# Asymptotic Solutions and Unsymmetric Solutions of a Fifth Order Ordinary Differential Equation 

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#### Abstract

The objective of this paper is to construct the asymptotic solutions and unsymmetric solutions of a fifth order model equation for steady capillary-gravity waves over a smooth compact bump with the Froude number near 1 and the Bond number near $1 / 3$.


Key-Words: - Steady capillary-gravity wave, fifth order model equation, bump, Green's function, asymptotic solution, unsymmetric Solutions

## 1 Introduction

Progressive capillary-gravity waves on an irrotional incompressible inviscid fluid of constant density with surface tension in a two-dimensional channel of finite depth have been studied since nineteen century. Assume that a coordinate system moving with the wave at a speed is chosen so that in reference to it the wave motion is steady. Let H be the depth of water, $g$ the acceleration of gravity, T the coefficient of surface tension, and $\rho$ the constant density of the fluid. Then there are two nondimensional numbers which are important and defined as $F=c^{2} /(g H)$, the Froude number, and $\tau=T /\left(\rho g H^{2}\right)$, the Bond number

When $F$ is not close to 1 , the linear theory of water waves is applicable. But when $F$ approaches to 1 , the solutions of linearized equations of water waves will grow to infinity (Peters and Stoker [11]). Therefore for $F$ close to 1 nonlinear effect must be taken into account and thus $F=1$ is a critical value. The first study of a solitary wave on water with surface tension is due to Korteweg and DeVries [10] after whom the K-dV equation with surface tension effect is named. A stationary $\mathrm{K}-\mathrm{dV}$ equation with Bond number not near $1 / 3$ can also be formally derived by different approaches. However, if $\tau$ is close to 1 , the formal derivation of the stationary K dV equation fails. Thus $\tau=1 / 3$ is also a critical value.

It becomes apparent that the problems for $F$ near 1 and for $\tau$ near $1 / 3$ depend on each other and are difficult because they are not only strongly nonlinear, but also very delicate. Since the full nonlinear equations for the water waves are too
complicated to study, it is of interest to study model equations. In Hunter and Vanden-Broeck's work [8], a fifth order ordinary differential equation considered as a perturbed stationary K-dV equation was obtained with the assumption that $F=1+F_{2}{ }^{2}$, $\tau=1 / 3+\tau_{1} \epsilon$ and $\epsilon$ is a small positive parameter. By integrating the fifth order ordinary differential equation once and set the con-stant of integration to be zero, then the model equation becomes

$$
\begin{equation*}
2 F_{2} \eta-\frac{3}{2} \eta^{2}+\tau_{1} \eta_{x x}-\frac{1}{45} \eta_{x x x}=0 \tag{1}
\end{equation*}
$$

Equation (1) has been studied extensively by many authors $[1-7,9]$ and several types of solutions have been found, such as periodic solutions $[1,5,6$, 7], solitary wave solutions [2-7], generalized solitary wave solutions (solitary waves with osciallatory tails at infinity) in the parameter region $\tau_{1}<0$ and $F_{2}>0$ $[1,9]$, etc.

We add a bump $y=b(x)$ at the bottom of the two-dimensional ideal fluid flow and then derive a forced model equation (56),

$$
2 F_{2} \eta-\frac{3}{2} \eta^{2}+\tau_{1} \eta_{x x}-\frac{1}{45} \eta_{x x x x}=\mathbf{b}
$$

Equation (56) has been studied in [12-16] and several types of solutions have been found, such as unsymmetric solitary wave solutions [14], solitary wave solutions [15], and generalized solitary wave solutions [16].

However, for Case 0,2,3 and $\mathbf{4}$ in section 3, the proof of existence of bounded solutions of (56) is not available at present; we construct asymptotic solutions in section 4 for these four Cases by
$\operatorname{assuming}_{\eta}$ and bump b to be small. In section 5, we also construct unsymmetric solitary wave solutions for Case 1, 5, 6,7 and 8.

## 2 Derivation of the model equation

We consider the two-dimensional flow of an irrotional incompressible inviscid fluid of constant density $\rho^{*}$ with surface tension $T^{*}$ in a two-dimensional channel of finite depth. A rectangular coordinate $\operatorname{system}\left(x^{*}, y^{*}\right)$ is chosen such that the flow is bounded above by the free surface $y^{*}=\eta^{*}\left(x^{*}, t^{*}\right)$ and below by the rigid horizontal bottom with a bump $y^{*}=-H+\mathbf{b}^{*}\left(x^{*}\right)$.

The governing equations are:
In $-\infty<x^{*}<\infty,-H+\mathbf{b}^{*}\left(x^{*}\right)<y^{*}<\eta^{*}$

$$
\begin{equation*}
\phi_{x^{*} x^{*}}^{*}+\phi_{y^{\prime} y^{*}}^{*}=0, \tag{2}
\end{equation*}
$$

at the free surface, $y^{*}=\eta^{*}$

$$
\begin{gather*}
\eta_{t^{*}}^{*}+\phi_{x^{*}}^{*} \eta_{x^{*}}^{*}-\phi_{y^{*}}^{*}=0,  \tag{3}\\
\phi_{t}^{*}+\frac{1}{2}\left(\phi_{x^{*}}^{* 2}+\phi_{y^{*}}^{* 2}\right)+g \eta^{*}-\frac{T^{*} \eta_{x^{*} x^{*}}^{*}}{\rho^{*}\left(1+\eta_{x^{*}}^{* 2}\right)^{\frac{3}{2}}}=\frac{B^{* 2}}{2} . \tag{4}
\end{gather*}
$$

at the bottom, $y^{*}=-H+b^{*}\left(x^{*}\right)$

$$
\begin{equation*}
\phi_{y^{*}}^{*}-\phi_{x_{x}^{*}}^{*} \mathbf{b}_{x^{*}}^{*}=0 \tag{5}
\end{equation*}
$$

Where $\phi^{*}\left(x^{*}, y^{*}, t^{*}\right)$ is the potential function, $B^{*}$ is an arbitrary constant, and $H$ is the depth when the bump $\mathbf{b}^{*}$ is zero. In order to investigate long waves and derive asymptoyic solutions, it is conventient to introduce the following dimensionless variables:

$$
\left.\begin{array}{l}
x=\frac{x^{*}}{L}, y=\frac{y^{*}}{H}, t=\left(\frac{H}{L}\right)^{4} \frac{(g H)^{\frac{1}{2}}}{L} t^{*} \\
\eta(x, t)=\frac{\eta^{*}\left(x^{*}, t^{*}\right)}{A}, B=\frac{B^{*}}{(g H)^{\frac{1}{2}}} \\
\phi(x, y, t)=\frac{H}{L A(g H)^{\frac{1}{2}}} \phi^{*}\left(x^{*}, y^{*}, t^{*}\right)  \tag{6}\\
\tau=\frac{\tau^{*}}{\rho^{*} g H^{2}}, \mathbf{b}(x)=\frac{(H / L)^{-2 M}}{H} \mathbf{b}^{*}\left(x^{*}\right)
\end{array}\right\}
$$

where $M$ is a positive integer to be chosen later.
In terms of the nondimensional variables (6), (2)-(5) become:

In $-\infty<x<\infty,-1+\beta^{M} \mathbf{b}(x)<y<\alpha \eta$

$$
\begin{equation*}
\beta \phi_{x x}+\phi_{y y}=0 \tag{7}
\end{equation*}
$$

At the free surface, $y=\alpha \eta$

$$
\begin{gather*}
\beta^{2} \eta_{t}+\alpha \phi_{x} \eta_{x}-\beta^{-1} \phi_{y}=0  \tag{8}\\
\beta^{2} \phi_{t}+\frac{\alpha}{2}\left(\phi_{x}^{2}+\beta^{-1} \phi_{y}^{2}\right)+\eta-\frac{\beta \tau \eta_{x x}}{\left(1+\alpha^{2} \beta \eta_{x}^{2}\right)^{\frac{3}{2}}}=\frac{B^{2}}{2 \alpha} \tag{9}
\end{gather*}
$$

at the bottom, $y=-1+\beta^{M} \mathbf{b}(x)$

$$
\begin{equation*}
\phi_{y}-\beta^{M+1} \phi_{x} \mathbf{b}_{x}=0 . \tag{10}
\end{equation*}
$$

