t) = \frac{1}{3}\eta_{0xxx} - \eta_{1x} + B_1\eta_{0x} \quad (34)$$

$$= R_{1xx}(x,t). \quad (35)$$

$O(\epsilon^2)$**:**

$$\phi_{1xx}(x,y,t) + \phi_{2yy}(x,y,t) = 0, \quad (36)$$

$$\eta_{0t} + B_0\eta_{2x} + B_1\eta_{1x} + (B_2 + \phi_{0x})\eta_{0x}$$
$$-\eta_0\phi_{1yy} - \phi_{3y} = 0 \text{ at } y=0, \quad (37)$$

$$\phi_{0t} + \phi_{2x} + B_1\phi_{1x} - B_2\eta_0 + \tfrac{\eta_0^2}{2} + \eta_2 - \tfrac{1}{3}\eta_{1xx}$$
$$-\tau_1\eta_{0xx} = 0 \text{ at } y=0, \quad (38)$$

$$\phi_{2y}(x,-1,t) = 0. \quad (39)$$

From (36), (39) and by (31), we found that

$$R_2(x,t) = -\frac{1}{3}\phi_{0xxxx}(x,t) - R_{1xx}(x,t), \quad (40)$$

$$\phi_{2y}(x,y,t) = \phi_{0xxxx}(x,t)(\frac{y^3}{6}+\frac{y^2}{2}+\frac{y}{3})$$
$$+R_2(x,t)(y+1), \quad (41)$$

and

$$\phi_2(x,y,t) = \phi_{0xxxx}(x,t)(\frac{y^4}{24}+\frac{y^3}{6}+\frac{y^2}{6})$$
$$+R_2(x,t)(\frac{y^2}{2}+y) + R_3(x,t) \quad (42)$$

From (27) and by (32),(41)

$$R_2(x,t) = \eta_{1x} + B_1\eta_{0x}. \quad (43)$$

From (37) and by (30),(32) ,(33),

$$\eta_{2x} = -\eta_{0t} - B_1\eta_{1x} - (B_2 - 2\eta_0)\eta_{0x} + \phi_{3y}(x,0,t) \quad (44)$$

Differentiating (38) about x and by (33) , (35) , (42)

$$\eta_{2x} = \eta_{0t} - R_{3xx} - B_1R_{1xx} + (B_2 - \eta_0)\eta_{0x}$$
$$+\frac{1}{3}\eta_{1xxx} + \tau_1\eta_{0xxx}. \quad (45)$$

By (34), (35), (40), and (43)

$$B_1 = 0. \quad (46)$$

By (44), (45), and (46)

$$\frac{1}{3}\eta_{1xxx} = -2\eta_{0t} - 2B_2\eta_{0x} + 3\eta_0\eta_{0x}$$
$$-\tau_1\eta_{0xxx} + R_{3xx} + \phi_{3y}(x,0,t) \quad (47)$$

$O(\epsilon^3)$**:**

$$\phi_{2xx}(x,y,t) + \phi_{3yy}(x,y,t) = 0, \quad (48)$$

$$\phi_{3y}(x,-1,t) = B_0 b_x. \quad (49)$$

From(48), (49) and by (42), we obtain

$$\phi_{3y}(x,-1,t) = \frac{1}{45}\phi_{0xxxxxx}(x,t) - \frac{1}{3}R_{2xx}(x,t)$$
$$+R_{3xx}(x,t) + \phi_{3y}(x,0,t) \quad (50)$$

By (32), (33), (43), (46), and (50), we have

$$2\eta_{0t} + 2B_2\eta_{0x} - 3\eta_0\eta_{0x}$$
$$+\tau_1\eta_{0xxx} - \frac{1}{45}\eta_{0xxxxx} = \mathbf{b}_x. \quad (51)$$

The Froude number F is defined and expanded as

$$F = F_0 + \epsilon F_1 + \epsilon^2 F_2 + O(\epsilon^3)$$
$$= \frac{\epsilon^2}{\lambda}\int_0^\lambda (\frac{B_0 + \epsilon B_1 + \epsilon^2 B_2 + O(\epsilon^3)}{\epsilon^2} + \phi_{0x} + O(\epsilon))dx$$
$$= B_0 + \epsilon B_1 + \epsilon^2 B_2 + \frac{\epsilon^2}{\lambda}\int_0^\lambda \phi_{0x}dx + O(\epsilon^3). \quad (52)$$