In (7)-(10), $\alpha, \beta$, and $\tau$ are nondimensional parameters

$$
\begin{equation*}
\alpha=\frac{A}{H}, \quad \beta=\left(\frac{H}{L}\right)^{2}, \quad \tau=\frac{T^{*}}{\rho^{*} g H^{2}} . \tag{11}
\end{equation*}
$$

We seek solutions for periodic water waves of wavelength $\lambda^{*}$, and introduce the dimensionless wavelength

$$
\begin{equation*}
\lambda=\frac{\lambda^{*}}{L} \tag{12}
\end{equation*}
$$

The Froude number $F$ is defined as

$$
\begin{equation*}
F=\frac{c}{(g H)^{\frac{1}{2}}}=\frac{\alpha}{\lambda} \int_{0}^{\lambda} \phi_{x} d x . \tag{13}
\end{equation*}
$$

Since we are interested in small amplitude and shallow-water waves with $\tau$ near $\frac{1}{3}$, in (7)-(10), we take

$$
\begin{equation*}
\alpha=\epsilon^{2}, \quad \beta=\epsilon \tag{14}
\end{equation*}
$$

and expand $\eta, \phi, \tau$, and $B$ as

$$
\left.\begin{array}{l}
\eta=\eta_{0}+\epsilon \eta_{1}+\epsilon^{2} \eta_{2}+\cdots  \tag{15}\\
\phi=\frac{B x}{\epsilon}+\phi_{0}+\epsilon \phi_{1}+\epsilon^{2} \phi_{2}+\cdots \\
\tau=\frac{1}{3}+\epsilon \tau_{1}+\epsilon^{2} \tau_{2}+\cdots \\
B=B_{0}+\epsilon B_{1}+\epsilon^{2} B_{2}+\cdots \\
F=F_{0}+\epsilon F_{1}+\epsilon^{2} F_{2}+\cdots
\end{array}\right\}
$$

Substituting (14) and (15) into (7)-(10), taking $M$ $=4$ in (10), and expanding at the boundary condition $y=0$ and $y=-1$, we obtain in $-\infty<x<\infty,-1<y<0$

$$
\begin{align*}
\epsilon\left(\phi_{0_{x x}}\right. & \left.+\epsilon \phi_{1_{x x}}+\epsilon^{2} \phi_{2_{x x}}+O\left(\epsilon^{3}\right)\right)+\left(\phi_{0_{y y}}+\epsilon \phi_{1 y y}\right. \\
& \left.+\epsilon^{2} \phi_{2_{y y}}+O\left(\epsilon^{3}\right)\right)=0, \tag{16}
\end{align*}
$$

at $y=0$,

$$
\begin{align*}
& \epsilon^{2}\left(\eta_{0 t}+\epsilon \eta_{1 t}+O\left(\epsilon^{2}\right)\right) \\
& +\epsilon^{2}\left\{\left(\frac{B_{0}+\epsilon B_{1}+\epsilon^{2} B_{2}+O\left(\epsilon^{3}\right)}{\epsilon^{2}}+\phi_{0_{x}}(x, 0, t)+O(\epsilon)\right)\right\} \\
& \quad\left(\eta_{0 x}+\epsilon \eta_{1 x}+\epsilon^{2} \eta_{2 x}+O\left(\epsilon^{3}\right)\right) \\
& -\epsilon^{-1}\left\{\left(\phi_{0 y}(x, 0, t)+\epsilon^{2}\left(\eta_{0}+\epsilon \eta_{1}+O\left(\epsilon^{2}\right)\right) \phi_{0 y y}(x, 0, t)\right.\right. \\
& +\epsilon\left(\phi_{1 y}(x, 0, t)+\epsilon^{2}\left(\eta_{0}+\epsilon \eta_{1}+O\left(\epsilon^{2}\right)\right) \phi_{1 y y}(x, 0, t)\right. \\
& \left.+O\left(\epsilon^{4}\right)\right)+\epsilon^{2}\left(\phi_{2 y}(x, 0, t)+O\left(\epsilon^{2}\right)\right)+\epsilon^{3}\left(\phi_{3 y}(x, 0, t)\right. \\
& \left.\left.\left.\quad+O\left(\epsilon^{2}\right)\right)+O\left(\epsilon^{4}\right)\right)\right\}=0,  \tag{17}\\
& \epsilon^{2}\left(\phi_{0 t}(x, 0, t)+\epsilon \phi_{1 t}(x, 0, t)+O\left(\epsilon^{2}\right)\right)
\end{align*}
$$

$$
\begin{align*}
& +\frac{\epsilon^{2}}{2}\left\{\left\{\frac{B_{0}+\epsilon B_{1}+\epsilon^{2} B_{2}+O\left(\epsilon^{3}\right)}{\epsilon^{2}}\right.\right. \\
& +\left(\phi_{0 x}(x, 0, t)+\epsilon^{2} \eta_{0} \phi_{0 x y}(x, 0, t)\right)+\epsilon \phi_{1 x}(x, 0, t) \\
& \left.+\epsilon^{2} \phi_{2 x}(x, 0, t)+O\left(\epsilon^{3}\right)\right\}^{2}+\epsilon^{-1}\left\{\left(\phi_{0 y}+\epsilon^{2} \phi_{0 y y}\right)+\epsilon \phi_{1 y}\right. \\
& \left.\left.+\epsilon^{2} \phi_{2 y}+O\left(\epsilon^{3}\right)\right\}^{2}\right\}+\left(\eta_{0}+\epsilon \eta_{1}+\epsilon^{2} \eta_{2}+O\left(\epsilon^{3}\right)\right) \\
& -\epsilon\left(\frac{1}{3}+\epsilon \tau_{1}+\epsilon^{2} \tau_{2}+O\left(\epsilon^{3}\right)\right)\left(\eta_{0 x x}+\epsilon \eta_{1 x x}+\epsilon^{2} \eta_{2 x x}\right. \\
& \left.+O\left(\epsilon^{3}\right)\right)\left(1+O\left(\epsilon^{10}\right)\right) \\
& =\frac{\left(B_{0}+\epsilon B_{1}+\epsilon^{2} B_{2}+\epsilon^{3} B_{3}+\epsilon^{4} B_{4}+O\left(\epsilon^{5}\right)\right)^{2}}{2 \epsilon^{2}}, \tag{18}
\end{align*}
$$

at $y=-1$

$$
\begin{gather*}
\left(\phi_{0 y}(x,-1, t)+\epsilon \phi_{1 y}(x,-1, t)+\epsilon^{2} \phi_{2 y}(x,-1, t)\right. \\
\left.+\epsilon^{3} \phi_{3 y}(x,-1, t)+O\left(\epsilon^{4}\right)\right)-\epsilon^{5}\left(\frac{B_{0}+\epsilon B_{1}+\epsilon^{2} B_{2}+O\left(\epsilon^{3}\right)}{\epsilon^{2}}\right) \\
\left.+\phi_{0 x}(x,-1, t)+O(\epsilon)\right) \mathbf{b}_{x}=0 . \tag{19}
\end{gather*}
$$

From (16) to (19), we have
$O\left(\epsilon^{-1}\right):$

$$
\begin{equation*}
\phi_{0 y}(x, 0, t)=0 . \tag{20}
\end{equation*}
$$

$O(1):$

$$
\begin{array}{r}
\phi_{0 y y}(x, y, t)=0, \\
B_{0} \eta_{0 x}-\phi_{1 y}(x, 0, t)=0, \\
B_{0} \phi_{0 x}(x, 0, t)+\eta_{0}=0, \\
\phi_{0 y}(x,-1, t)=0 . \tag{24}
\end{array}
$$

From (21) and by (22) or (24), it follows that

$$
\begin{equation*}
\phi_{0 y}=0, \quad \phi_{0}(x, y, t)=\phi_{0}(x, t) . \tag{25}
\end{equation*}
$$

$O(\epsilon):$

$$
\begin{array}{r}
\phi_{0 x x}(x, t)+\phi_{1 y y}(x, y, t)=0, \\
B_{0} \eta_{1 x}+B_{1} \eta_{0 x}-\phi_{2 y}(x, 0, t)=0, \\
B_{0} \phi_{1 x}(x, 0, t)+B_{1} \phi_{0 x}(x, 0, t) \\
+\frac{\phi_{y}^{2}(x, 0, t)}{2}+\eta_{1}-\frac{1}{3} \eta_{0 x x}=0, \\
\phi_{1 y}(x,-1, t)=0 . \tag{29}
\end{array}
$$

From (26) and by (29), we found that

$$
\begin{equation*}
\phi_{1 y}(x, y, t)=-\phi_{0 x x}(x, t)(y+1), \tag{30}
\end{equation*}
$$

and

$$
\begin{equation*}
\phi_{1 x}(x, y, t)=-\phi_{0 x x x}(x, t)\left(\frac{y^{2}}{2}+y\right)+R_{1 x}(x, t), \tag{31}
\end{equation*}
$$

From (22), (23), and by (30), we obtain

$$
\begin{align*}
& B_{0}=1,  \tag{32}\\
& \phi_{0 x}=-\eta_{0} . \tag{33}
\end{align*}
$$

From (28) and by (25), (31), and (32), it follows that

$$
\begin{align*}
\phi_{1 x x}(x, 0, t) & =\frac{1}{3} \eta_{0 x x}-\eta_{1 x}+B_{1} \eta_{0 x}  \tag{34}\\
& =R_{1 x x}(x, t) . \tag{35}
\end{align*}
$$

$O\left(\epsilon^{2}\right):$

$$
\begin{align*}
& \phi_{1 x x}(x, y, t)+\phi_{2 y y}(x, y, t)=0  \tag{36}\\
& \eta_{0 t}+B_{0} \eta_{2 x}+B_{1} \eta_{1 x}+\left(B_{2}+\phi_{0 x}\right) \eta_{0 x} \\
&-\eta_{0} \phi_{1 y y}-\phi_{3 y}=0 \text { at } y=0,  \tag{37}\\
& \phi_{0 t}+\phi_{2 x}+B_{1} \phi_{1 x}-B_{2} \eta_{0}+\frac{\eta_{o}^{2}}{2}+\eta_{2}-\frac{1}{3} \eta_{1 x x} \\
&-\tau_{1} \eta_{0 x x}=0 \text { at } y=0,  \tag{38}\\
& \phi_{2 y}(x,-1, t)=0 \tag{39}
\end{align*}
$$

From (36), (39) and by (31), we found that

$$
\begin{align*}
R_{2}(x, t)= & -\frac{1}{3} \phi_{0 x x x}(x, t)-R_{1 x x}(x, t),  \tag{40}\\
\phi_{2 y}(x, y, t)= & \phi_{0 x x x x}(x, t)\left(\frac{y^{3}}{6}+\frac{y^{2}}{2}+\frac{y}{3}\right) \\
& +R_{2}(x, t)(y+1), \tag{41}
\end{align*}
$$

and

$$
\begin{array}{r}
\phi_{2}(x, y, t)=\phi_{0 x x x}(x, t)\left(\frac{y^{4}}{24}+\frac{y^{3}}{6}+\frac{y^{2}}{6}\right) \\
+R_{2}(x, t)\left(\frac{y^{2}}{2}+y\right)+R_{3}(x, t) \tag{42}
\end{array}
$$

From (27) and by (32),(41)

$$
\begin{equation*}
R_{2}(x, t)=\eta_{1 x}+B_{1} \eta_{0 x} \tag{43}
\end{equation*}
$$

From (37) and by (30),(32),(33),

$$
\begin{equation*}
\eta_{2 x}=-\eta_{0 t}-B_{1} \eta_{1 x}-\left(B_{2}-2 \eta_{0}\right) \eta_{0 x}+\phi_{3 y}(x, 0, t) \tag{44}
\end{equation*}
$$

Differentiating (38) about $x$ and by (33), (35), and (42)

$$
\begin{align*}
\eta_{2 x}=\eta_{0 t}-R_{3 x x} & -B_{1} R_{1 x x}+\left(B_{2}-\eta_{0}\right) \eta_{0 x} \\
& +\frac{1}{3} \eta_{1 x x x}+\tau_{1} \eta_{0 x x x} . \tag{45}
\end{align*}
$$

By (34), (35), (40), and (43)

$$
\begin{equation*}
B_{1}=0 \tag{46}
\end{equation*}
$$

By (44), (45), and (46)

$$
\begin{align*}
\frac{1}{3} \eta_{1 x x x}= & -2 \eta_{0 t}-2 B_{2} \eta_{0 x}+3 \eta_{0} \eta_{0 x} \\
& -\tau_{1} \eta_{0 x x x}+R_{3 x x}+\phi_{3 y}(x, 0, t) \tag{47}
\end{align*}
$$

$O\left(\epsilon^{3}\right):$

$$
\begin{align*}
\phi_{2 x x}(x, y, t)+\phi_{3 y y}(x, y, t) & =0,  \tag{48}\\
\phi_{3 y}(x,-1, t) & =B_{0} b_{x} . \tag{49}
\end{align*}
$$

From(48), (49) and by (42), we obtain

$$
\begin{align*}
\phi_{3 y}(x,-1, t)= & \frac{1}{45} \phi_{0 x x x x x}(x, t)-\frac{1}{3} R_{2 x x}(x, t) \\
& +R_{3 x x}(x, t)+\phi_{3 y}(x, 0, t) \tag{50}
\end{align*}
$$

By (32), (33), (43), (46), and (50), we have
$2 \eta_{0 t}+2 B_{2} \eta_{0 x}-3 \eta_{0} \eta_{0 x}+\tau_{1} \eta_{0 x x x}-\frac{1}{45} \eta_{0 x x x x x}=\mathbf{b}_{x}$.
The Froude number $F$ is defined and expanded as

$$
\begin{align*}
F & =F_{0}+\epsilon F_{1}+\epsilon^{2} F_{2}+O\left(\epsilon^{3}\right) \\
& =\frac{\epsilon^{2}}{\lambda} \int_{0}^{\lambda}\left(\frac{B_{0}+\epsilon B_{1}+\epsilon^{2} B_{2}+O\left(\epsilon^{3}\right)}{\epsilon^{2}}+\phi_{0 x}+O(\epsilon)\right) d x \\
& =B_{0}+\epsilon B_{1}+\epsilon^{2} B_{2}+\frac{\epsilon^{2}}{\lambda} \int_{0}^{\lambda} \phi_{0 x} d x+O\left(\epsilon^{3}\right) . \tag{52}
\end{align*}
$$

By (33) and the mean value of periodic solution over a period is zero, we found that

$$
\int_{0}^{\lambda} \phi_{0 x} d x=-\int_{0}^{\lambda} \eta_{0} d x=0
$$

If $\eta_{0}$ is a solitary wave solution with the properity that

$$
\begin{equation*}
\int_{0}^{\infty} \phi_{0 x} d x=-\int_{0}^{\infty} \eta_{0} d x<\infty, \tag{53}
\end{equation*}
$$

then, with $\lambda=\infty$, the term

$$
\frac{1}{\lambda} \int_{0}^{\lambda} \phi_{0 x} d x
$$

in (52) will be zero. We shall see that all the solitary wave solutions discovered in the following chapters will satisfy (53).