By (33) and the mean value of periodic solution over a period is zero, we found that

$$\int_0^\lambda \phi_{0x}dx = -\int_0^\lambda \eta_0 dx = 0 \cdot$$

If η_0 is a solitary wave solution with the properity that

$$\int_0^\infty \phi_{0x}dx = -\int_0^\infty \eta_0 dx < \infty, \quad (53)$$

then, with $\lambda = \infty$, the term

$$\frac{1}{\lambda}\int_0^\lambda \phi_{0x}dx$$

in (52) will be zero. We shall see that all the solitary wave solutions discovered in the following chapters will satisfy (53).

Therefore, we have

$$B_0 = F_0, \quad B_1 = F_1, \quad B_2 = F_2.$$

and then (51) becomes

$$2\eta_{0t} + 2F_2\eta_{0x} - 3\eta_0\eta_{0x} + \tau_1\eta_{0xxx} - \frac{1}{45}\eta_{0xxxxx} = \mathbf{b}_x. \quad (54)$$

Next, we assume $\eta_{0t} = 0$ in equation (54), integrate (54) once and set the constant of integration to be zero, then we have the following model equation

$$2F_2\eta_0 - \frac{3}{2}\eta_{0x}^2 + \tau_1\eta_{0xx} - \frac{1}{45}\eta_{0xxxx} = \mathbf{b}. \quad (55)$$

In the following sections, we shall use η for η_0 in equation (55), that is,

$$2F_2\eta - \frac{3}{2}\eta^2 + \tau_1\eta_{xx} - \frac{1}{45}\eta_{xxxx} = \mathbf{b}. \quad (56)$$

and disscuss the solutions of the model equation (56).

3 Problem Formulation

We follow Zufiria [17] to construct a Hamiltonian associated to (56).

When $\mathbf{b} = 0$, we rewrite (56) as

$$\eta_{xxxx} - 45\tau_1\eta_{xx} - 90F_2\eta + \frac{135}{2}\eta^2 = 0. \quad (57)$$

We multiply $-\eta_x$ to (57) and integrate the resulting equation, then equation (57) has first integral as

$$H = 45F_2\eta^2 + \frac{1}{2}\eta_{xx}^2 - \eta_{xxx}\eta_x + \frac{45}{2}\tau_1\eta_x^2 - \frac{45}{2}\eta^3, \quad (58)$$

where H is a constant. Introducing the change of variables

$$\left.\begin{aligned} q_1 &= \eta, \quad p_1 = \eta_{xxx} - 45\tau_1\eta_x, \\ q_2 &= \eta_{xx}, \quad p_2 = \eta_x, \end{aligned}\right\}$$

then (58) becomes

$$H(q_1, q_2, p_1, p_2) = 45F_2 q_1^2 + \frac{1}{2}q_2^2 - p_1 p_2 - \frac{45}{2}\tau_1 p_2^2 - \frac{45}{2}q_1^3, \quad (59)$$

and we have

$$\frac{dz}{dx} = J\nabla_z H(z) = Az + g(z) \equiv f(z, \mu), \quad (60)$$

where $\mu = (\tau_1, F_2) \in \mathbf{R}^2$,

$$z = \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} \in \mathbf{R}^4, \quad J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad (61)$$

and

$$A = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & -45\tau_1 \\ -90F_2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, g(z) = \begin{pmatrix} 0 \\ 0 \\ \frac{135}{2}q_1^2 \\ 0 \end{pmatrix}. \quad (62)$$

Therefore (59) is a two degree of freedom Hamiltonian with two parameters τ_1 and F_2. Because different parameters (τ_1, F_2) in (59) give rise to different eigenvalues λ for the linearized system of (60) at the origin, we divide the parameter plane (τ_1, F_2) into following nine cases

Case 0 $(\tau_1 = 0, F_2 = 0)$: $\lambda = 0,0,0,0$.

Case 1 $(\tau_1 \in \mathbf{R}, F_2 > 0)$: $\lambda = \pm r, \pm wi;\ r, w > 0$.

Case 2 $(\tau_1 < 0, F_2 = 0)$: $\lambda = 0, 0, \pm wi;\ w > 0$.