Therefore, we have

$$
B_{0}=F_{0}, B_{1}=F_{1}, B_{2}=F_{2} .
$$

and then (51) becomes

$$
\begin{equation*}
2 \eta_{0 t}+2 F_{2} \eta_{0 x}-3 \eta_{0} \eta_{0 x}+\tau_{1} \eta_{0 x x x}-\frac{1}{45} \eta_{0 x x x x}=\mathbf{b}_{x} . \tag{54}
\end{equation*}
$$

Next, we assume $\eta_{0 t}=0$ in equation (54), integrate (54) once and set the constant of integration to be zero, then we have the following model equation

$$
\begin{equation*}
2 F_{2} \eta_{0}-\frac{3}{2} \eta_{0 x}^{2}+\tau_{1} \eta_{0 x x}-\frac{1}{45} \eta_{0 x x x x}=\mathbf{b} \tag{55}
\end{equation*}
$$

In the following sections, we shall use $\eta$ for $\eta_{0}$ in equation (55), that is,

$$
\begin{equation*}
2 F_{2} \eta-\frac{3}{2} \eta^{2}+\tau_{1} \eta_{x x}-\frac{1}{45} \eta_{x x x x}=\mathbf{b} \tag{56}
\end{equation*}
$$

and disscuss the solutions of the model equation (56).

## 3 Problem Formulation

We follow Zufiria [17] to construct a Hamiltonian associated to (56).

When $\mathbf{b}=0$, we rewrite (56) as

$$
\begin{equation*}
\eta_{x x x}-45 \tau_{1} \eta_{x x}-90 F_{2} \eta+\frac{135}{2} \eta^{2}=0 \tag{57}
\end{equation*}
$$

We multiply $-\eta_{x}$ to (57) and integrate the resulting equation, then equation (57) has first integral as

$$
\begin{equation*}
H=45 F_{2} \eta^{2}+\frac{1}{2} \eta_{x x}^{2}-\eta_{x x} \eta_{x}+\frac{45}{2} \tau_{1} \eta_{x}^{2}-\frac{45}{2} \eta^{3}, \tag{58}
\end{equation*}
$$

where $H$ is a constant. Introducing the change of variables

$$
\left.\begin{array}{ll}
q_{1}=\eta, & p_{1}=\eta_{x x x}-45 \tau_{1} \eta_{x,} \\
q_{2}=\eta_{x x}, & p_{2}=\eta_{x,}
\end{array}\right\}
$$

then (58) becomes

$$
\begin{align*}
H\left(q_{1}, q_{2}, p_{1}, p_{2}\right)= & 45 F_{2} q_{1}^{2}+\frac{1}{2} q_{2}^{2} \\
& -p_{1} p_{2}-\frac{45}{2} \tau_{1} p_{2}^{2}-\frac{45}{2} q_{1}^{3} \tag{59}
\end{align*}
$$

and we have

$$
\begin{equation*}
\frac{d z}{d x}=J \nabla_{z} H(z)=A z+g(z) \equiv f(z, \mu) \tag{60}
\end{equation*}
$$

where $\mu=\left(\tau_{1}, F_{2}\right) \in \mathbf{R}^{2}$,

$$
z=\left(\begin{array}{l}
q_{1}  \tag{61}\\
q_{2} \\
p_{1} \\
p_{2}
\end{array}\right) \in \mathbf{R}^{4}, \quad J=\left(\begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{array}\right),
$$

and

$$
A=\left(\begin{array}{cccc}
0 & 0 & 0 & -1  \tag{62}\\
0 & 0 & -1 & -45 \tau_{1} \\
-90 F_{2} & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{array}\right), g(z)=\left(\begin{array}{c}
0 \\
0 \\
\frac{135}{2} q_{1}^{2} \\
0
\end{array}\right) .
$$

Therefore (59) is a two degree of freedom Hamiltonian with two parameters $\tau_{1}$ and $F_{2}$. Because different parameters $\left(\tau_{1}, F_{2}\right)$ in (59) give rise to different eigenvalues $\lambda$ for the linearized system of (60) at the origin, we divide the parameter plane $\left(\tau_{1}, F_{2}\right)$ into following nine cases

Case 0 ( $\tau_{1}=0, F_{2}=0$ ): $\lambda=0,0,0,0$.
Case $1\left(\tau_{1} \in \mathbf{R}, F_{2}>0\right): \lambda= \pm r, \pm w i ; r, w>0$.
Case $2\left(\tau_{1}<0, F_{2}=0\right): \lambda=0,0, \pm w i ; w>0$.
Case $3\left(\tau_{1}<0, F_{2}<0,\left(45 \tau_{1}\right)^{2}+360 F_{2}>0\right)$ :

$$
\lambda= \pm w_{1} i, \pm w_{2} i ; w_{1}>w_{2}>0 .
$$

Case $4\left(\tau_{1}<0, F_{2}<0,\left(45 \tau_{1}\right)^{2}+360 F_{2}=0\right)$ :

$$
\lambda= \pm w i, \pm w i ; w>0
$$

Case $5\left(\tau_{1} \in \mathbf{R}, F_{2}<0,\left(45 \tau_{1}\right)^{2}+360 F_{2}<0\right)$ : $\lambda= \pm a \pm b i ; a, b>0$

Case $6\left(\tau_{1}>0, F_{2}<0,\left(45 \tau_{1}\right)^{2}+360 F_{2}=0\right)$ :

$$
\lambda= \pm r, \pm r ; r>0
$$

Case $7\left(\tau_{1}>0, F_{2}<0,\left(45 \tau_{1}\right)^{2}+360 F_{2}>0\right)$ : $\lambda= \pm r_{1}, \pm r_{2} ; r_{1}>r_{2}>0$
Case $8\left(\tau_{1}>0, F_{2}=0\right): \lambda=0,0, \pm r ; r>0$.
We construct asymptotic solutions in Case 0,2,3, and 4.

## 4 Problem Solution

In this section, we rewrite (56) as follows,

$$
\begin{equation*}
\left.\eta_{x x x x}-45 \tau_{1} \eta_{x x}-90 F_{2} \eta=-45(\mathbf{b}(x))+\frac{3}{2} \eta^{2}\right) \equiv f \tag{63}
\end{equation*}
$$

and assume that there exist small-norm bounded solutions of equation (63) with small-norm bump $\mathbf{b}=\delta b_{1}$ in Case $\mathbf{0 , 2 , 3}$ and 4, and then construct its asymptotic solutions by expanding the solution as

$$
\begin{equation*}
\eta(x)=\delta \eta_{1}(x)+\delta^{2} \eta_{2}(x)+\ldots \tag{64}
\end{equation*}
$$

where $\delta$ is a small parameter. Substituting (64) in (63), we obtain
$O(\delta)$ :

$$
\begin{equation*}
\eta_{l_{\max }}-45 \tau_{1} \eta_{1_{x}}-90 F_{2} \eta_{1}=-45 b_{1}, \tag{65}
\end{equation*}
$$

and $O\left(\delta^{n}\right)$ :

$$
\begin{align*}
& \eta_{n_{\max }}-45 \tau_{1} \eta_{n_{x}}-90 F_{2} \eta_{n} \\
& = \begin{cases}-135\left(\sum_{i=1}^{n-1} 1-1\right. \\
\left.\eta_{i} \eta_{n-i}\right) & , n \text { odd } \\
-135\left(\sum_{i=1}^{-1} \eta_{i} \eta_{n-i}+\frac{1}{2} \eta_{\frac{2}{2}}^{2}\right) & , n \text { even }\end{cases} \tag{66}
\end{align*}
$$

Where $n=2,3$
In the following, we shall construct bounded solutions for the first approximation $\eta_{1}$.