Case 3 $(\tau_1 < 0, F_2 < 0,\ (45\tau_1)^2 + 360F_2 > 0)$:

$\lambda = \pm w_1 i, \pm w_2 i;\ w_1 > w_2 > 0$.

Case 4 $(\tau_1 < 0, F_2 < 0,\ (45\tau_1)^2 + 360F_2 = 0)$:

$\lambda = \pm wi, \pm wi;\ w > 0$

Case 5 $(\tau_1 \in \mathbf{R}, F_2 < 0,\ (45\tau_1)^2 + 360F_2 < 0)$:

$\lambda = \pm a \pm bi;\ a, b > 0$

Case 6 $(\tau_1 > 0, F_2 < 0,\ (45\tau_1)^2 + 360F_2 = 0)$:

$\lambda = \pm r, \pm r;\ r > 0$

Case 7 $(\tau_1 > 0, F_2 < 0,\ (45\tau_1)^2 + 360F_2 > 0)$:

$\lambda = \pm r_1, \pm r_2;\ r_1 > r_2 > 0$

Case 8 $(\tau_1 > 0, F_2 = 0)$: $\lambda = 0, 0, \pm r;\ r > 0$.

We construct asymptotic solutions in **Case 0,2,3,** and **4**.

4 Problem Solution

In this section, we shall construct asymptotic solutions of the model equation (56) for **Case 0**,**2**,**3,** and **4**. In what follows except specified otherwise, the bump **b**(x) is sufficiently smooth and has a compact support on the inteval [x_1, x_2] with **b**(x_1)=**b**(x_2) = 0. We rewrite (56) as follows,

$$\eta_{xxxx} - 45\tau_1\eta_{xx} - 90F_2\eta = -45(\mathbf{b}(x)) + \frac{3}{2}\eta^2) \equiv f, \quad (63)$$

The idea is first to show the existence of solutions, $\eta_L(x), \eta_C(c)$, and $\eta_R(x)$ on intervals $(-\infty, x_1)$, $[x_1, x_2]$, and (x_2, ∞), respectively, then match these solutions to be a solution of equation (63).

We assume that there exist small-norm bounded solutions of equation (63) with small-norm bump $\mathbf{b} = \delta b_1$ in **Case 0**,**2**,**3** and **4**, and then construct its asymptotic solutions by expanding the solution as

$$\eta(x) = \delta\eta_1(x) + \delta^2\eta_2(x) + ... \quad (64)$$

where δ is a small parameter. Substituting (64) in (9), we obtain

$O(\delta)$:

$$\eta_{1_{xxxx}} - 45\tau_1\eta_{1_{xx}} - 90F_2\eta_1 = -45b_1, \quad (65)$$

and

$O(\delta^n)$:

$$\eta_{n_{xxxx}} - 45\tau_1\eta_{n_{xx}} - 90F_2\eta_n = \begin{cases} -135(\Sigma_{i=1}^{\frac{n+1}{2}-1}\eta_i\eta_{n-i}) & ,n \text{ odd} \\ -135(\Sigma_{i=1}^{\frac{n}{2}-1}\eta_i\eta_{n-i} + \frac{1}{2}\eta_{\frac{n}{2}}^2) & ,n \text{ even} \end{cases} \quad (66)$$

Where n = 2,3

In the following, we shall construct bounded solutions for the first approximation η_1.

3.1 Case 0

In **Case 0**, $\tau_1 = 0$ and $F_2 = 0$. Thus, equation (65) becomes

$$\eta_{1_{xxxx}} = -45b_1. \quad (67)$$

We solve the initial value problem of (67) subject to

$$\eta_1(x_1) = P, \quad \eta_{1_x}(x_1) = Q, \quad \eta_{1_{xx}}(x_1) = R, \quad \eta_{1_{xxx}}(x_1) = S \quad (68)$$

Where P, Q, R, S are constants and obtain

$$\eta_1(x) = \frac{1}{6}(6P - 6Qx_1 + 3Rx_1^2 - Sx_1^3) + \frac{1}{2}(2Q - 2Rx_1 + Sx_1^2)x + \frac{1}{2}(R - Sx_1)x^2 + \frac{S}{6}x^3 + \int_{x_1}^{x} G(x,t)(-45b_1(t))dt \quad (69)$$

where

$$G(x,t) = \frac{1}{6}(x-t)^3 \quad (70)$$

is the causal Green's function of (67) subject to (68) with $P = Q = R = S = 0$.

For $\eta_1(x)$ to be bounded for $x \le x_1$, the only case is that $\eta_1(x) = P$ for $x \le x_1$ with $Q = R = S = 0$. The integral term in (69) is bounded since $b_1(x)$ is compact on the interval $[x_1, x_2]$. Therefore, to have $\eta_1(x)$ to be bounded for $x \ge x_2$, we also need $\eta_1(x)$ to be a constant, that is, $\eta_1'(x_2) = \eta_1''(x_2) = \eta_1'''(x_2) = 0$. After some algebra, we obtain

$$\int_{x_1}^{x_2} b_1(t)dt = 0, \int_{x_1}^{x_2} tb_1(t)dt = 0, \int_{x_1}^{x_2} t^2 b_1(t)dt = 0 . (71)$$

Thus, if the bump $b_1(t)$ satisfies (71), then we can rewrite bounded η_1 as

$$\eta_1(x) = \begin{cases} P, & x \le x_1, \\ P + \int_{x_1}^{x} G(x,t)(-45b_1(t))dt, & x_1 < x < x_2, \\ P + \int_{x_1}^{x_2} G(x_2,t)(-45b_1(t))dt, & x_2 \le x, \end{cases} \quad (72)$$

which is constant for $x_1 \ge x$ and $x_2 \le x$ .There are infinitely many b_1 that satisfy (71) and $b_1(-1) = b_1(1) = 0$.

3.2 Case 2

In **Case 2**, $\tau_1 < 0$ and $F_2 = 0$. Thus, equation (65) becomes

$$\eta_{1_{xxxx}} - 45\tau_1\eta_{1_{xx}} = -45b_1 \cdot \quad (73)$$

We solve the initial value problem of (73) subject to initial conditions (68) and obtain

$$\eta_1(x) = P + \frac{R-(S+Qw^2)a}{w^2} + (Q+\frac{S}{w^2})x - \frac{R}{w^2}cos(w(x-a)) - \frac{S}{w^3}sin(w(x-a)) + \int_a^x G(x,t)\,(-45b_1(t))dt \quad (74)$$

where

$$G(x,t) = \frac{x-t}{w^2} - \frac{sin(w(x-t))}{w^3}. \quad (75)$$

is the causal Green's function of (73) subject to (68) with $P = Q = R = S = 0$.

The integral term in (74) is bounded since $b_1(x)$ is compact on the interval $[x_1, x_2]$. Therefore, to have $\eta_1(x)$ to be bounded for all $x \ge a$, we need the cofficient of *x* term in (74) to be zero, thus

$$Qw^2 + S = 0. \quad (76)$$

In addition, *R* must be zero if we require $\eta_1(x)$ to be bounded when $a \to -\infty$. Now, we rewrite the bounded η_1 as

$$\eta_1(x) = \begin{cases} P - \frac{S}{w^3}sin(w(x-a)) & ,x \le x_1, \\ P - \frac{S}{w^3}sin(w(x-a)) + \int_a^x G(x,t)(-45b_1(t))dt & ,x_1 < x < x_2, \\ P - \frac{S}{w^3}sin(w(x-a)) + \int_a^x G(x_2,t)(-45b_1(t))dt & ,x_2 \le x, \end{cases} \quad (77)$$

Which is periodic for $x_1 \ge x$ and $x_2 \le x$ and s atisfy the initial conditions at $x = -\infty$ with t he following properties :

$$\eta_1'(-\infty)w^2 + \eta_1'''(-\infty) = 0, \eta_1''(-\infty) = 0, \quad (78)$$

and $\eta_1(-\infty) = P$, an arbitrary constant.

3.3 Case 3 with w_1/w_2 rational

In this subsection, the eigenvalues of linearization of equation (63) in **Case 3** are $\pm w_1 i$ and $\pm w_2 i$ and we assume that the ratio $w_1/w_2 = n_1/n_2$ is a rational number with $(n_1, n_2) = 1$, $n_1, n_2 \in N$. To obtain an asymptotic solution, we solve the initial value problem of (65) subject to initial conditions (68) and obtain