### 4.1 Case 0

In Case 0, $\tau_{1}=0$ and $F_{2}=0$. Thus, equation (65) becomes

$$
\begin{equation*}
\eta_{l_{\max }}=-45 b_{1} . \tag{67}
\end{equation*}
$$

We solve the initial value problem of (67) subject to

$$
\begin{equation*}
\eta_{1}\left(x_{1}\right)=P, \quad \eta_{1_{x}}\left(x_{1}\right)=Q, \quad \eta_{1_{x}}\left(x_{1}\right)=R, \quad \eta_{l_{1 x x}}\left(x_{1}\right)=S \tag{68}
\end{equation*}
$$

Where $P, Q, R, S$ are constants and obtain

$$
\begin{align*}
\eta_{1}(x)= & \frac{1}{6}\left(6 P-6 Q x_{1}+3 R x_{1}^{2}-S x_{1}^{3}\right) \\
& +\frac{1}{2}\left(2 Q-2 R x_{1}+S x_{1}^{2}\right) x+\frac{1}{2}\left(R-S x_{1}\right) x^{2} \\
& +\frac{S}{6} x^{3}+\int_{x_{1}}^{x} G(x, t)\left(-45 b_{1}(t)\right) d t \tag{69}
\end{align*}
$$

where

$$
\begin{equation*}
G(x, t)=\frac{1}{6}(x-t)^{3} \tag{70}
\end{equation*}
$$

is the causal Green's function of (67) subject to (68) with $P=Q=R=S=0$.

For $\eta_{1}(x)$ to be bounded for $x \leq x_{1}$, the only case is that $\eta_{1}(x)=P$ for $x \leq x_{1}$ with $Q=R=S=0$. The integral term in (69) is bounded since $b_{1}(x)$ is compact on the interval $\left[x_{1}, x_{2}\right]$. Therefore, to have $\eta_{1}(x)$ to be bounded for $x \geq x_{2}$, we also need $\eta_{1}(x)$ to be a constant, that is, $\eta_{1}^{\prime}\left(x_{2}\right)=\eta_{1}^{\prime \prime}\left(x_{2}\right)=\eta_{1}^{\prime \prime \prime}\left(x_{2}\right)=0$. After some algebra, we obtain $\int_{x_{1}}^{x_{2}} b_{1}(t) d t=0, \int_{x_{1}}^{x_{2}} t b_{1}(t) d t=0, \int_{x_{1}}^{x_{2}} t^{2} b_{1}(t) d t=0$.
Thus, if the bump $b_{1}(t)$ satisfies (71), then we can rewrite bounded $\eta_{1}$ as

$$
\eta_{1}(x)=\left\{\begin{array}{lc}
P, & x \leq x_{1},  \tag{72}\\
P+\int_{x_{1}}^{x} G(x, t)\left(-45 b_{1}(t)\right) d t, & x_{1}<x<x_{2}, \\
P+\int_{x_{1}}^{x_{2}} G\left(x_{2}, t\right)\left(-45 b_{1}(t)\right) d t, & x_{2} \leq x,
\end{array}\right.
$$

which is constant for $x_{1} \geq x$ and $x_{2} \leq x$. There are infinitely many $b_{1}$ that satisfy $(71)$ and $b_{1}(-1)=b_{1}(1)=0$.

### 4.2 Case 2

In Case 2, $\tau_{1}<0$ and $F_{2}=0$. Thus, equation (65)
becomes

$$
\begin{equation*}
\eta_{l_{\max }}-45 \tau_{1} \eta_{l_{1 x}}=-45 b_{1} . \tag{73}
\end{equation*}
$$

We solve the initial value problem of (73) subject to initial conditions (68) and obtain

$$
\begin{align*}
\eta_{1}(x)= & P+\frac{R-\left(S+Q w^{2}\right) a}{w^{2}}+\left(Q+\frac{S}{w^{2}}\right) x-\frac{R}{w^{2}} \cos (w(x-a)) \\
& -\frac{S}{w^{3}} \sin (w(x-a))+\int_{a}^{x} G(x, t)\left(-45 b_{1}(t)\right) d t \tag{74}
\end{align*}
$$

where

$$
\begin{equation*}
G(x, t)=\frac{x-t}{w^{2}}-\frac{\sin (w(x-t))}{w^{3}} . \tag{75}
\end{equation*}
$$

is the causal Green's function of (73) subject to (68) with $P=Q=R=S=0$.

The integral term in (74) is bounded since $b_{1}(x)$ is compact on the interval $\left[x_{1}, x_{2}\right]$. Therefore, to have $\eta_{1}(x)$ to be bounded for all $x \geq a$, we need the cofficient of $x$ term in (74) to be zero, thus

$$
\begin{equation*}
Q w^{2}+S=0 . \tag{76}
\end{equation*}
$$

In addition, $R$ must be zero if we require $\eta_{1}(x)$ to be bounded when $a \rightarrow-\infty$. Now, we rewrite the bounded $\eta_{1}$ as

$$
\eta_{1}(x)= \begin{cases}P-\frac{s}{w^{3}} \sin (w(x-a)) & , x \leq x_{1},  \tag{77}\\ P-\frac{s}{w^{3}} \sin (w(x-a))+\int_{a}^{x} G(x, t)\left(-45 b_{1}(t)\right) d t, & x_{1}<x<x_{2}, \\ P-\frac{s}{w^{3}} \sin (w(x-a))+\int_{a}^{x} G\left(x_{2}, t\right)\left(-45 b_{1}(t)\right) d t, x_{2} \leq x,\end{cases}
$$

which is periodic for $x_{1} \geq x$ and $x_{2} \leq x$ and satisfy the initial conditions at $x=-\infty$ with the following properties :

$$
\begin{equation*}
\eta_{1}^{\prime}(-\infty) w^{2}+\eta_{1}^{\prime \prime \prime}(-\infty)=0, \eta_{1}^{\prime \prime}(-\infty)=0, \tag{78}
\end{equation*}
$$

and $\eta_{1}(-\infty)=P$, an arbitrary constant.

### 4.3 Case 3 with $w_{1} / w_{2}$ rational

In this subsection, the eigenvalues of linearization of equation (63) in Case 3 are $\pm w_{1} i$ and $\pm w_{2} i$ and we assume that the ratio $w_{1} / w_{2}=n_{1} / n_{2}$ is a rational number with $\left(n_{1}, n_{2}\right)=1, n_{1}, n_{2} \in N$. To obtain an asymptotic solution, we solve the initial value problem of (65) subject to initial conditions (68) and obtain

$$
\begin{equation*}
\eta_{1}(x)=Y(x)+\int_{x_{1}}^{x} G(x, t)\left(-45 b_{1}(t)\right) d t . \tag{79}
\end{equation*}
$$

Where

$$
\begin{aligned}
& Y(x)=A \cos \left(w_{2} x\right)+B \sin \left(w_{2} x\right)+C \cos \left(w_{1} x\right)+D \sin \left(w_{1} x\right) \\
& w_{1}=\left(-\left(45 \tau_{1}-\left(\left(45 \tau_{1}\right)^{2}+360 F_{2}\right)^{\frac{1}{2}}\right)\right)^{\frac{1}{2}} \\
& w_{2}=\left(-\left(45 \tau_{1}+\left(\left(45 \tau_{1}\right)^{2}+360 F_{2}\right)^{\frac{1}{2}}\right)\right)^{\frac{1}{2}}, \\
& A=\frac{1}{\left(w_{1}^{2}-w_{2}^{2}\right)}\left\{\frac{-\left(Q w_{1}^{2}+S\right) \sin \left(a w_{2}\right)}{w_{2}}+\left(P w_{1}^{2}+R\right) \cos \left(a w_{2}\right)\right\}, \\
& B=\frac{1}{\left(w_{1}^{2}-w_{2}^{2}\right)}\left\{\frac{\left(Q w_{1}^{2}+S\right) \cos \left(a w_{2}\right)}{w_{2}}+\left(P w_{1}^{2}+R\right) \sin \left(a w_{2}\right)\right\}, \\
& C=\frac{1}{\left(w_{1}^{2}-w_{2}^{2}\right)}\left\{\frac{\left(Q w_{2}^{2}+S\right) \sin \left(a w_{1}\right)}{w_{1}}-\left(P w_{2}^{2}+R\right) \cos \left(a w_{1}\right)\right\}, \\
& D=\frac{1}{\left(w_{1}^{2}-w_{2}^{2}\right)}\left\{\frac{-\left(Q w_{2}^{2}+S\right) \cos \left(a w_{1}\right)}{w_{1}}-\left(P w_{2}^{2}+R\right) \sin \left(a w_{1}\right)\right\}
\end{aligned}
$$

and the causal Green's is

$$
G(x, t)=\left\{w_{2} \sin \left(w_{1}(x-t)\right)-w_{1} \sin \left(w_{2}(x-t)\right)\right\} /\left(w_{2} w_{1}\left(w_{2}^{2}-w_{1}^{2}\right)\right) .
$$

Note that $\eta_{1}(x)$ in (79) is bounded since $b_{1}(x)$ is compact on the interval $\left[x_{1}, x_{2}\right]$.