$$\eta_1(x) = Y(x) + \int_{x_1}^x G(x,t)(-45b_1(t))dt \cdot \quad (79)$$

Where

$$Y(x) = Acos(w_2x) + Bsin(w_2x) + Ccos(w_1x) + Dsin(w_1x)$$
$$w_1 = (-(45\tau_1 - ((45\tau_1)^2 + 360F_2)^{\frac{1}{2}}))^{\frac{1}{2}}$$
$$w_2 = (-(45\tau_1 + ((45\tau_1)^2 + 360F_2)^{\frac{1}{2}}))^{\frac{1}{2}},$$
$$A = \frac{1}{(w_1^2 - w_2^2)}\left\{\frac{-(Qw_1^2+S)sin(aw_2)}{w_2} + (Pw_1^2 + R)cos(aw_2)\right\},$$
$$B = \frac{1}{(w_1^2 - w_2^2)}\left\{\frac{(Qw_1^2+S)cos(aw_2)}{w_2} + (Pw_1^2 + R)sin(aw_2)\right\},$$
$$C = \frac{1}{(w_1^2 - w_2^2)}\left\{\frac{(Qw_2^2+S)sin(aw_1)}{w_1} - (Pw_2^2 + R)cos(aw_1)\right\},$$
$$D = \frac{1}{(w_1^2 - w_2^2)}\left\{\frac{-(Qw_2^2+S)cos(aw_1)}{w_1} - (Pw_2^2 + R)sin(aw_1)\right\}$$

and the causal Green's is

$$G(x,t) = \{w_2sin(w_1(x-t)) - w_1sin(w_2(x-t))\}/(w_2w_1(w_2^2 - w_1^2)).$$

Note that $\eta_1(x)$ in (79) is bounded since $b_1(x)$ is compact on the interval $[x_1, x_2]$.

Now, we rewrite the bounded η_1 as

$$\eta_1(x) = \begin{cases} Y(x) & ,x \le x_1, \\ Y(x) + \int_{x_1}^x G(x,t)(-45b_1(t))dt & ,x_1 < x < x_2, \\ Y(x) + \int_{x_1}^{x_2} G(x_2,t)(-45b_1(t))dt & ,x_2 \le x, \end{cases} \quad (80)$$

which is periodic for $x_1 \ge x$ and $x_2 \le x$ with period $T = 2n_1\pi/w_1 = 2n_2\pi/w_2$.

3.4 Case 4

In **Case 4**, we solve the initial value problem of (65) subject to initial conditions (68) and obtain

$$\eta_1(x) = Y(x) + \int_{x_1}^{x} G(x,t)(-45b_1(t))dt. \quad (81)$$

where

$$Y(x) = (A + Bx)\,cos(rx) + (C + Dx)\,sin(rx)$$

with

$$\begin{aligned}
r &= \sqrt{-45\tau_1/2},\\
A &= M\,cos(x_1 r) - N\,sin(x_1 r),\\
B &= -\left\{(Qr_2 + S)\,cos(x_1 r)/r + (Pr^2 + R)\,sin(x_1 r)\right\}/r,\\
C &= M\,sin(x_1 r) + N\,cos(x_1 r),\\
D &= -\left\{(Qr_2 + S)\,sin(x_1 r)/r + (Pr^2 + R)\,cos(x_1 r)\right\}/r,\\
M &= P + x_1(Qr^2 + S)/(2r^2),\\
N &= \left\{(Qr^2 + S)/(2r^2) - x_1(Pr^2 + R)/2 + Q\right\}/r,
\end{aligned}$$

and

$$G(x,t) = (sin(r(x-t)) - r(x-t)\cos(r(x-t)))/(2r^3).$$

The integral term in (81) is bounded since $b_1(x)$ is compact on the interval $[x_1, x_2]$, To have bounded $\eta(x)$ in (81), we need $B = D = 0$. After some algebra, we obtain

$$R + r^2P = 0, \quad S + r^2Q = 0. \quad (82)$$

Now, we rewrite the bounded η_1 as

$$\eta_1(x) = \begin{cases} Y(x) & ,x \le x_1,\\ Y(x) + \int_{x_1}^{x} G(x,t)(-45b_1(t))dt & ,x_1 < x < x_2,\\ Y(x) + \int_{x_1}^{x_2} G(x_2,t)(-45b_1(t))dt & ,x_2 \le x, \end{cases} \quad (83)$$

which is periodic for $x_1 \ge x$ and $x_2 \le x$ if (82) is satisfied.

5 Numerical Experiment

In this section, we shall give asymptotic solution numerically of equation (63) by using classical fourth-order Runge-Kutta method. (See Figure 1-3).

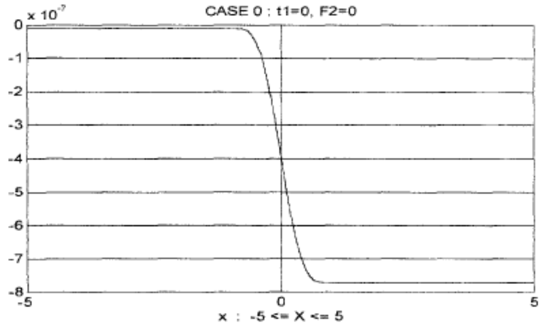

Figure 1: An asymptotic solution of equation (63) obtained by using classical fourth-order Runge-Kutta method in equation (67) for **Case 0** with compact bump $b(x) = 10^{-6}(7x^5 - 10x^3 + 3x)$ on interval $(-1,1)$.

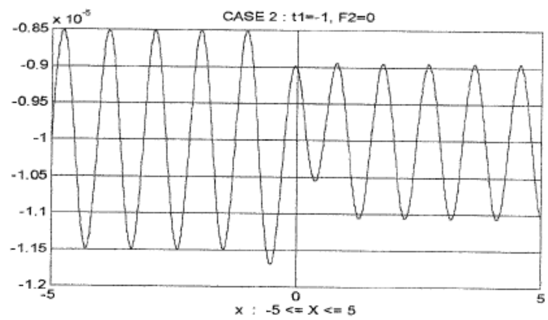

Figure 2: An asymptotic solution of equation (63) obtained by using classical fourth-order Runge-Kutta method in equation (73) for **Case 2** with $\tau_1 = -1$, $F_2 = 0$, and compact bump $b(x) = 10^{-5}(7x^5 - 10x^3 + 3x)$ on interval $(-1,1)$.

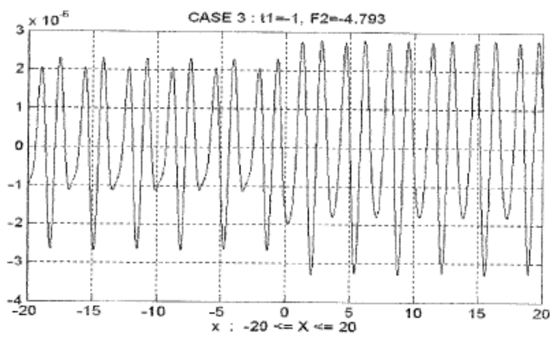

Figure 3: An asymptotic solution of equation (63) obtained by using classical fourth-order Runge-Kutta method in equation (65) for **Case 3** with $\tau_1 = -1$, $F_2 = -810/169 \approx -4.793$ which such that $w_1/w_2 = 3/2$ is a rational number, and compact bump $b(x) = 10^{-5}\exp\ (1/(x^2 - 1))$ on interval $(-1,1)$.

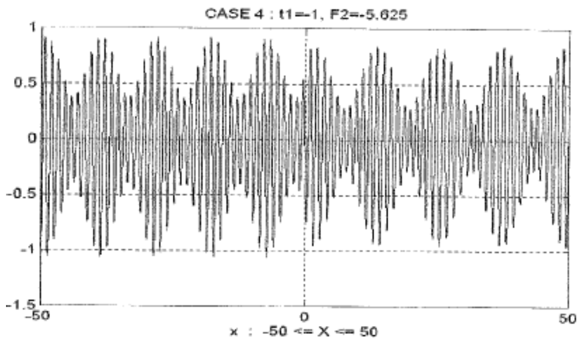

Figure 4: An asymptotic solution of equation (63) obtained by using classical fourth-order Runge-Kutta method in equation (65) for **Case 4** with $\tau_1 = -1,\ \ F_2 = -45/8 = -5.625$, and compact bump $b(x) = 10^{-5} \exp \quad (1/(x^2 - 1))$ on interval $(-1,1)$.

6 Conclusion

We constructed asymptotic solutions of model equation (56) for a sufficiently smooth compact bump $\mathbf{b}(x)$ and has a compact support on the inteval $[x_1,\ x_2]$ with $\mathbf{b}(x_1) = \mathbf{b}(x_2) = 0$. The numerical experiment in section 5 confirm the the constructed asymptotic solutions in section 4.

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