Now, we rewrite the bounded $\eta_{1}$ as

$$
\eta_{1}(x)= \begin{cases}Y(x) & , x \leq x_{1},  \tag{80}\\ Y(x)+\int_{x_{1}}^{x} G(x, t)\left(-45 b_{1}(t)\right) d t & , x_{1}<x<x_{2}, \\ Y(x)+\int_{x_{1}}^{x_{2}} G\left(x_{2}, t\right)\left(-45 b_{1}(t)\right) d t, & , x_{2} \leq x,\end{cases}
$$

which is periodic for $x_{1} \geq x$ and $x_{2} \leq x$ with period $T=2 n_{1} \pi / w_{1}=2 n_{2} \pi / w_{2}$.

### 4.4 Case 4

In Case 4, we solve the initial value problem of (65) subject to initial conditions (68) and obtain

$$
\begin{equation*}
\eta_{1}(x)=Y(x)+\int_{x_{1}}^{x} G(x, t)\left(-45 b_{1}(t)\right) d t . \tag{81}
\end{equation*}
$$

where

$$
Y(x)=(A+B x) \cos (r x)+(C+D x) \sin (r x)
$$

with
$r=\sqrt{-45 \tau_{1} / 2}$,
$A=M \cos \left(x_{1} r\right)-N \sin \left(x_{1} r\right)$,
$B=-\left\{\left(Q r_{2}+S\right) \cos \left(x_{1} r\right) / r+\left(P r^{2}+R\right) \sin \left(x_{1} r\right)\right\} / r$,
$C=M \sin \left(x_{1} r\right)+N \cos \left(x_{1} r\right)$,
$D=-\left\{\left(Q r_{2}+S\right) \sin \left(x_{1} r\right) / r+\left(P r^{2}+R\right) \cos \left(x_{1} r\right)\right\} / r$,
$M=P+x_{1}\left(Q r^{2}+S\right) /\left(2 r^{2}\right)$,
$N=\left\{\left(Q r^{2}+S\right) /\left(2 r^{2}\right)-x_{1}\left(P r^{2}+R\right) / 2+Q\right\} / r$,
and
$G(x, t)=(\sin (r(x-t))-r(x-t) \cos (r(x-t))) /\left(2 r^{3}\right)$.
The integral term in (81) is bounded since $b_{1}(x)$ is compact on the interval $\left[x_{1}, x_{2}\right]$, To have bounded $\eta(x)$ in (81), we need $B=D=0$. After some algebra, we obtain

$$
\begin{equation*}
R+r^{2} P=0, \quad S+r^{2} Q=0 \tag{82}
\end{equation*}
$$

Now, we rewrite the bounded $\eta_{1}$ as

$$
\eta_{1}(x)= \begin{cases}Y(x) & , x \leq x_{1}  \tag{83}\\ Y(x)+\int_{x_{1}}^{x} G(x, t)\left(-45 b_{1}(t)\right) d t & , x_{1}<x<x_{2} \\ Y(x)+\int_{x_{1}}^{x_{2}} G\left(x_{2}, t\right)\left(-45 b_{1}(t)\right) d t & , x_{2} \leq x\end{cases}
$$

which is periodic for $x_{1} \geq x$ and $x_{2} \leq x$ if (82) is satisfied.

## 5 Unsymmetric solitary wave solutions for Case $1,5,6,7$, and 8

In this section, we shall construct unsymmetric solitary wave solutions of the model equation (63) for Case 1, 5, 6,7 and 8.

Our idea is to investgate the solutions of equation (63) on three different intervals $\left(-\infty,-T_{1}\right),\left[-T_{1}, T_{2}\right]$, and $\left(T_{2}, \infty\right)$, where $T_{1}$ and $T_{2}$ are positive constants and will be specified later. On intervals $\left(-\infty,-T_{1}\right)$ and $\left(T_{2}, \infty\right)$, we try
to show that equation (64) with initial values at $x=-T_{1}$ on $\left(-\infty,-T_{1}\right)$ and initial values at $x=T_{2}$ on $\left(T_{2}, \infty\right)$ has bounded solutions $\eta_{L}(x)$ and respectively, which decay to zero exponentially at negative and positive infinity by using a theorem from [6]. On $\left[-T_{1}, T_{2}\right]$, we shall use Schauder fixed point theorem to prove there exist bounded solutions $\eta_{C}(x)$ of equation (64) subject to initial values $\left(\eta_{L}\left(-T_{1}\right),-\eta_{L}^{\prime}\left(-T_{1}\right),-\eta_{L}^{\prime \prime \prime}\left(-T_{1}\right)\right)$ at $x=-T_{1}$. Then we combine $\eta_{C}(x), \eta_{L}(x)$, and $\eta_{R}(x)$ to obtain a solution of equation (63).

### 5.1 Solutions on $\left(-\infty,-T_{1}\right)$ and $\left(T_{2}, \infty\right)$

On interval $\left(T_{2}, \infty\right)$, we rewrite (64) as a system of first order differential equations,

$$
\begin{equation*}
\frac{d z}{d x}=A z+g(z) \tag{84}
\end{equation*}
$$

where $z(x)=\left(\eta(x), \eta^{\prime}(x), \eta^{\prime \prime}(x), \eta^{\prime \prime \prime}(x)\right)^{t}$,

$$
A=\left(\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
90 F_{2} & 0 & 45 \tau_{1} & 0
\end{array}\right)
$$

and

$$
g(x, z)=\left(\begin{array}{c}
0  \tag{85}\\
0 \\
0 \\
-\frac{135}{2} \eta^{2}-45 \mathrm{~b}(x)
\end{array}\right)
$$

In the following, we shall use a theorem from [6] to prove that (84) with some restriction on the initial values at $x=T_{2}$ has bounded solutions. The theorem is stated as follows :

We consider the asymptotic behavior of the solutions of equation

$$
\begin{equation*}
\frac{d z}{d x}=A z+f(x, z) \tag{86}
\end{equation*}
$$

where $A$ is a constant matrix and $f$ is a continuous vector function defined for $x \geq x_{0},|z|<c$. Then the underlying vector space $X$ can be uniquely represented as the direct sum of three suspaces $X_{-1}, X_{0}, X_{1}$ invariant under $A$ on which all characteristic roots of $A$ have real parts respectively less than, equal to, greater than $\mu$. We shall denote by $P_{i}$ the corresponding projection of $X$ onto $X_{i}(i=-1,0,1)$.

Theorem 1 Suppose that at least one characteristic root of $A$ has real part $\mu<0$ and

$$
\begin{equation*}
f(x, z)=o(|z|) \quad \text { for } \quad x \rightarrow \infty,|z| \rightarrow 0 \tag{87}
\end{equation*}
$$

holds.
Then there exist positive constants $k, K$ depending only on $A$ and positive constants $T, \rho$ depending also on $f$ such that if $x^{*} \geq T$ and if $\xi_{-1} \in X_{-1}, \xi_{0} \in X_{0}$ satisfy

$$
\begin{equation*}
\left|\xi_{-1}\right| \leq k\left|\xi_{0}\right|, \quad 0<\left|\xi_{0}\right|<\frac{\rho}{2 K}, \tag{88}
\end{equation*}
$$

then the equation (86) has at least one solution $z(x)$ for $x \geq x^{*}$ satisfying

$$
\begin{equation*}
P_{-1} z\left(x^{*}\right)=\xi_{-1}, \quad P_{0} z\left(x^{*}\right)=\xi_{0}, \tag{89}
\end{equation*}
$$

$|z(x)| \leq \rho$ for $x \geq x^{*}$ and

$$
\begin{equation*}
\mu=\lim _{x \rightarrow \infty} x^{-1} \log |z(x)| \cdot \tag{90}
\end{equation*}
$$

For each of Case 1, 5, 6, 7, and 8, there exists at least one eigenvalue with negative real part and $g(x$, $z$ ) in (85) satisfies (87) since $\mathbf{b}(x)$ is compact on $\left[x_{1}, x_{2}\right]$. Hence, by Theorem 1, there are bounded solutions $z_{R}(x)$ of equation (84) subject to the initial values $z_{R}\left(T_{2}\right)$ that satisfy (88) and (89) with $T_{2} \geq T$. Then we have $\eta_{R}(x)$, the first component of $z_{R}(x)$, as the solution of (63) subject to the initial values $z_{R}\left(T_{2}\right)=\left(\eta_{R}\left(T_{2}\right), \quad \eta_{R}^{\prime}\left(T_{2}\right), \quad \eta_{R}^{\prime \prime}\left(T_{2}\right), \quad \eta_{R}^{\prime \prime \prime}\left(T_{2}\right)\right)^{t}$ on interval $\left(T_{2}, \infty\right)$.

For interval $\left(-\infty,-T_{1}\right)$, we let $x=-\hat{x}$ and put it in (63), then equation (63) does not change except that the independent variable is replaced by $\hat{x}$. Thus, by Theorem 1 again, there exist bounded solutions $z_{L}(\hat{x})$ of equation (84) subject to the initial value $z_{L}\left(T_{1}\right)=\left(\eta_{L}\left(T_{1}\right), \eta_{L}^{\prime}\left(T_{1}\right), \eta_{L}^{\prime \prime}\left(T_{1}\right), \eta_{L}^{\prime \prime \prime}\left(T_{1}\right)\right)$ that satisfy (88) and (89) with $T_{1} \geq T$. Hence, by substituting $\hat{x}=-x$, we obtain $\eta_{L}(x)$, the first component of $z_{L}(x)$, to be the solution of (63) subject to the initial values $z_{L}\left(-T_{1}\right)=\left(\eta_{L}\left(-T_{1}\right), \quad-\eta_{L}^{\prime}\left(-T_{1}\right), \quad \eta_{L}^{\prime \prime}\left(-T_{1}\right), \quad-\eta_{L}^{\prime \prime \prime}\left(-T_{1}\right)\right)^{t} \quad$ on interval $\left(-\infty,-T_{1}\right)$.

Next, we shall prove there is a bounded solution $\eta_{C}(x)$ of (63) subject to initial value $\left(\eta_{L}\left(-T_{1}\right)\right.$, $\left.-\eta_{L}^{\prime}\left(-T_{1}\right), \quad \eta_{L}^{\prime \prime}\left(-T_{1}\right), \quad-\eta_{L}^{\prime \prime \prime}\left(-T_{1}\right)\right)$ at $x=-T_{1}$ on interval $\left[-T_{1}, T_{2}\right]$ and the end point value, $\left(\eta_{C}\left(T_{2}\right), \eta_{C}^{\prime}\left(T_{2}\right)\right.$, $\left.\eta_{C}^{\prime \prime}\left(T_{2}\right), \eta_{C}^{\prime \prime \prime}\left(T_{2}\right)\right)$, which also satisfies (88) and (89).

### 5.2 Solutions on [ $-T_{1}, T_{2}$ ]

From (63) and posing initial values at $x=-T_{1}$, we have:

$$
\begin{gather*}
\eta_{x x x}-45 \tau_{1} \eta_{x x}-90 F_{2} \eta=-45\left(\mathbf{b}(x)+\frac{3}{2} \eta^{2}\right) \equiv f(\eta), \quad x \geq-T_{1}, \\
\eta\left(-T_{1}\right)=P, \quad \eta_{x}\left(-T_{1}\right)=Q \\
\eta_{x x}\left(-T_{1}\right)=R, \quad \eta_{x x x}\left(-T_{1}\right)=S \tag{91}
\end{gather*}
$$

where $P=\eta_{L}\left(-T_{1}\right), Q=-\eta_{L}^{\prime}\left(-T_{1}\right), R=\eta_{L}^{\prime \prime}\left(-T_{1}\right), S=-\eta_{L}^{\prime \prime \prime}\left(-T_{1}\right)$.
To analyze the solutions of (91), we transform the ordinary differential equation (91) to an integral equation. First we solve the homogeneous equation of (91) :

$$
\begin{gather*}
Y_{x x x}-45 \tau_{1} Y_{x x}-90 F_{2} Y=0, \quad x \geq-T_{1} \\
Y\left(-T_{1}\right)=P, Y_{x}\left(-T_{1}\right)=Q, Y_{x x}\left(-T_{1}\right)=R, Y_{x x x}\left(-T_{1}\right)=S \tag{92}
\end{gather*}
$$

Next, we use $Y(x)$ in (92) and let $\eta=S+Y$ to convert equation (91) as follows :

$$
\begin{gather*}
S_{x x x}-45 \tau_{1} S_{x x}-90 F_{2} S=f, \quad x \geq-T_{1} \\
S\left(-T_{1}\right)=0, S_{x}\left(-T_{1}\right)=0, S_{x x}\left(-T_{1}\right)=0, S_{x x x}\left(-T_{1}\right)=0 \tag{93}
\end{gather*}
$$

Let the causal Green's function of equation (93) be $G(x, t)$, then we have

$$
\begin{equation*}
S(x)=\int_{-T_{1}}^{\infty} G(x, t) f(\eta(t)) d t \tag{94}
\end{equation*}
$$

Thus we transform the differential equation (91) to the integral equation:

$$
\begin{align*}
\eta(x)=Y(x)+ & \int_{-T_{1}}^{x} G(x-t) \\
& \left\{-45\left(\mathbf{b}(t)+\frac{3}{2} \eta^{2}(t)\right)\right\} d t=Q(\eta)(x) \tag{95}
\end{align*}
$$

To prove the existence of a bounded solution of equation (63) initiating at $x=-T_{1}$ on the interval $\left[-T_{1}, T_{2}\right]$, we need to show that the operator defined by the right-hand side of (95) has a fixed point. In other words, we try to find a function $\hat{\eta}$ such that $Q(\hat{\eta})(x)=\hat{\eta}(x)$ for all $x \in\left[-T_{1}, T_{2}\right]$. We take the domain of $Q$ to be

$$
\begin{equation*}
K=\left\{p_{2} \in C\left(\left[-T_{1}, T_{2}\right] ; \mathbf{R}\right)| | \eta(x) \mid \leq M \text { for } x \in\left[-T_{1}, T_{2}\right]\right\} \text {, } \tag{96}
\end{equation*}
$$

where $M$ is some positive real number and should be chosen in such a way that $Q$ maps $K$ into itself.

It is clear that the function $x \mapsto Q(\eta)(x)$ is continuous. In order to prove that $Q$ maps $K$ into itself it remains only to analyze the size of $|Q(\eta)(x)|$. If $\eta \in K$, then we have for all $x \in\left[-T_{1}, T_{2}\right]$

$$
\begin{equation*}
|Q(\eta)(x)| \leq M_{Y}+M_{G} M_{x}\left(M_{\mathbf{b}}+\frac{3}{2} M^{2}\right) \tag{97}
\end{equation*}
$$

where

$$
M_{Y}=\max _{x \in\left[-T_{1}, T_{2}\right]}|Y(x)|, \quad M_{G}=\max _{x, t \in\left[-T_{1}, T_{2}\right]}|G(x, t)|
$$

and

$$
M_{x}=x_{2}-x_{1}, \quad M_{\mathbf{b}}=\sup _{x \in\left[-T_{1}, T_{2}\right]}|\mathbf{b}(x)| \cdot
$$

If we assume that the right-hand side of $(97) \leq M$, then we have

$$
\frac{3}{2} M_{G}\left(M-M^{+}\right)\left(M-M^{-}\right) \leq 0
$$

where

$$
\begin{equation*}
M^{ \pm}=\frac{1 \pm\left(1-6 M_{G} M_{x}\left(M_{G} M_{\mathbf{b}} M_{x}+M_{Y}\right)\right)^{\frac{1}{2}}}{3 M_{G} M_{x}} \tag{98}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(1-6 M_{G} M_{x}\left(M_{G} M_{\mathbf{b}} M_{x}+M_{Y}\right)\right) \geq 0 \tag{99}
\end{equation*}
$$

The inequality (99) can be satisfied if we choose bump $\mathbf{b}$ and the initial values in (91) such that both $M_{\mathbf{b}}$ and $M_{Y}$ are sufficiently small. Hence, if we take $M \in\left[M^{-}, M^{+}\right]$and inequality (99) is also satisfyied, then $|Q(\eta)(x)| \leq M$ for all $-T_{1} \leq x \leq T_{2}$, and $Q$ maps $K$ into itself.

The set $K$ is a bounded, closed, and convex subset of the Banach space $C\left(\left[-T_{1}, T_{2}\right]\right)$. To apply Schauder's theorem it suffices, therefore, to show that $Q$ is a compact map of $K$ into itself. By the Arzelà-Ascoli Theorem and by what we have already proved, this amounts to showing that the set $\{Q(\eta) \mid \eta \in K\}$ is equicontinuous. The following simple estimate accomplishes the task. Let $-T_{1} \leq \xi \leq x$, then

$$
\begin{aligned}
& \mid Q(\eta)(x)-Q(\eta)(\xi)|\leq|Y(x)-Y(\xi)| \\
&+\left|\int_{\xi}^{x} G(x-t) f(\eta(t)) d t\right| \\
&+\left|\int_{-T_{1}}^{\xi}(G(x-t)-G(\xi-t)) f(\eta(t)) d t\right| \\
& \quad \leq|Y(x)-Y(\xi)|+ \\
& \quad \sup _{|\eta| \leqslant M}|f(\eta)|\left\{\int_{0}^{x-\xi}|G(x-\xi-t)| d t\right\}+ \\
&\left.\quad \sup _{|\eta| \leqslant M}|f(\eta)|\left\{\int_{0}^{\xi+T_{1}} \mid G(x-\xi+t)-G(t)\right) \mid d t\right\} .
\end{aligned}
$$

Since the function $Y$ and $G$ are continuous, we conclude that the set $\{Q(\eta) \mid \eta \in K\}$ is equicontinuous on $\left[-T_{1}, T_{2}\right]$. An application of the Schauder Theorem tells us that there exists a fixed point $\eta_{C}$ of $Q$.

To combine $\eta_{L}(x), \eta_{C}(x)$ and $\eta_{R}(x)$ to be a solution of equation (63), it requires that the end point values, $\left(\eta_{C}\left(T_{2}\right), \eta_{C}^{\prime}\left(T_{2}\right), \eta_{C}^{\prime \prime}\left(T_{2}\right), \eta_{C}^{\prime \prime \prime}\left(T_{2}\right)\right)$ which will be used as the initial values of $z_{R}(x)$ on $\left(x_{2}, \infty\right)$, satisfy (88) and (89) in Theorem 1. This needs the right hand side of (97) to be small and this could be done by having $M_{Y}, M_{\mathrm{b}}$ and $M$ sufficiently small. Observing (98), the positive number $M^{-}$could be as
small as we want by choosing sufficiently $\operatorname{small}_{M_{\mathrm{b}}}$ and $M_{Y}$, and thus $M$ could be as small as required. Therefore, we obtain an unsymmetric solitary wave solution of (63) in Case 1, 5, 6, 7, and 8.

## 6 Numerical Experiment

In this section, we shall give asymptotic solution numerically of equation (63) by using classical fourthorder Runge-Kutta method. (See Figure 1-3).


Figure 1: An asymptotic solution of equation (63) obtained by using classical fourth-order Runge-Kutta method in equation (67) for Case $\mathbf{0}$ with compact bump $b(x)=10^{-6}\left(7 x^{5}-10 x^{3}+3 x\right)$ on interval $(-1,1)$.


Figure 2: An asymptotic solution of equation (63) obtained by using classical fourth-order Runge-Kutta method in equation (73) for Case 2 with $\tau_{1}=-1, F_{2}=0$, and compact bump $b(x)=10^{-5}\left(7 x^{5}-10 x^{3} \quad+3 x\right)$ on interval $(-1,1)$.


Figure 3: An asymptotic solution of equation (63) obtained by using classical fourth-order Runge-Kutta method in equation (65) for Case 3 with $\tau_{1}=-1$, $F_{2}=-810 / 169 \approx-4.793$ which such that $w_{1} / w_{2}=3 / 2$ is a rational number, and compact bump $b(x)=10^{-5} \exp$ $\left(1 /\left(x^{2}-1\right)\right)$ on interval $(-1,1)$.


Figure 4: An asymptotic solution of equation (63) obtained by using classical fourth-order RungeKutta method in equation (65) for Case 4 with $\tau_{1}=-1, \quad F_{2}=-45 / 8=-5.625$, and compact bump $b(x)=10^{-5} \exp \left(1 /\left(x^{2}-1\right)\right)$ on interval $(-1,1)$.


Figure 5: An unsymmetric solution of equation (63) for Case 1 with $\tau_{1}=-1, F_{2}=1$, and compact bump $b(x)=10^{-18} \exp \left(1 /\left(x^{2}-1\right)\right)$ on interval $(-1,1)$.


Figure 6: An unsymmetric solution of equation (63) for Case 5 with $\tau_{1}=-1, F_{2}=-6$, and compact bump $b(x)=\exp \left(1 /\left(x^{2}-1\right)\right) \sin (x)$ on interval $(-1,1)$.


Figure 7: An unsymmetric solution of equation (63) for Case 6 with $\tau_{1}=1, F_{2}=-45 / 8=-5.625$, and compact bump $b(x)=\exp \left(1 /\left(x^{2}-1\right)\right) \sin (x)$ on interval $(-1,1)$.


Figure 8: An unsymmetric solution of equation (63) for Case 7 with $\tau_{1}=1, F_{2}=-810 / 169 \approx-4.793$, and compact bump $b(x)=\exp \left(1 /\left(x^{2}-1\right)\right) \sin (x)$ on interval ( $-1,1$ ).


Figure 9: An unsymmetric solution of equation (63) for Case 8 with $\tau_{1}=-1, F_{2}=0$, and compact bump $b(x)=10^{-19} \exp \left(1 /\left(x^{2}-1\right)\right) \sin (x)$ on interval $(-1,1)$.

## 7 Conclusion

We constructed asymptotic solutions and unsymmetric solutions of model equation (56) for a sufficiently smooth compact bump $\mathbf{b}(x)$ and has a compact support on the inteval $\left[x_{1}, x_{2}\right]$ with $\mathbf{b}\left(x_{1}\right)=$ $\mathbf{b}\left(x_{2}\right)=0$. The numerical experiment in section 6 confirm the the constructed asymptotic solutions and unsymmetric solutions in section 4 and 5.

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